Korean J. Math. **32** (2024), No. 4, pp. 703–725
https://dx.doi.org/10.11568/kjm.2024.32.4.703

ON TAUBERIAN CONDITIONS FOR WEIGHTED GENERATORS OF TRIPLE SEQUENCES

Asif Hussain Jan* and Tanweer Jalal

Abstract. This paper introduces a novel perspective on how the $(\bar{N}, p, q, r)$ method relates to P-convergence for triple sequences. Our main objective is to establish Tauberian conditions that govern the behavior of the weighted generator sequence (z_{lmn}) concerning the sequences (P_l), (Q_m), and (R_n), aiming to offer a fresh interpretation. These conditions manage the O_L- and O-oscillatory properties and establish a link from $(\bar{N}, p, q, r)$ summability to P-convergence, contingent upon specific constraints on the weight sequences. Furthermore, we demonstrate that particular instances, such as the O_L-condition of Landau type and the O-condition of Hardy type concerning (P_l), (Q_m), and (R_n), serve as Tauberian conditions for $(\bar{N}, p, q, r)$ summability under additional conditions. Thus, our findings encompass traditional Tauberian theorems, including conditions related to gradual decline and slow oscillation in specific scenarios.

1. Introduction

Understanding the origins and development of ideas from the late nineteenth century that expanded summability theory from single to multiple sequences is a challenging endeavor. Before 1990, when Pringsheim published his article "Zur Theorie der zweifach unendlichen Zahlenfolgen" (On the Theory of Doubly Infinite Sequences) [26], it seemed that no researchers were actively working on the theory of multiple sequences. In his paper, Pringsheim introduced the concept of P-convergence, which was further explored by Hardy [10] and Bromwich [5] in their detailed study of double sequences. This work significantly advanced research on this new type of sequences. The earliest known contribution to the application of weighted mean methods to double sequences comes from Baron and Stadtmüller [1]. They examined the relationship between the $(\bar{N}, p, q)$ method and P-convergence for double sequences, identifying necessary conditions for the (bounded) P-convergence of a double sequence that is (boundedly) $(\bar{N}, p, q, r)$ summable, expressed through Hardy-type O-conditions relative to $P = (P_l)$, $Q = (Q_m)$, and $R = (R_n)$:

$$\sup_{mn\in\mathbb{N}} (\Delta_{100} z_{lmn}) = O\left(\frac{p_l}{P_l}\right), \sup_{ln\in\mathbb{N}} (\Delta_{010} z_{lmn}) = O\left(\frac{q_m}{Q_m}\right) \text{ and } \sup_{lm\in\mathbb{N}} (\Delta_{001} z_{lmn}) = O\left(\frac{r_n}{R_n}\right)$$

Received July 16, 2024. Revised September 23, 2024. Accepted September 29, 2024.
2010 Mathematics Subject Classification: 40A05, 40E05, 40G99.
Key words and phrases: Triple sequences, $(\bar{N}, p, q, r)$ summability, Slowly decreasing sequences, Slowly oscillating sequences, Weighted generator sequences, Tauberian conditions and theorems, Weighted mean summability method.
∗ Corresponding author.

with the sequences (P_l), (Q_m), and (R_n) being regularly varying. Stadtmüller [29] built on this by generalizing O_L-Tauberian conditions given by Móricz [23] for the $(C,1,1)$ method, showing that these conditions could be relaxed. Subsequently, Mishra et al. [20–22], Chen and Hsu [6] developed Tauberian theorems for double sequences, addressing the implications of $(\bar{N},p,q)$ summability to P-convergence under Landau-type conditions, Schmidt-type slow decrease conditions, and more general frameworks involving deferred means. Móricz and Stadtmüller [24] reduced the assumptions previously made by Stadtmüller [29], investigating the necessary conditions for (boundedly) $(\bar{N},p,q)$ summable double sequences to be (boundedly) P-convergent using classes Λ_z and Λ_ℓ based on non-factorable weights. Belen [2] introduced the concept of double weighted generator sequences and demonstrated that certain conditions involving these sequences, such as

$$\Delta_{10}V_{mn}^{11^{(0)}}\left(\Delta_{11}(u)\right) = O_L\left(\frac{p_m}{P_{m-1}}\right) \text{ and } \Delta_{01}V_{mn}^{11^{(0)}}\left(\Delta_{11}(u)\right) = O_L\left(\frac{q_n}{Q_{n-1}}\right)$$

constitute Tauberian conditions for the $(\bar{N},p,q)$ method, with additional conditions on the weight sequences (p_m) and (q_n). Önder et al. [25] introduced weighted generator of double sequences and its Tauberian conditions..

This paper investigates the relationship between the $(\bar{N},p,q,r)$ method and P-convergence for triple sequences. Our goal is to establish Tauberian conditions governing the behavior of the weighted generator sequence (z_{lmn}) relative to (P_l), (Q_m), and (R_n), using O_L- and O-oscillation. We explore the transition from $(\bar{N},p,q,r)$ summability to P-convergence, imposing specific restrictions on the weight sequences. Within this framework, we demonstrate that certain conditions, such as the O_L-condition of Landau type relative to (P_l), (Q_m), and (R_n), and the O-condition of Hardy type relative to (P_m), (Q_m), and (R_n), can be seen as Tauberian conditions for $(\bar{N},p,q,r)$ summability under additional conditions. These findings encompass classical Tauberian theorems, including those related to slow decrease and slow oscillation in specific contexts.

2. Preliminaries

In this section, we will start by providing basic definitions and notations related to double sequences and their weighted means. We will then introduce the weighted generator sequences and the weighted Kronecker identities, which are based on the sequence (z_{lmn}). We will discuss their weighted means and generator ones in certain senses, as well as the weighted de la Vallée Poussin means for triple sequences.

Additionally, we will introduce the concepts of slow decrease relative to (P_l), (Q_m), and (R_n), as well as slow oscillation relative to (P_m), (Q_m), and (R_n) for triple sequences. We will demonstrate how a relationship exists between these newly described concepts.

Finally, we will conclude this section by identifying the class SVA, providing its characterization, and discussing two of its subclasses.

Let $\mathbb{K} := \mathbb{R}$ or $\mathbb{K} := \mathbb{C}$ be the field of all real or complex numbers, respectively. Further, let $\mathbb{N}$ be the set of all nonnegative integers.

The function $X : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(C)$ allows for the creation of a sequence of triples consisting of real or complex numbers. Initially, Sahiner et al. [28] introduced and examined various concepts related to triple sequences and their statistical convergence. For the further information on triple sequence, refer ([8, 15–19, 23, 24, 26, 30, 31]).

If the set $w^3(\mathbb{K})$ denotes the set of all triple sequences, then $w^3(\mathbb{K})$ together with coordinate-wise addition and scalar multiplication defined by $((z_{lmn}), (\omega_{lmn})) \to (z_{lmn} + \omega_{lmn})$ and $(\lambda, (z_{lmn})) \to (\lambda z_{lmn})$,

$$w^3(\mathbb{K}) = \mathbb{K}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} := \{z = (z_{lmn}) \mid z : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{K}, (l,m,n) \to z(l,m,n) := z_{lmn}\}$$

is a linear space over $\mathbb{K}$. Each linear subspace of $w^3(\mathbb{K})$ is called a triple sequence space. Besides, the following subsets of $w^3(\mathbb{K})$ are obviously triple sequences spaces:

$$c^3(\mathbb{K}) := \left\{ z = (z_{lmn}) \mid (z_{lmn}) \text{ is convergent in Pringsheim's sense, that is, } P - \lim_{l,m,n\to\infty} z_{lmn} \text{ has a finite value} \right\},$$

or equivalently,

$$\begin{aligned} c^3(\mathbb{K}) = &\{z = (z_{lmn}) \mid \forall\, \epsilon > 0,\ \exists\, n_0 = n_0(\epsilon) \in \mathbb{N} \text{ such that } |z_{lmn} - \ell| < \epsilon \\ &\text{holds for all } l,m,n \geq n_0\}, \\ \ell^3_\infty(\mathbb{K}) := &\{z = (z_{lmn}) \mid \|z_{lmn}\|_\infty = \sup| z_{lmn} | < \infty \text{ for all } l,m,n \in \mathbb{N}\}. \end{aligned}$$

These spaces represent the set of all P-convergent triple sequences and the set of all bounded triple sequences, respectively.

Note that (z_{lmn}) may converge without (z_{lmn}) being a bounded function of l, m and n. To put it more explicitly, P-convergence of (z_{lmn}) may not imply boundedness of its term in contrast to the case in single sequences. For instance, the sequence (z_{lmn}) defined by

$$z_{lmn} = \begin{cases} 5^n & \text{if } l = 1, m, n \in \mathbb{N} \\ 5^{l+m+3} & \text{if } n = 3, l, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

is P-convergent, but it is unbounded.

Some notations that will be used in places throughout this paper are given below.

NOTATION 2.1: Let (z_{lmn}) be a triple sequence.

(1) The symbol $z_{lmn} = O\,(1)$ means that $|z_{lmn}| \leq H$ for some constant $H > 0$ and each $l, m, n \geq n_0$.
(2) The symbol $z_{lmn} = O_L\,(1)$ means that $z_{lmn} \geq M$ for some constant $M > 0$ and each $l, m, n \geq n_0$.
(3) The symbol $z_{lmn} = o(1)$ means that $z_{lmn} \to 0$ as $l, m, n \to \infty$.

Let $z = (z_{lmn}) \in w^3(\mathbb{K})$ and let $(p_l), (q_m), (r_n) \in w\,(\mathbb{R}^{>0})$ such that

(2.1)
$$P_l := \sum_{i=0}^{l} p_i \to \infty, Q_m := \sum_{j=0}^{m} q_j \to \infty \quad \text{and} \quad R_n := \sum_{k=0}^{n} q_j \to \infty \quad \text{as } l,m,n \to \infty$$

where $w\,(\mathbb{R}^{>0})$ represents the set of all single sequences of positive real numbers.

The weighted means of (z_{lmn}) determined by the sequences of weights (p_l), (q_m) and (r_n) are defined by

$$\sigma^{111}_{lmn} := \frac{1}{P_l Q_m R_n} \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{r=0}^{n} p_l q_j r_k z_{ijk},$$

$$\sigma^{100}_{lmn} := \frac{1}{P_l} \sum_{i=0}^{l} p_l z_{imn}, \quad \sigma^{010}_{lmn} := \frac{1}{Q_m} \sum_{j=0}^{m} q_j z_{ljn},$$

$$\sigma^{001}_{lmn} := \frac{1}{R_n} \sum_{k=0}^{n} r_k z_{lmk}$$

for all $(l,m,n) \in \mathbb{N} \times \mathbb{N}$ and $P_l Q_m R_n > 0$.
A triple sequence (z_{lmn}) is called $(\bar{N},p,q,r)$ summable to ℓ if $P - \lim \sigma^{111}_{lmn} = \ell$. Similarly, $(\bar{N},p,*,r)$, $(\bar{N},*,q,r)$ and $(\bar{N},p,q,*)$ summable sequences are defined via triple sequences (σ^{10}_{lmn}), (σ^{010}_{lmn}) and (σ^{001}_{lmn}), respectively. It can be easily seen that necessary and sufficient condition for regularity of the $(\bar{N},p,q,r)$ method is condition (2.1). To put it another way, $(z_{lmn}) \in c^3(\mathbb{K}) \cap \ell^3_\infty(\mathbb{K})$ is also $(\bar{N},p,q,r)$ summable to same number under condition (2.1). Nevertheless, the opposite of this proposition is not true in general. The question of whether some conditions on the terms z_{lmn} under which its $(\bar{N},p,q,r)$ summability implies its P-convergence exist comes to mind at this point. The condition $T\{z_{lmn}\}$ making such a situation possible is called a Tauberian condition. The resulting theorem stating that P-convergence follows from its $(\bar{N},p,q,r)$ summability and $T\{z_{lmn}\}$ is called a Tauberian Theorem.
In conjunction with the weighted means, there are many special means occurring depends on choosing of the sequences of weights (p_l), (q_m) and (r_n). Included by the weighted means and also commonly used by researchers in literature, some means are listed as follows.

(a) In case $p_l = q_m = r_n = 1$, it leads to the arithmetic means (or called Cesàro means of order $(1,1,1)$) of a triple sequence where $P_l = l+1$, $Q_m = m+1$ and $R_n = n+1$ for all $l,m,n \in \mathbb{N}$.
(b) In case $p_l = 1/(l+1)$, $q_m = 1/(m+1)$ and $r_n = 1/(n+1)$, it leads to the harmonic means (or called the logarithmic means) of a triple sequence where $P_l \sim \log l$, $Q_m \sim \log m$ and $R_n \sim \log n$ for all $l,m,n \in \mathbb{N}$.
(c) In case $p_l = 1/((l+1)\log(l+1))$, $q_m = 1/((m+1)\log(m+1))$ and $r_n = 1/((n+1)\log(n+1))$, it leads to the harmonic means of second order (or called the iterated logarithmic means) of a double sequence where $P_l \sim \log l$, $Q_m \sim \log m$ and $R_n \sim \log n$ for all $l,m,n \in \mathbb{N}$.
(d) In case $p_l = (l+1)^\alpha$, $q_m = (m+1)^\beta$ and $r_n = (n+1)^\gamma$ with $\alpha,\beta,\gamma > -1$, it leads to untitled means of a double sequence where $P_l \sim (l+1)^{\alpha+1}/(\alpha+1)$, $Q_m \sim (m+1)^{\beta+1}/\ (\beta+1)$ and $r_n \sim (n+1)^{\gamma+1}/\ (\gamma+1)$ for all $l,m,n \in \mathbb{N}$.

For $(z_{lmn}) \in w^3(\mathbb{K})$, we define

$$\begin{aligned}
\Delta_{111} z_{lmn} &:= \Delta_{100}\Delta_{010}\Delta_{001} z_{lmn} = \Delta_{100}\left(\Delta_{010}\Delta_{001} z_{lmn}\right) = \Delta_{010}\left(\Delta_{100}\Delta_{001} z_{lmn}\right) \\
&= z_{l,mn} - z_{l,m,n-1} - z_{l-1,m,n} - z_{l,m-1,n} + z_{l-1,m-1,n-1}, \\
\Delta_{100} z_{lmn} &:= z_{lmn} - z_{l,m,n-1}, \\
\Delta_{010} z_{lmn} &:= z_{lmn} - z_{l,m-1,n} \\
\Delta_{001} z_{lmn} &:= z_{lmn} - z_{l-1,m,n}
\end{aligned}$$

for all $l, m, n \in \mathbb{N}$.
The weighted Kronecker identities for a sequence (z_{lmn}) are defined by:

$$z_{lmn} - \sigma^{100}_{lmn}(z) = \frac{1}{P_l}\sum_{i=1}^{l} P_{i-1}\Delta_{100} z_{imn} =: V^{10o^{(0)}}_{lmn}(\Delta_{100} z),$$

$$z_{lmn} - \sigma^{010}_{lmn}(z) = \frac{1}{Q_m}\sum_{j=1}^{m} Q_{j-1}\Delta_{010} u_{ljn} =: V^{01o^{(0)}}_{lmn}(\Delta_{010} z)$$

and

$$z_{lmn} - \sigma^{001}_{lmn}(z) = \frac{1}{R_n}\sum_{j=1}^{m} R_{j-1}\Delta_{001} z_{lmk} =: V^{010^{(0)}}_{lmn}(\Delta_{0010} z)$$

for all $l, m, n \in \mathbb{N}$.

The sequence $\left(V^{100^{(0)}}_{lmn}(\Delta_{100} z)\right)$ is the $(\bar{N}, p, *, *)$ mean of $(P_{l-1}\Delta_{100} z_{lmn})$ and called the weighted generator sequence of (z_{lmn}) in the sense $(1, 0, 0)$. Concordantly, the sequence $\left(V^{010^{(0)}}_{lmn}(\Delta_{010} z)\right)$ is the $(\bar{N}, *, q, *)$ mean of $(Q_{m-1}\Delta_{010} z_{lmn})$ and $\left(V^{001^{(0)}}_{lmn}(\Delta_{001} z)\right)$ is the $(\bar{N}, *, *, r)$ mean of $(R_{n-1}\Delta_{001} z_{lmn})$ and called the weighted generator sequence of (z_{lmn}) in the sense $(0, 1, 0)$ and $(0, 0, 1)$ respectively. More generally, the triple weighted Kronecker identity for a sequence (z_{lmn}) are defined via $\left(V^{100^{(0)}}_{lmn}(\Delta_{100} z)\right)$, $\left(V^{010^{(0)}}_{lmn}(\Delta_{010} z)\right)$ and $\left(V^{001^{(0)}}_{lmn}(\Delta_{001} z)\right)$ as follows:

$$z_{lmn} - \sigma^{111}_{lmn}(z) = V^{111^{(0)}}_{lmn}(\Delta_{111} z)$$

where
$V^{111^{(0)}}_{lmn}(\Delta_{111} z) :=$

$$V^{100^{(0)}}_{lmn}(\Delta_{100} z) + V^{010^{(0)}}_{lmn}(\Delta_{010} z) + V^{001^{(0)}}_{lmn}(\Delta_{001} z) - \frac{1}{P_l Q_m R_m}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} P_{i-1} Q_{j-1} R_{k-1}\Delta_{111} z_{ijk}$$

for all $l, m, n \in \mathbb{N}$.
The sequence $\left(V^{111^{(0)}}_{lmn}(\Delta_{111} z)\right)$ is called the weighted generator sequence of (z_{lmn}) in the sense $(1, 1, 1)$.
In addition, the $(\bar{N}, p, q, r)$ means of order $v \in \mathbb{N}$ of sequences (z_{lmn}) and $\left(V^{111^{(0)}}_{lmn}(\Delta_{111} z)\right)$ are defined by

$$\sigma^{111^{(v)}}_{lmn}(z) := \begin{cases} \frac{1}{P_l Q_m R_n}\sum_{i=0}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} P_i Q_j R_k \sigma^{111^{(v-1)}}_{ijk}(z) & \text{if } v \geq 1 \\ z_{lmn} & \text{if } v = 0 \end{cases}$$

and

$$V^{111^{(v)}}_{lmn}(\Delta_{111} z) := \begin{cases} \frac{1}{P_l Q_m R_n}\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} p_i q_j r_k V^{111^{(v-1)}}_{ijk}(\Delta_{111} z) & \text{if } v \geq 1 \\ V^{111^{(0)}}_{lmn}(\Delta_{111} z) & \text{if } v = 0 \end{cases}$$

respectively.

Throughout this paper, σ_{lmn}^{111} and $V_{lmn}^{111^{(1)}}$ will be used instead of $\sigma_{lmn}^{111^{(1)}}(z)$ and $V_{lmn}^{111^{(1)}}(\Delta_{111}z)$ for the sake of convenience.
The weighted de la Vallée Poussin means of (z_{lmn}) are defined by:

$$\tau_{lmn}^{\mu\eta\chi}(z) := \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sum_{i=l+1}^{\mu} \sum_{j=m+1}^{\eta} \sum_{k=n+1}^{\chi} p_i q_j r_k z_{ijk}, \quad \mu > l, \eta > m, \chi > n$$

and

$$\tau_{\mu\eta\chi}^{lmn}(z) := \frac{1}{(P_l - P_\mu)(Q_m - Q_\eta)(R_m - R_\chi)} \sum_{i=\mu+1}^{l} \sum_{j=\eta+1}^{m} \sum_{j=\chi+1}^{n} p_i q_j r_k z_{ijk}, \quad \mu < l, \eta < m, \chi < n$$

for all $l, m, n \in \mathbb{N}$.
At present, we define concepts of slow decrease relative to (P_l), (Q_m) and (R_n) and slow oscillation relative to (P_l), (Q_m) and (R_n) for triple sequences. In the wake of defining of that, we mention a relation between them.
A sequence $(z_{lmn}) \in w^3(\mathbb{R})$ is said to be slowly decreasing relative to both (P_l), (Q_m) and (R_n) provided that

$$\lim_{\substack{\lambda\to 1^+\\ \kappa\to 1^+\\ \delta\to 1^+}} \liminf_{l,m,n\to\infty} \min_{\substack{P_l \le P_i \le \lambda P_l\\ Q_m \le Q_j \le \kappa\\ R_n \le R_k \le \delta}} (z_i - z_{lmn}) \ge 0; \tag{2.2}$$

that is, for each $\epsilon > 0$ there exist $n_0 = n_0(\epsilon) \in \mathbb{N}, \lambda = \lambda(\epsilon) > 1$, $\kappa = \kappa(\epsilon)$ and $\delta = \delta(\epsilon) > 1$ such that

$$z_{ijk} - z_{lmn} \ge -\epsilon \text{ whenever } n_0 \le l \le i, n_0 \le m \le j, n_0 \le n \le k$$
$$\text{and } 1 \le \frac{P_i}{P_l} \le \lambda, 1 \le \frac{Q_j}{Q_m} \le \kappa, 1 \le \frac{R_k}{R_n} \le \delta.$$

Condition (2.2) is equivalent to

$$\lim_{\substack{\lambda\to 1^+\\ \kappa\to 1^+\\ \delta\to 1^+}} \liminf_{l,m,n\to\infty} \min_{\substack{P_l \le P_i \le \lambda P_l\\ Q_m \le Q_j \le \kappa\\ R_n \le R_k \le \delta}} (u_{mn} - u_{ij}) \ge 0.$$

The set of all slowly decreasing sequences relative to both (P_l), (Q_m) and (R_n) is denoted by $\mathscr{S}\mathscr{D}_{(P,Q,R)}$.
A sequence $(z_{lmn}) \in w^3(\mathbb{C})$ is said to be slowly oscillating relative to both (P_l), (Q_m) and (R_n) provided that

$$\lim_{\substack{\lambda\to 1^+\\ \kappa\to 1^+\\ \delta\to 1^+}} \lim_{l,m,n\to\infty} \sup_{\substack{P_l \le P_i \le \lambda P_l\\ Q_m \le Q_j \le \kappa Q_m\\ R_n \le R_k \le \delta R_n}} |z_{ijk} - u_{lmn}| = 0 \tag{2.3}$$

that is, for each $\epsilon > 0$ there exist $n_0 = n_0(\epsilon) \in \mathbb{N}, \lambda = \lambda(\epsilon) > 1$, $\kappa = \kappa(\epsilon) > 1$ and $\delta = \delta(\epsilon) > 1$ such that

$$|z_{ijk} - z_{lmn}| \le \epsilon \text{ whenever } n_0 \le l \le i, n_0 \le m \le j, n_0 \le n \le k,$$
$$\text{and} \quad 1 \le \frac{P_i}{P_l} \le \lambda, 1 \le \frac{Q_j}{Q_m} \le \kappa, 1 \le \frac{R_k}{R_n} \le \delta.$$

Condition (2.3) is equivalent to

The set of all slowly oscillating sequences relative to both (P_l), (Q_m) and (R_n) is denoted by $\mathscr{SO}_{(P,Q,R)}$.
A sequence $(z_{lmn}) \in w^3(\mathbb{R})$ is said to be slowly decreasing relative to (P_l) provided that

$$\lim_{\lambda\to 1^+} \liminf_{l,m,n\to\infty} \min_{P_l \le P_i \le \lambda P_l} (z_{imn} - z_{lmn}) \ge 0, \tag{2.4}$$

or equivalently,

$$\lim_{\lambda\to 1^-} \liminf_{l,m,n\to\infty} \min_{\lambda P_l < P_i \le P_l} (z_{lmn} - z_{imn}) \ge 0.$$

The set of all slowly decreasing sequences relative to (P_l) is denoted by $\mathscr{S}_{\mathscr{D}_{(P)}}$. Besides, a sequence (z_{lmn}) is said to be slowly decreasing relative to (P_l) in the strong sense if (2.4) is satisfied with

$$\min_{\substack{P_l \le P_i \le \lambda P_l \\ Q_m \le Q_j \le \kappa Q_m \\ R_n \le R_k \le \delta R_n}} (z_{ijk} - z_{ljk}) \quad \text{instead of} \quad \min_{P_l \le P_i \le \lambda P_l} (z_{imn} - z_{lmn}).$$

The set of all slowly decreasing sequences relative to (P_l) in the strong sense is denoted by $\mathscr{SDST}(P)$.
A sequence $(z_{lmn}) \in w^3(\mathbb{C})$ is said to be slowly oscillating relative to (P_l) provided that

$$\lim_{\lambda\to 1^+} \limsup_{l,m,n\to\infty} \max_{P_l \le P_i \le \lambda P_l} |z_{imn} - z_{lmn}| = 0 \tag{2.5}$$

or equivalently,

$$\lim_{\lambda\to 1^-} \limsup_{l,m,n\to\infty} \max_{\lambda P_l < P_i \le P_l} |z_{lmn} - z_{imn}| = 0.$$

Besides, a sequence (z_{lmn}) is said to be slowly oscillating relative to (P_l) in the strong sense if (2.5) is satisfied with

$$\max_{\substack{P_l \le P_i \le \lambda P_l \\ Q_m \le Q_j \le \kappa Q_m \\ R_n \le R_k \le \delta R_n}} |z_{ijk} - z_{ljk}| \quad \text{instead of} \quad \max_{P_l \le P_i \le \lambda P_l} |z_{imn} - z_{lmn}|.$$

The set of all slowly oscillating sequences relative to (P_l) in the strong sense is denoted by $\mathscr{SOST}(P)$.
Similarly, the sets $\mathscr{SD}_{(Q)}, \mathscr{SDST}(Q), \mathscr{SO}_{(Q)}$, and $\mathscr{SOST}(Q)$ can be analogously defined.
Indeed, for all large enough l, m and n, that is, $l, m, n \ge n_0, \lambda > 1$, $\kappa > 1$ and $\delta > 1$, we find

$$\begin{aligned}
\min_{\substack{P_l \le P_i \le \lambda P_l \\ Q_m \le Q_j \le \kappa Q_m \\ R_n \le R_k \le \delta R_n}} (z_{ijk} - z_{lmn}) &= \min_{\substack{P_l \le P_i \le \lambda P_l \\ Q_m \le Q_j \le \kappa Q_m \\ R_n \le R_k \le \delta R_n}} (z_{ijk} - u_{ljk} + u_{ljk} - u_{lmn}) \\
&\ge \min_{\substack{P_l \le P_i \le \lambda P_l \\ Q_m \le Q_j \le \kappa Q_m \\ R_m \le R_j \le \kappa R_n}} (z_{ijk} - u_{ljk}) + \min_{Q_m \le j \le \kappa Q_m} (u_{ljk} - z_{lmn}).
\end{aligned} \tag{2.6}$$

Taking lim inf and limit of both sides of (2.6) as $l, m, n \to \infty$ and $\lambda, \kappa, \delta \to 1^+$respectively, we get that the terms on right-hand side of (2.6) are greater than 0 . Therefore, we reach $(z_{lmn}) \in \mathscr{SD}_{(P,Q)}$.

It can be also said that if $(z_{lmn}) \in \mathscr{SDST}(Q) \cap \mathscr{SD}_{(P)} \cap \mathscr{SD}_{(R)}$, then $(z_{lmn}) \in \mathscr{SD}_{(P,Q,R)}$. Similarly, if $(z_{lmn}) \in \mathscr{SOST}(P) \cap \mathscr{S}_{(Q)}$ or $(z_{lmn}) \in \mathscr{SOST}(Q) \cap \mathscr{S}_{(P)} \cap \mathscr{SD}_{(P)}$, then $(z_{lmn}) \in \mathscr{SO}_{(P,Q,R)}$.
In the remainder of this section, we mention the classes including all positive sequences (p_l) whose sequence of partial sum (P_l) is

(i) a regularly varying sequence of positive index,
(ii) a rapidly varying sequence of index ∞ (see [3] for more details).

Let $p = (p_l)$ be a sequence that satisfies $(p_l) = (P_l - P_{l-1})$, where $P_{-1} = 0$ and $P_l \neq 0$ for all $l \in \mathbb{N}$.
(i) A sequence (P_m) of positive numbers is said to be regularly varying if for all $\lambda > 0$

$$\lim_{m\to\infty} \frac{P_{\lambda_m}}{P_m} = \varphi(\lambda) \text{ exists,}$$

where $0 < \varphi(\lambda) < \infty$ (cf. [4]).
In spite of the fact that this definition has been used by many authors as a starting point for studies including regularly varying sequences, these sequences possess quite useful properties, the most important of which is probably the following characterization theorem.

Theorem 2.2 [18] (Characterization Theorem) *The following statements are equivalent:*

(a) A sequence (P_l) of positive numbers is a regularly varying sequence.
(b) There exists a real number $\alpha > 0$ such that $\varphi(\lambda) = \lambda^\alpha$ for all $\lambda > 0$.
(c) The sequence (P_l) has the form $P_l = (l+1)^\alpha L(l)$ for $l \geq 0$ with constant $\alpha \geq 0$ and slowly varying function $L(.)on(0,\infty)$, i.e. the function $L(.)ispositive$, measurable, and satisfies

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1 \textit{ for all } \lambda > 0.$$

To emphasize such α, a sequence (P_l) is called a regularly varying sequence of positive index α, as well. Note that a regularly varying sequence of index $\alpha = 0$ corresponds to a slowly varying sequence.
The set of all sequences of positive numbers (p_l) with $p_0 > 0$ satisfying (c) is denoted by $SVA_{\text{reg}(\alpha)}$.
Here, it is useful to give the following implication proved by Bojanic and Seneta [4].

Lemma 2.3 [4] *If a sequence $P = (P_l)$ of positive numbers is regularly varying, then $\frac{P_{l-1}}{P_l} \to 1$ as $l \to \infty$*
(ii) A sequence (P_l) of positive numbers is said to be rapidly varying of index ∞ if

$$\frac{P_{\lambda_l}}{P_l} \to \begin{cases} 0 & \text{if } 0 < \lambda < 1, \\ 1 & \text{if } \lambda = 1, \\ \infty & \text{if } \lambda > 1 \end{cases} \qquad \text{as } m \to \infty. \tag{2.7}$$

The set of all sequences of positive numbers (p_m) satisfying (2.7) is denoted by SVA_{rap}. In addition, it may be written conventionally as λ^∞ because the right hand side of (2.7) is the limit of λ^α as $\alpha \to \infty$.

3. Auxiliary results

In this section, we state and prove some auxiliary results to be benefited in the proofs of our main results. The next lemma presents two representations of difference between general terms of (z_{lmn}) and (σ^{111}_{lmn}) and it can be proved when it is make convenient modification in Lemma 1.2 proved by Fekete [9].

Lemma 3.1: *Let $z = (z_{lmn})$ be a triple sequence.*

(i) For sufficiently large $\mu > l$, $\eta > m$ and $\chi > n$, we have

$$\begin{aligned} z_{lmn} - \sigma^{111}_{lmn} =& \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \left(\sigma^{111}_{\mu\eta\chi} - \sigma^{111}_{\mu mn} - \sigma^{111}_{l\eta n} + \sigma^{111}_{lmn}\right) \\ &+ \frac{P_\mu}{P_\mu - P_l}\left(\sigma^{111}_{\mu mn} - \sigma^{111}_{lmn}\right) + \frac{Q_\eta}{Q_\eta - Q_m}\left(\sigma^{111}_{l\eta n} - \sigma^{111}_{lmn}\right) + \frac{R_\chi}{Q_\chi - Q_n}\left(\sigma^{111}_{lm\chi} - \sigma^{111}_{lmn}\right) \\ &- \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - Q_n)} \sum_{i=l+1}^{\mu}\sum_{j=m+1}^{\eta}\sum_{k=n+1}^{\chi} p_i q_j r_k \left(z_{ijk} - z_{lmn}\right). \end{aligned} \tag{3.1}$$

(ii) For sufficiently large $\mu < l$, $\eta < m$ and $\chi < n$, we have

$$\begin{aligned} z_{lmn} - \sigma^{111}_{lmn} =& \frac{P_\mu Q_\eta R_\chi}{(P_l - P_\mu)(Q_m - Q_\eta)(R_n - R_\chi)} \left(\sigma^{111}_{lmn} - \sigma^{111}_{\mu mn} - \sigma^{111}_{l\eta n} + \sigma^{111}_{\mu\eta\chi}\right) \\ &+ \frac{P_\mu}{P_l - P_\mu}\left(\sigma^{111}_{\mu mn} - \sigma^{111}_{lmn}\right) + \frac{Q_\eta}{Q_m - Q_\eta}\left(\sigma^{111}_{l\eta n} - \sigma^{111}_{lmn}\right) + \frac{R_\chi}{Q_n - Q_\chi}\left(\sigma^{111}_{lm\chi} - \sigma^{111}_{lmn}\right) \\ &- \frac{1}{(P_l - P_\mu)(Q_m - Q_\eta)(R_n - Q_\chi)} \sum_{i=\mu+1}^{l}\sum_{j=\eta+1}^{m}\sum_{k=\chi+1}^{n} p_i q_j r_k \left(z_{lmn} - z_{ijk}\right). \end{aligned} \tag{3.2}$$

Interpreted differently from the statement given in Lemma 3.1, the following lemma points out two representations of difference between the general terms of (z_{lmn}) and (σ^{111}_{lmn}) via the weighted de la Vallée Poussin means of (z_{lmn}).

Lemma 3.2: *Let $z = (z_{lmn})$ be a triple sequence.*

(i) For sufficiently large $\mu > l$, $\eta > m$ and $\chi > n$ we have

$$\begin{aligned} z_{lmn} - \sigma^{111}_{lmn} =& \frac{P_\mu Q_\eta R_\chi}{P_l Q_m R_n}\left(\sigma^{111}_{\mu\eta\chi} - \tau^{\mu\eta\chi}_{lmn}\right) - \frac{P_\mu}{P_l}\left(\sigma^{111}_{\mu mn} - \tau^{\mu\eta\chi}_{lmn}\right) \\ &- \frac{Q_\eta}{Q_m}\left(\sigma^{111}_{l\eta n} - \tau^{\mu\eta\chi}_{lmn}\right) - \frac{R_\chi}{Q_n}\left(\sigma^{111}_{ln\chi} - \tau^{\mu\eta\chi}_{lmn}\right) - \left(\tau^{\mu\eta\chi}_{lmn} - z_{lmn}\right). \end{aligned}$$

(ii) For sufficiently large $\mu < l$, $\eta < m$ and $\chi < n$, we have

$$\begin{aligned} z_{lmn} - \sigma^{111}_{lmn} =& -\frac{P_\mu Q_\eta R_\chi}{P_l Q_m R_n}\left(\tau^{lmn}_{\mu\eta\chi} - \sigma^{111}_{\mu\eta\chi}\right) + \frac{P_\mu}{P_l}\left(\tau^{lmn}_{\mu\eta\chi} - \sigma^{111}_{\mu mn}\right) \\ &+ \frac{Q_\eta}{Q_m}\left(\tau^{lmn}_{\mu\eta\chi} - \sigma^{111}_{l\eta n}\right) + + \frac{R_\eta}{R_n}\left(\tau^{lmn}_{\mu\eta\chi} - \sigma^{111}_{ln\chi}\right) + \left(z_{lmn} - \tau^{lmn}_{\mu\eta\chi}\right). \end{aligned}$$

Proof. (i) For sufficiently large $\mu > l$, $\eta > m$ and $\chi > n$ we have from definition of the weighted de la Vallée Poussin means of (z_{lmn})

$$\tau_{lmn}^{\mu\eta\chi}(z) = \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sum_{i=l+1}^{\mu} \sum_{j=m+1}^{\eta} \sum_{k=n+1}^{\chi} p_i q_j r_k z_{ijk}$$
$$= \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \left[\left(\sum_{i=0}^{\mu} - \sum_{i=0}^{l} \right) \left(\sum_{j=0}^{\eta} - \sum_{j=0}^{m} \right) \left(\sum_{k=0}^{\chi} - \sum_{k=0}^{n} \right) \right] p_i q_j r_k z_{ijk}$$
$$= \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \Bigg[\sum_{i=0}^{\mu}\sum_{j=0}^{\eta}\sum_{k=0}^{\chi} - \sum_{i=0}^{\mu}\sum_{j=0}^{\eta}\sum_{k=0}^{n} - \sum_{i=0}^{\mu}\sum_{j=0}^{m}\sum_{k=0}^{\chi} + \sum_{i=0}^{\mu}\sum_{i=0}^{m}\sum_{j=0}^{n}$$
$$- \sum_{i=0}^{l}\sum_{j=0}^{\eta}\sum_{k=0}^{\chi} + \sum_{i=0}^{l}\sum_{j=0}^{\eta}\sum_{k=0}^{n} + \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{\chi} - \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \Bigg] p_i q_j r_k z_{ijk}$$
$$= \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sigma_{\mu\eta\chi}^{111} - \frac{P_n}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sigma_{\mu\eta n}^{111}$$
$$- \frac{P_l Q_m R_n}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sigma_{l\eta n}^{11} + \frac{P_\mu}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sigma_{lmn}^{111}. \tag{3.3}$$

It follows from Eq. (3.3) that

$$- \sigma_{mn}^{11}(u)$$
$$= \frac{P_\mu Q_\eta R_\chi}{P_l Q_m R_n} \sigma_{\mu\eta}^{11} - \frac{P_\mu}{P_l} \sigma_{\mu mn}^{111} - \frac{Q_\eta}{Q_m} \sigma_{l\eta n}^{111} - \frac{R_\eta}{R_n} \sigma_{lm\chi}^{111} - \frac{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)}{P_l Q_m R_n} \tau_{lmn}^{\mu\eta\chi}$$
$$= \frac{P_\mu Q_\eta R_\chi}{P_l Q_m R_n} \sigma_{\mu\eta\chi}^{111} - \frac{P_\mu}{P_l} \sigma_{\mu mn}^{111} - \frac{Q_\eta}{Q_m} \sigma_{l\eta n}^{111} - \frac{R_\eta}{R_n} \sigma_{lm\chi}^{111} - \left[\frac{P_\mu Q_\eta R_\chi}{P_l Q_m R_n} - \frac{P_\mu}{P_l} - \frac{Q_\eta}{Q_m} - \frac{R_\chi}{Q_n} + 1 \right] \tau_{lmn}^{\mu\eta\chi}$$
$$= \frac{P_\mu Q_\eta R_\chi}{P_l Q_m R_n} \left(\sigma_{\mu\eta\chi}^{111} - \tau_{lmn}^{\mu\eta\chi} \right) - \frac{P_\mu}{P_l} \left(\sigma_{\mu mn}^{111} - \tau_{lmn}^{\mu\eta\chi} \right) - \frac{Q_\eta}{Q_m} \left(\sigma_{l\eta n}^{111} - \tau_{lmn}^{\mu\eta\chi} \right)$$
$$- \frac{R_\chi}{R_n} \left(\sigma_{lm\chi}^{111} - \tau_{lmn}^{\mu\eta\chi} \right) - \tau_{lmn}^{\mu\eta\chi}. \tag{3.4}$$

If we implicate in the term u_{mn} to both sides of equality (3.4), then we can observe that the proof of (i) is completed.

(ii) The proof is similar to that of part (i) of Lemma 3.2. So, we omit it.
To be also commented as a result of Lemma 3.1, the mentioned representations below give the difference between the weighted de la Vallée Poussin means and the weighted means of (u_{mn}). □

Lemma 3.3 *Let $z = (z_{lmn})$ be a triple sequence.*
(i) For sufficiently large $\mu > l$, $\eta > m$ and $\chi > n$, we have

$$\tau_{lmn}^{\mu\eta\chi}(z) - \sigma_{lmn}^{111}(z) = \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \left(\sigma_{\mu\eta\chi}^{111} - \sigma_{\mu mn}^{111} - \sigma_{m\eta n}^{111} - \sigma_{lm\chi}^{111} + \sigma_{lmn}^{111} \right)$$
$$+ \frac{P_\mu}{P_\mu - P_l} \left(\sigma_{\mu mn}^{11} - \sigma_{lmn}^{111} \right) + \frac{Q_\eta}{Q_\eta - Q_m} \left(\sigma_{l\eta n}^{111} - \sigma_{lmn}^{111} \right)$$
$$+ \frac{R_\chi}{R_\chi - R_n} \left(\sigma_{lm\chi}^{111} - \sigma_{lmn}^{111} \right).$$

(ii) For sufficiently large $\mu < l$, $\eta < m$ and $\chi < n$, we have

$$\begin{aligned}\tau_{\mu\eta\chi}^{lmn}(z) - \sigma_{lmn}^{111}(z) =& \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)}\left(\sigma_{lmn}^{111} - \sigma_{\mu mn}^{111} - \sigma_{l\eta n}^{111} - \sigma_{lm\chi}^{111} + \sigma_{\mu\eta\chi}^{111}\right)\\ &+ \frac{P_\mu}{P_m - P_\mu}\left(\sigma_{mn}^{11} - \sigma_{\mu mn}^{111}\right) + \frac{Q_\eta}{Q_m - Q_\eta}\left(\sigma_{lmn}^{111} - \sigma_{l\eta n}^{111}\right)\\ &+ \frac{R_\eta}{R_n - R_\chi}\left(\sigma_{lmn}^{111} - \sigma_{lm\chi}^{111}\right).\end{aligned}$$

In [7], Çanak pointed out that a generator sequence $\left(V_m^{(0)}(\Delta z)\right)$ converges under some proper conditions. Getting inspired the one for single sequences, we demonstrate under which conditions the weighted generator sequence of (z_{lmn}) in sense $(1,1,1)$ is P-convergent.

LEMMA 3.4: *For a sequence $z = (z_{lmn}) \in w^3(\mathbb{R})$ and $(p_l), (q_m)(r_n), \in SVA_{\mathrm{reg}(\alpha)}$, let the hypotheses*

$$\lim_{\lambda\to 1^+}\liminf_{l,m,n\to\infty}\left(\sigma_{\mu mn}^{111} - \tau_{lmn}^{\mu\eta\chi}\right) \geq 0, \quad \lim_{\kappa\to 1^+}\liminf_{l,m,n\to\infty}\left(\sigma_{l\eta n}^{111} - \tau_{lmn}^{\mu\eta\chi}\right) \geq 0,$$

$$\lim_{\delta\to 1^+}\liminf_{l,m,n\to\infty}\left(\sigma_{lm\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}\right) \geq 0,$$

$$\lim_{\lambda,\kappa,\delta\to 1^+}\operatorname{limsup}_{l,m,n\to\infty}\left(\sigma_{\mu\eta\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}\right) \leq 0 \tag{3.5}$$

for $\mu > l$, $\eta > m$ and $\chi > n$ and

$$\lim_{\lambda\to 1^-}\limsup_{\mu,\eta,\chi\to\infty}\left(\sigma_{\mu mn}^{111} - \tau_{\mu\eta\chi}^{lmn}\right) \leq 0, \quad \lim_{\kappa\to 1^-}\limsup_{\mu,\eta,\chi\to\infty}\left(\sigma_{l\eta n}^{111} - \tau_{\mu\eta\chi}^{lmn}\right) \leq 0,$$

$$\lim_{\lambda,\kappa\to 1^-}\liminf_{\mu,\eta,\chi\to\infty}\left(\sigma_{\mu\eta\chi}^{111} - \tau_{\mu\eta\chi}^{lmn}\right) \geq 0 \tag{3.6}$$

$\mu < l$, $\eta < m$ and $\chi < n$ hold.
If conditions

$$\lim_{\lambda,\kappa\to 1^+}\liminf_{l,m,n\to\infty}\left(\tau_{lmn}^{\mu\eta} - z_{lmn}\right) \geq 0 \tag{3.7}$$

and

$$\lim_{\lambda,\kappa\to 1^-}\liminf_{\mu,\eta,\chi\to\infty}\left(z_{lmn} - \tau_{\mu\eta\chi}^{lmn}\right) \geq 0 \tag{3.8}$$

are satisfied, then the weighted generator sequence $\left(V_{lmn}^{111^{(0)}}(\Delta_{111}u)\right)$ is P-convergent to 0 .

Proof. Suppose that (3.5)-(3.8) are satisfied. To prove that

$$P - \lim_{l,m,n\to\infty}\left(V_{lmn}^{111^{(0)}}(\Delta_{111}z)\right) = 0,$$

we investigate $z_{lmn} - \sigma_{lmn}^{111}(z)$ in two cases $\mu > l, \eta > m, \chi > n$ and $\mu < l, \eta < m, \chi < n$.

Firstly, we consider the case $\mu > l$, $\eta > m$ and $\chi > n$. Putting

$$\mu = \operatorname{argmin}\{P_i \geq \lambda P_l\} = \min\{i > m : P_i \geq \lambda P_l\},$$

$$\eta = \operatorname{argmin}\{Q_j \geq \kappa Q_m\} = \min\{j > m : Q_j \geq \kappa Q_m\}$$

and

$$\chi = \operatorname{argmin}\{R_k \geq \lambda P_n\} = \min\{k > n : R_k \geq \delta R_n\}$$

with $\lambda, \kappa, \delta > 1$, we observe by Lemma 2.3 that $p_l/P_l \to 0$, q_m/Q_m and $r_n/R_n \to 0$ as $l, m, n \to \infty$ and hence

$$\frac{P_\mu}{P_l} \geq \frac{\lambda P_l}{P_l} = \lambda \quad \text{and} \quad \frac{P_\mu}{P_l} = \frac{P_{\mu-1}}{P_l} + \frac{p_\mu}{P_l} \leq \lambda + \frac{p_\mu}{P_\mu}\frac{P_\mu}{P_l} = \lambda(1 + o(1)), \tag{3.9}$$

$$\frac{Q_\eta}{Q_m} \geq \frac{\kappa Q_m}{Q_m} = \kappa \text{ and } \frac{Q_\eta}{Q_m} = \frac{Q_{\eta-1}}{Q_m} + \frac{q_\eta}{Q_m} \leq \kappa + \frac{q_\eta}{Q_\eta}\frac{Q_\eta}{Q_m} = \kappa(1 + o(1)) \tag{3.10}$$

and

$$\frac{R_\chi}{Q_n} \geq \frac{\delta R_n}{R_n} = \delta \text{ and } \frac{R_\chi}{R_n} = \frac{R_{\chi-1}}{R_n} + \frac{q_\chi}{R_n} \leq \delta + \frac{q_\chi}{R_\eta}\frac{R_\chi}{R_n} = \delta(1 + o(1)) \tag{3.11}$$

which mean

$$\frac{P_\mu}{P_l} \to \lambda, \quad \frac{Q_\eta}{Q_m} \to \kappa \quad \text{and } \frac{R_\chi}{R_n} \to \delta \quad \text{as } l, m, n \to \infty \tag{3.12}$$

respectively. If we get lim sup of both sides of identity (3.1) as $l, m, n \to \infty$, then we arrive by (3.12)

$$\begin{aligned}
&\limsup_{l,m,n\to\infty} \left(z_{lmn} - \sigma^{111}_{lmn}\right) \\
&\leq \limsup_{l,m,n\to\infty} \frac{P_\mu Q_\eta R_\chi}{P_m Q_m} \limsup_{l,m,n\to\infty} \left(\sigma^{111}_{\mu\eta\chi} - \tau^{\mu\eta\chi}_{lmn}\right) + \limsup_{l\to\infty} \frac{P_\mu}{P_l} \limsup_{l,m,n\to\infty} \left(-\left(\sigma^{111}_{\mu mn} - \tau^{\mu\eta\chi}_{lmn}\right)\right) \\
&\quad + \limsup_{m\to\infty} \frac{Q_\eta}{Q_m} \limsup_{l,m,n\to\infty} \left(-\left(\sigma^{111}_{l\eta n} - \tau^{\mu\eta\chi}_{lmn}\right)\right) + \limsup_{l,m,n\to\infty} \left(-\left(\tau^{\mu\eta\chi}_{lmn} - z_{lmn}\right)\right) \\
&= \lambda\kappa\delta \limsup_{l,m,n\to\infty} \left(\sigma^{111}_{\mu\eta\chi} - \tau^{\mu\eta\chi}_{lmn}\right) - \lambda \liminf_{l,m,n\to\infty} \left(\sigma^{11}_{\mu mn} - \tau^{\mu\eta}_{lmn}\right) - \kappa \liminf_{l,m,n\to\infty} \left(\sigma^{111}_{l\eta n} - \tau^{\mu\eta\chi}_{lmn}\right) \\
&\quad - \delta \liminf_{l,m,n\to\infty} \left(\sigma^{111}_{lm\chi} - \tau^{\mu\eta\chi}_{lmn}\right) \quad - \liminf_{l,m,n\to\infty} \left(\tau^{\mu\eta\chi}_{lmn} - z_{lmn}\right).
\end{aligned}$$

If we get limit of both sides of last inequality as $\lambda, \kappa, \delta \to 1^+$, we obtain

$$\begin{aligned}
&\limsup_{l,m,n\to\infty} \left(z_{lmn} - \sigma^{111}_{lmn}\right) \leq \lim_{\lambda,\kappa,\delta\to 1^+} \limsup_{l,m,n\to\infty} \left(\sigma^{111}_{\mu\eta\chi} - \tau^{\mu\eta\chi}_{lmn}\right) - \lim_{\lambda\to 1^+} \liminf_{l,m,n\to\infty} \left(\sigma^{111}_{\mu mn} - \tau^{\mu\eta\chi}_{lmn}\right) \\
&\quad - \lim_{\kappa\to 1^+} \liminf_{l,m,n\to\infty} \left(\sigma^{111}_{l\eta n} - \tau^{\mu\eta\chi}_{lmn}\right) \quad - \lim_{\delta\to 1^+} \liminf_{l,m,n\to\infty} \left(\sigma^{111}_{lm\chi} - \tau^{\mu\eta\chi}_{lmn}\right) \\
&- \lim_{\lambda,\kappa,\delta\to 1^+} \liminf_{l,m,n\to\infty} \left(\tau^{\mu\eta\chi}_{lmn} - z_{lmn}\right).
\end{aligned}$$

From hypotheses in (3.5) and (3.7), it follows that

$$\limsup_{l,m,n\to\infty} \left(z_{lmn} - \sigma^{111}_{lmn}\right) \leq 0. \tag{3.13}$$

On the other hand, we consider the case $\tilde{\mu} < l$, $\tilde{\eta} < m$ and $\tilde{\chi} < n$. Putting

$$\tilde{\mu} = \operatorname{argmax}\{P_l \geq \lambda P_i\} = \max\{l > i : P_m \geq \lambda P_i\},$$

$$\tilde{\eta} = \operatorname{argmax}\{Q_m \geq \kappa Q_j\} = \max\{m > j : Q_m \geq \kappa Q_j\}$$

and

$$\tilde{\chi} = \operatorname{argmax}\{R_n \geq \delta R_k\} = \max\{n > k : R_n \geq \delta R_k\}$$

with $\lambda, \kappa, \delta > 1$, we observe by Lemma 2.3 that $p_{\tilde{\mu}}/P_{\tilde{\mu}} \to 0$, $q_{\tilde{\eta}}/Q_{\tilde{\eta}} \to 0$ and $r_{\tilde{\chi}}/R_{\tilde{\chi}} \to 0$ as $\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \to \infty$ and hence

$$\frac{P_{\tilde{\mu}}}{P_l} \leq \frac{1}{\lambda} \quad \text{and} \quad \frac{P_{\tilde{\mu}}}{P_l} = \frac{P_{\tilde{\mu}+1}}{P_m} - \frac{p_{\tilde{\mu}+1}}{P_l} > \frac{1}{\lambda} - \frac{p_{\tilde{\mu}+1}}{P_{\tilde{\mu}+1}} \frac{P_{\tilde{\mu}+1}}{P_{\tilde{\mu}}} \frac{P_{\tilde{\mu}}}{P_l}$$

$$= \frac{1}{\lambda} - \frac{1}{\lambda} o(1)(1 + o(1)) = \frac{1}{\lambda}(1 - o(1)), \tag{3.14}$$

$$\frac{Q_{\tilde{\eta}}}{Q_m} \leq \frac{1}{\kappa} \text{ and } \quad \frac{Q_{\tilde{\eta}}}{Q_m} = \frac{Q_{\tilde{\eta}+1}}{Q_m} - \frac{q_{\tilde{\eta}+1}}{Q_m} > \frac{1}{\kappa} - \frac{q_{\tilde{\eta}+1}}{Q_{\tilde{\eta}+1}} \frac{Q_{\tilde{\eta}+1}}{Q_{\tilde{\eta}}} \frac{Q_{\tilde{\eta}}}{Q_m}$$

$$= \frac{1}{\kappa} - \frac{1}{\kappa} o(1)(1 + o(1)) = \frac{1}{\kappa}(1 - o(1)) \tag{3.15}$$

and

$$\frac{R_{\tilde{\chi}}}{R_n} \leq \frac{1}{\delta} \text{ and } \quad \frac{R_{\tilde{\chi}}}{R_n} = \frac{R_{\tilde{\chi}+1}}{R_n} - \frac{q_{\tilde{\chi}+1}}{R_n} > \frac{1}{\delta} - \frac{q_{\tilde{\chi}+1}}{R_{\tilde{\chi}+1}} \frac{R_{\tilde{\chi}+1}}{R_{\tilde{\chi}}} \frac{R_{\tilde{\chi}}}{R_n}$$

$$= \frac{1}{\delta} - \frac{1}{\delta} o(1)(1 + o(1)) = \frac{1}{\delta}(1 - o(1)) \tag{3.16}$$

which mean

$$\frac{P_{\tilde{\mu}}}{P_l} \to \frac{1}{\lambda}, \quad \frac{Q_{\tilde{\eta}}}{Q_m} \to \frac{1}{\kappa} \text{ and } \frac{R_{\tilde{\chi}}}{R_n} \to \frac{1}{\delta} \quad \text{as } \tilde{\mu}, \tilde{\eta}, \tilde{\chi} \to \infty,$$

respectively. If we get lim inf of both sides of identity (3.2) as $\tilde{\mu}, \tilde{\eta}, \tilde{\chi}, \to \infty$, then we arrive by (3.12)

$$\begin{aligned}
&\liminf_{l,m,n\to\infty} \left(z_{lmn} - \sigma^{111}_{lmn}\right) \\
&\geq \liminf_{\tilde{\mu},\tilde{\eta},\tilde{\chi}\to\infty} \frac{P_{\tilde{\mu}} Q_{\tilde{\eta}} R_{\tilde{\chi}},}{P_l Q_m R_n} \liminf_{\tilde{\mu},\tilde{\eta},\tilde{\chi}\to\infty} \left(\sigma^{111}_{\tilde{\mu}\tilde{\eta}} - \tau^{mn}_{\mu\eta}\right) + \liminf_{\tilde{\mu}\to\infty} \frac{P_{\tilde{\mu}}}{P_l} \liminf_{\tilde{\mu},\tilde{\eta},\tilde{\chi}\to\infty} \left(-\left(\sigma^{111}_{\tilde{\mu}mn} - \tau^{lmn}_{\mu\eta\chi}\right)\right) \\
&+ \liminf_{\tilde{\eta}\to\infty} \frac{Q_{\tilde{\eta}}}{Q_m} \liminf_{\tilde{\mu},\tilde{\eta},\tilde{\chi}\to\infty} \left(-\left(\sigma^{111}_{l\tilde{\eta}n} - \tau^{lmn}_{\mu\eta\chi}\right)\right) + \liminf_{\tilde{\chi}\to\infty} \frac{R_{\tilde{\chi}}}{R_n} \liminf_{\tilde{\mu},\tilde{\eta},\tilde{\chi}\to\infty} \left(-\left(\sigma^{111}_{lm\tilde{\chi}} - \tau^{lmn}_{\mu\eta\chi}\right)\right) \\
&+ \liminf_{l,m,n\to\infty} \left(z_{lmn} - \tau^{lmn}_{\mu\eta\chi}\right) \\
&= \frac{1}{\lambda\kappa\delta} \liminf_{\tilde{\mu},\tilde{\eta},\tilde{\chi}\to\infty} \left(\sigma^{111}_{\tilde{\mu}\tilde{\eta}\tilde{\chi}} - \tau^{lmn}_{\mu\eta\chi}\right) - \frac{1}{\lambda} \limsup_{\tilde{\mu},\tilde{\eta},\tilde{\chi}\to\infty} \left(\sigma^{111}_{\tilde{\mu}mn} - \tau^{lmn}_{\mu\eta\chi}\right) - \frac{1}{\kappa} \limsup_{\tilde{\mu},\tilde{\eta},\tilde{\chi}\to\infty} \left(\sigma^{111}_{l\tilde{\eta},n} - \tau^{lmn}_{\mu\eta\chi}\right) \\
&- \frac{1}{\delta} \limsup_{\tilde{\mu},\tilde{\eta},\tilde{\chi}\to\infty} \left(\sigma^{111}_{lm\tilde{\chi}} - \tau^{lmn}_{\mu\eta\chi}\right) + \liminf_{l,m,n\to\infty} \left(z_{lmn} - \tau^{lmn}_{\mu\eta\chi}\right).
\end{aligned}$$

If we get limit of both sides of last inequality as $\lambda, \kappa, \delta \to 1^-$, we obtain

$$\begin{aligned}
&\liminf_{l,m,n\to\infty} \left(z_{lmn} - \sigma^{111}_{lmn}\right) \\
&\geq \lim_{\lambda,\kappa,\delta\to 1^-} \liminf_{\tilde{\mu},\tilde{\eta},\tilde{\chi}\to\infty} \left(\sigma^{111}_{\tilde{\mu}\tilde{\eta}\tilde{\chi}} - \tau^{lmn}_{\mu\eta\chi}\right) - \lim_{\lambda\to 1^-} \limsup_{\tilde{\mu},\tilde{\eta},\tilde{\chi}\to\infty} \left(\sigma^{111}_{\tilde{\mu}mn} - \tau^{lmn}_{\mu\eta\chi}\right) \\
&- \lim_{\kappa\to 1^-} \limsup_{\tilde{\mu},\tilde{\eta},\tilde{\chi}\to\infty} \left(\sigma^{111}_{l\tilde{\eta}n} - \tau^{lmn}_{\mu\eta\chi}\right) + \liminf_{l,m,n\to\infty} \left(z_{lmn} - \tau^{lmn}_{\mu\eta\chi}\right).
\end{aligned}$$

From hypotheses in (3.6) and (3.8), it follows that

$$\liminf_{l,m,n\to\infty} \left(z_{lmn} - \sigma^{111}_{lmn}(z)\right) \geq 0. \tag{3.17}$$

If we combine inequalities (3.13) with (3.17), we reach

$$\lim_{l,m,n\to\infty}\left(z_{lmn}-\sigma^{111}_{lmn}(z)\right)=0$$

which means by the triple weighted Kronecker identity that $\left(V^{111^{(0)}}_{lmn}(\Delta_{111}z)\right)$ is $P-$ convergent to 0 . □

4. Main results for the $(\bar{N},p,q,r)$ summable triple sequences

In this section, we introduce several Tauberian theorems concerning triple sequences where the concept of P-convergence can be derived from the summability condition of $(\bar{N},p,q,r)$, given certain requirements on the weighted generator sequence $\left(V^{111^{(0)}}_{lmn}(\Delta_{111}z)\right)$ in terms of slow decrease or slow oscillation. Additionally, certain conditions are imposed on the sequences (p_l), (q_m), and (r_n). Subsequently, we provide some related corollaries based on these outcomes.

THEOREM 4.1 *Let* $(z_{lmn})\in\ell^3_\infty(\mathbb{R})$ *and* $(p_l),(q_m),(r_n)\in SVA_{\mathrm{reg}(\alpha)}$. *If a sequence* (z_{lmn}) *is* $(\bar{N},p,q,r)$ *summable to a number* ℓ *and*

$$\left(V^{111^{(0)}}_{lmn}(\Delta_{111}z)\right)\in\mathscr{SD}_{(P)}\cap\mathscr{SD}_{(Q)}\cap\mathscr{SD}_{(R)}\cap\mathscr{SDST}(Q)$$

or

$$\left(V^{111^{(0)}}_{lmn}(\Delta_{111}z)\right)\in\mathscr{S}_{(P)}\cap\mathscr{SD}_{(Q)}\cap\mathscr{SD}_{(R)}\cap\mathscr{SDST}_{(P)},$$

or

$$\left(V^{111^{(0)}}_{lmn}(\Delta_{111}z)\right)\in\mathscr{S}_{(P)}\cap\mathscr{SD}_{(Q)}\cap\mathscr{SD}_{(R)}\cap\mathscr{SDST}_{(R)},\tag{4.1}$$

then (z_{lmn}) *is* P*-convergent to* ℓ.

Proof. Without loss of generality, assume that $(z_{lmn})\in\ell^3_\infty(\mathbb{R})$ is $(\bar{N},p,q,r)$ summable to ℓ and $\left(V^{111^{(0)}}_{lmn}(\Delta_{111}z)\right)\in\mathscr{SD}_{(P)}\cap\mathscr{SD}_{(Q)}\cap\mathscr{SD}_{(R)}\cap\mathscr{SDST}_{(P)}$. To prove that (z_{lmn}) is P-convergent to ℓ, we demonstrate that $\left(V^{111^{(0)}}_{lmn}(\Delta_{111}z)\right)$ is P-convergent to 0 . Because $(\sigma^{111}_{lmn}(z))$ is P-convergent to ℓ and the $(\bar{N},p,q,r)$ method is regular under the boundedness condition of (z_{lmn}), we attain that $\left(\sigma^{111^{(2)}}_{lmn}(z)\right)$ is also P-convergent to the same number. It follows from the triple weighted Kronecker identity that $\left(V^{111^{(1)}}_{lmn}(\Delta_{111}z)\right)$ is P - convergent to 0 . For $\mu>l$, $\eta>$ and $\chi>n$, if we replace z_{lmn} by $V^{111^{(0)}}_{lmn}(\Delta_{111}z)$ in Lemma 3.1(i), we obtain

$$
\begin{aligned}
&V^{111^{(0)}}_{lmn}-V^{111^{(1)}}_{lmn}\\
&=\frac{P_\mu Q_\eta R_\chi}{(P_\mu-P_l)(Q_\eta-Q_m)(R_\chi-R_n)}\left(V^{111^{(1)}}_{\mu\eta\chi}-V^{111^{(1)}}_{\mu mn}-V^{111^{(1)}}_{l\eta n}-V^{111^{(1)}}_{lm\chi}+V^{111^{(1)}}_{lmn}\right)\\
&+\frac{P_\mu}{P_\mu-P_l}\left(V^{111^{(1)}}_{\mu mn}-V^{111^{(1)}}_{lmn}\right)+\frac{Q_\eta}{Q_\eta-Q_m}\left(V^{111^{(1)}}_{l\eta n}-V^{111^{(1)}}_{lmn}\right)+\frac{R_\chi}{R\chi-R_m}\left(V^{111^{(1)}}_{lm\chi}-V^{111^{(1)}}_{lmn}\right)\\
&-\frac{1}{(P_\mu-P_l)(Q_\eta-Q_m)(R_\chi-R_n)}\sum_{i=l+1}^{\mu}\sum_{j=m+1}^{\eta}\sum_{k=n+1}^{\chi}p_iq_jr_k\left(V^{111^{(0)}}_{ijk}-V^{111^{(0)}}_{lmn}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \left(V^{111^{(1)}}_{\mu\eta\chi} - V^{111^{(1)}}_{\mu mn} - V^{111^{(1)}}_{l\eta n} - V^{111^{(1)}}_{lm\chi} + V^{111^{(1)}}_{lmn} \right) \\
&+ \frac{P_\mu}{P_\mu - P_l} \left(V^{111^{(1)}}_{\mu mn} - V^{111^{(1)}}_{lmn} \right) + \frac{Q_\eta}{Q_\eta - Q_m} \left(V^{111^{(1)}}_{l\eta n} - V^{111^{(1)}}_{lmn} \right) + \frac{R_\chi}{R\chi - R_m} \left(V^{111^{(1)}}_{lm\chi} - V^{111^{(1)}}_{lmn} \right) \\
&- \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sum_{i=l+1}^{\mu} \sum_{j=m+1}^{\eta} \sum_{k=n+1}^{\chi} p_i q_j r_k \left(V^{111^{(0)}}_{ijk} - V^{11^{(0)}}_{ljk} \right) \\
&- \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sum_{i=l+1}^{\mu} \sum_{j=m+1}^{\eta} \sum_{k=n+1}^{\chi} p_i q_j r_k \left(V^{111^{(0)}}_{ljk} - V^{111^{(0)}}_{lmn} \right) \\
&\leq \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \left(V^{111^{(1)}}_{\mu\eta\chi} - V^{111^{(1)}}_{\mu mn} - V^{111^{(1)}}_{l\eta n} - V^{111^{(1)}}_{lm\chi} + V^{111^{(1)}}_{lmn} \right) \\
&+ \frac{P_\mu}{P_\mu - P_l} \left(V^{111^{(1)}}_{\mu mn} - V^{111^{(1)}}_{lmn} \right) + \frac{Q_\eta}{Q_\eta - Q_m} \left(V^{111^{(1)}}_{l\eta n} - V^{111^{(1)}}_{lmn} \right) + \frac{R_\chi}{R\chi - R_m} \left(V^{111^{(1)}}_{lm\chi} - V^{111^{(1)}}_{lmn} \right) \\
&- \min_{\substack{m \leq i \leq \mu \\ n \leq j \leq \eta}} \left(V^{11^{(0)}}_{ij} - V^{11^{(0)}}_{mj} \right) - \min_{n \leq j \leq \eta} \left(V^{11^{(0)}}_{mj} - V^{11^{(0)}}_{mn} \right). \qquad (4.2)
\end{aligned}
$$

Putting $\mu = \operatorname{argmin}\{P_i \geq \lambda P_l\}$, $\eta = \operatorname{argmin}\{Q_j \geq \kappa Q_m\}$ and $\chi = \operatorname{argmin}\{Q_k \geq \delta R_n\}$ with $\lambda, \kappa, \delta > 1$, we can observe inequalities (3.9), (3.10) and, accordingly, (3.11). Grounding that $V^{111^{(1)}}_{lmn}(\Delta_{111} u) \to 0$ as $l, m, n \to \infty$, we could remark $V^{111^{(1)}}_{ijk}(\Delta_{111} z) - V^{111^{(1)}}_{lmn}(\Delta_{111} z) \to 0$ for $i = l$ or μ, $j = m$ or η and $k = n$ or χ as $l, m, n \to \infty$. Since the sequences (P_l), (Q_m) and (R_n) are strictly increasing sequences, we attain from (4.2) that

$$
\begin{aligned}
&V^{111^{(0)}}_{lmn} - V^{111^{(1)}}_{lmn} \\
&\leq \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_m)(Q_\eta - Q_n)} \left(V^{111^{(1)}}_{\mu\eta\chi} - V^{111^{(1)}}_{\mu mn} - V^{111^{(1)}}_{l\eta n} - V^{111^{(1)}}_{lm\chi} + V^{111^{(1)}}_{lmn} \right) \\
&+ \frac{P_\mu}{P_\mu - P_l} \left(V^{111^{(1)}}_{\mu mn} - V^{111^{(1)}}_{lmn} \right) + \frac{Q_\eta}{Q_\eta - Q_m} \left(V^{111^{(1)}}_{l\eta n} - V^{111^{(1)}}_{lmn} \right) + \frac{R_\chi}{R\chi - R_m} \left(V^{111^{(1)}}_{lm\chi} - V^{111^{(1)}}_{lmn} \right) \\
&- \min_{\substack{P_m \leq P_i \leq \lambda P_m \\ Q_n \leq Q_j \leq \kappa Q_n}} \left(V^{11^{(0)}}_{ij} - V^{11^{(0)}}_{mj} \right) - \min_{Q_n \leq Q_j \leq \kappa Q_n} \left(V^{11^{(0)}}_{mj} - V^{11^{(0)}}_{mn} \right) \\
&\leq \frac{\lambda\kappa\delta + \lambda(\kappa - 1) + (\lambda - 1)\kappa + \lambda(\delta - 1) + (\lambda - 1)\delta + \delta(\kappa - 1) + (\delta - 1)\kappa}{(\lambda - 1)(\kappa - 1)(\delta - 1)} (1 + o(1)) o(1) \qquad (4.3) \\
&\qquad - \min_{\substack{P_m \leq P_i \leq \lambda P_l \\ Q_m \leq Q_j \leq \kappa Q_m \\ R_n \leq R_j \leq \kappa R_n}} \left(V^{111^{(0)}}_{ijk} - V^{111^{(0)}}_{ljn} \right) - \min_{R_n \leq Q_j \leq \kappa Q_m} \left(V^{111^{(0)}}_{ljm} - V^{111^{(0)}}_{lmn} \right)
\end{aligned}
$$

where

$$
\frac{P_\mu}{P_\mu - P_l} = \frac{P_\mu / P_l}{P_\mu / P_l - 1} \leq \frac{\lambda}{(\lambda - 1)} (1 + o(1)),
$$

$$
\frac{Q_\eta}{Q_\eta - Q_m} = \frac{Q_\eta / Q_m}{Q_\eta / Q_m - 1} \leq \frac{\kappa}{(\kappa - 1)} (1 + o(1))
$$

and

$$\frac{R_\chi}{R_\chi - R_n} = \frac{R_\chi/R_n}{R_\chi/R_n - 1} \leq \frac{\delta}{(\delta-1)}(1+o(1)).$$

If we get lim sup of both sides of inequality (4.3) as $l, m, n \to \infty$, then we reach for any $\lambda, \kappa, \delta > 1$

$$\limsup_{l,m,n\to\infty}\left(V_{lmn}^{111^{(0)}} - V_{lmn}^{111^{(1)}}\right) \leq -\liminf_{l,m,n\to\infty} \min_{\substack{P_l \leq P_i \leq \lambda P_l \\ Q_m \leq Q_j \leq \kappa Q_m \\ R_n \leq R_k \leq \kappa R_n}} \left(V_{ijk}^{111^{(0)}} - V_{ljk}^{111^{(0)}}\right)$$
$$- \liminf_{l,m,n\to\infty} \min_{Q_m \leq Q_j \leq \kappa Q_m} \left(V_{ljk}^{111^{(0)}} - V_{lmn}^{111^{(0)}}\right).$$

If we get limit of both sides of last inequality as $\lambda, \kappa, \delta \to 1^+$, then we find

$$\limsup_{l,m,n\to\infty}\left(V_{lmn}^{111^{(0)}} - V_{lmn}^{111^{(1)}}\right) \leq 0 \tag{4.4}$$

due to $\left(V_{lmn}^{111^{(0)}}(\Delta_{111}z)\right) \in \mathscr{SD}_{(P)} \cap \mathscr{SD}_{(Q)} \cap \mathscr{SD}_{(R)} \cap \mathscr{SDST}_{(P)}$. Following a similar procedure to above for $\tilde{\mu} < l$, $\tilde{\eta} < m$ and $\tilde{\chi} < n$, if we replace z_{lmn} by $V_{lmn}^{111^{(0)}}(\Delta_{111}z)$ in Lemma 3.1(ii), we obtain

$$\begin{aligned}
&V_{lmn}^{111^{(0)}} - V_{lmn}^{111^{(1)}} \\
=&\frac{P_{\tilde{\mu}}Q_{\tilde{\eta}}R_{\tilde{\chi}}}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})}\left(V_{lmn}^{111^{(1)}} - V_{\tilde{\mu}mn}^{111^{(1)}} - V_{l\tilde{\eta}n}^{111^{(1)}} - V_{lm\tilde{\chi}}^{111^{(1)}} + V_{\tilde{\mu}\tilde{\eta}\tilde{\chi}}^{111^{(1)}}\right) \\
&+ \frac{P_{\tilde{\mu}}}{P_l - P_{\tilde{\mu}}}\left(V_{lmn}^{111^{(1)}} - V_{\tilde{\mu}mn}^{111^{(1)}}\right) + \frac{Q_{\tilde{\eta}}}{Q_m - Q_{\tilde{\eta}}}\left(V_{lmn}^{11^{(1)}} - V_{l\tilde{\eta}n}^{111^{(1)}}\right) + \frac{R_{\tilde{\chi}}}{R_n - R_{\tilde{\chi}}}\left(V_{lmn}^{111^{(1)}} - V_{lm\tilde{\chi}}^{111^{(1)}}\right) \\
&+ \frac{1}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})}\sum_{i=\tilde{\mu}+1}^{l}\sum_{j=\tilde{\eta}+1}^{m}\sum_{i=\tilde{\chi}+1}^{n} p_i q_j r_k \left(V_{lmn}^{111^{(0)}} - V_{ijk^{111^{(0)}}}\right) \\
=&\frac{P_{\tilde{\mu}}Q_{\tilde{\eta}}R_{\tilde{\chi}}}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})}\left(V_{lmn}^{111^{(1)}} - V_{\tilde{\mu}mn}^{111^{(1)}} - V_{l\tilde{\eta}n}^{111^{(1)}} - V_{lm\tilde{\chi}}^{111^{(1)}} + V_{\tilde{\mu}\tilde{\eta}\tilde{\chi}}^{111^{(1)}}\right) \\
&+ \frac{P_{\tilde{\mu}}}{P_l - P_{\tilde{\mu}}}\left(V_{lmn}^{111^{(1)}} - V_{\tilde{\mu}mn}^{111^{(1)}}\right) + \frac{Q_{\tilde{\eta}}}{Q_m - Q_{\tilde{\eta}}}\left(V_{lmn}^{11^{(1)}} - V_{l\tilde{\eta}n}^{111^{(1)}}\right) + \frac{R_{\tilde{\chi}}}{R_n - R_{\tilde{\chi}}}\left(V_{lmn}^{111^{(1)}} - V_{lm\tilde{\chi}}^{111^{(1)}}\right) \\
&+ \frac{m}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})}\sum_{i=\tilde{\mu}+1}^{l}\sum_{j=\tilde{\eta}+1}^{m}\sum_{i=\tilde{\chi}+1}^{n} p_i q_j r_k \left(V_{lmn}^{111^{(0)}} - V_{ljn}^{111^{(0)}}\right) \\
&+ \frac{1}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})}\sum_{i=\tilde{\mu}+1}^{l}\sum_{j=\tilde{\eta}+1}^{m}\sum_{i=\tilde{\chi}+1}^{n} p_i q_j r_k \left(V_{ljk}^{111^{(0)}} - V_{ijk}^{111^{(0)}}\right) \\
\geq& \frac{P_{\tilde{\mu}}Q_{\tilde{\eta}}R_{\tilde{\chi}}}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})}\left(V_{lmn}^{111^{(1)}} - V_{\tilde{\mu}mn}^{111^{(1)}} - V_{l\tilde{\eta}n}^{111^{(1)}} - V_{lm\tilde{\chi}}^{111^{(1)}} + V_{\tilde{\mu}\tilde{\eta}\tilde{\chi}}^{111^{(1)}}\right) \\
&+ \frac{P_{\tilde{\mu}}}{P_l - P_{\tilde{\mu}}}\left(V_{lmn}^{111^{(1)}} - V_{\tilde{\mu}mn}^{111^{(1)}}\right) + \frac{Q_{\tilde{\eta}}}{Q_m - Q_{\tilde{\eta}}}\left(V_{lmn}^{11^{(1)}} - V_{l\tilde{\eta}n}^{111^{(1)}}\right) + \frac{R_{\tilde{\chi}}}{R_n - R_{\tilde{\chi}}}\left(V_{lmn}^{111^{(1)}} - V_{lm\tilde{\chi}}^{111^{(1)}}\right) \\
&+ \min_{\tilde{\eta}\leq j\leq n}\left(V_{lmn}^{111^{(0)}} - V_{mj}^{111^{(0)}}\right) + \min_{\substack{\tilde{\mu}\leq i\leq l \\ \tilde{\eta}\leq j\leq m \\ \tilde{\chi}\leq k\leq n}}\left(V_{ljn}^{111^{(0)}} - V_{ijk}^{111^{(0)}}\right).
\end{aligned} \tag{4.5}$$

Putting $\tilde{\mu} = \operatorname{argmax}\{P_l \geq \lambda P_i\}$, $\tilde{\eta} = \operatorname{argmax}\{Q_m \geq \kappa Q_j\}$ and $\tilde{\chi} = \operatorname{argmax}\{R_n \geq \delta R_k\}$ with $\lambda, \kappa, \delta > 1$, we can observe (3.13), (3.14) and, accordingly, (3.11). Grounding that $V_{lmn}^{111^{(1)}}(\Delta_{111}z) \to 0$ as $l, m, n \to \infty$, we could remark $V_{lmn}^{111^{(1)}}(\Delta_{111}z) - V_{ij}^{111^{(1)}}(\Delta_{111}z) \to 0$ for $i = l$ or $\tilde{\mu}$, $j = m$ or $\tilde{\eta}$ and $k = n$ or $\tilde{\chi}$ as $\tilde{\mu}, \tilde{\eta}\tilde{\chi} \to \infty$. Since the sequences (P_l), (Q_m) and (R_n) are strictly increasing sequences, we attain from (4.5) that

$$\begin{aligned}
&V_{lmn}^{111^{(0)}} - V_{lmn}^{111^{(1)}} \\
&\geq \frac{P_{\tilde{\mu}} Q_{\tilde{\eta}} R_{\tilde{\chi}}}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})} \left(V_{lmn}^{111^{(1)}} - V_{\tilde{\mu}mn}^{111^{(1)}} - V_{l\tilde{\eta}n}^{111^{(1)}} - V_{lm\tilde{\chi}}^{111^{(1)}} + V_{\tilde{\mu}\tilde{\eta}\tilde{\chi}}^{111^{(1)}}\right) \\
&\qquad (4.6) \\
&+ \frac{P_{\tilde{\mu}}}{P_l - P_{\tilde{\mu}}}\left(V_{lmn}^{111^{(1)}} - V_{\tilde{\mu}mn}^{111^{(1)}}\right) + \frac{Q_{\tilde{\eta}}}{Q_m - Q_{\tilde{\eta}}}\left(V_{lmn}^{111^{(1)}} - V_{l\tilde{\eta}n}^{111^{(1)}}\right) + \frac{R_{\tilde{\chi}}}{R_n - R_{\tilde{\chi}}}\left(V_{lmn}^{111^{(1)}} - V_{lm\tilde{\chi}}^{111^{(1)}}\right) \\
&\geq \frac{\tilde{\lambda} + \tilde{\kappa} + \tilde{\delta} - \tilde{\lambda}\tilde{\kappa}\tilde{\lambda}}{(1-\tilde{\lambda})(1-\tilde{\kappa})(1-\tilde{\delta})}(1 - o(1))o(1) + \min_{\tilde{\kappa}Q_n < Q_j \leq Q_n}\left(V_{lmn}^{111^{(0)}} - V_{ljk}^{111^{(0)}}\right) \\
&+ \min_{\substack{\tilde{\lambda}P_l \leq P_i \leq P_l \\ \tilde{\kappa}Q_m \leq Q_j \leq Q_m \\ \tilde{\delta}R_m \leq R_k \leq R_n}} \left(V_{ijk}^{111^{(0)}} - V_{ijk}^{111^{(0)}}\right)
\end{aligned}$$

where

$$\frac{P_{\tilde{\mu}}}{P_l - P_{\tilde{\mu}}} = \frac{P_{\tilde{\mu}}/P_l}{1 - P_{\tilde{\mu}}/P_l} \geq \frac{\tilde{\lambda}(1 - o(1))}{1 - \tilde{\lambda}},$$

$$\frac{Q_{\tilde{\eta}}}{Q_m - Q_{\tilde{\eta}}} = \frac{Q_{\tilde{\eta}}/Q_m}{1 - Q_{\tilde{\eta}}/Q_m} \geq \frac{\tilde{\kappa}(1 - o(1))}{1 - \tilde{\kappa}},$$

and

$$\frac{R_{\tilde{\chi}}}{R_m - R_{\tilde{\chi}}} = \frac{R_{\tilde{\chi}}/R_n}{1 - R_{\tilde{\chi}}/R_n} \geq \frac{\tilde{\delta}(1 - o(1))}{1 - \tilde{\delta}},$$

for $1/\lambda = \tilde{\lambda}, 1/\kappa = \tilde{\kappa}, 1/\delta = \tilde{\delta}$, and $0 < \tilde{\lambda}, \tilde{\kappa}, \tilde{\delta} < 1$. If we get lim inf of both sides of inequality (4.6) as $\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \to \infty$, then we reach for any $0 < \tilde{\lambda}, \tilde{\kappa}, \tilde{\delta} < 1$

$$\begin{aligned}
\liminf_{l,m,n\to\infty}\left(V_{lmn}^{111^{(0)}} - V_{lmn}^{111^{(1)}}\right) &\geq \liminf_{l,m,n\to\infty} \min_{\tilde{\kappa}Q_m < Q_j \leq Q_m}\left(V_{lmn}^{111^{(0)}} - V_{ljk}^{111^{(0)}}\right) \\
&+ \liminf_{l,m,n\to\infty} \min_{\substack{\tilde{\lambda}P_l \leq P_i \leq P_l \\ \tilde{\kappa}Q_m \leq Q_j \leq Q_m \\ \tilde{\delta}R_n \leq R_k \leq Q_n}} \left(V_{ljk}^{111^{(0)}} - V_{ijk}^{11^{(0)}}\right).
\end{aligned}$$

If we get limit of both sides of last inequality as $\tilde{\lambda}, \tilde{\kappa}, \tilde{\delta} \to 1^-$, then we arrive

$$\liminf_{l,m,n\to\infty}\left(V_{lmn}^{111^{(0)}} - V_{lmn}^{111^{(1)}}\right) \geq 0 \tag{4.7}$$

due to $\left(V_{lmn}^{111^{(0)}}(\Delta_{111}z)\right) \in \mathscr{SD}_{(P)} \cap \mathscr{SD}_{(Q)} \cap \mathscr{SD}_{(R)} \cap \mathscr{SDST}_{(P)}$. If we combine inequalities (4.4) with (4.7), we reach

$$\lim_{l,m,n\to\infty} V_{lmn}^{111^{(0)}} = \lim_{l,m,n\to\infty} V_{lmn}^{111^{(1)}}$$

which means that $\left(V_{lmn}^{111^{(0)}}(\Delta_{111}z)\right)$ is P-convergent to 0 . Therefore, we conclude from the triple weighted Kronecker identity that (z_{lmn}) is P-convergent to ℓ. □

In regard to Theorem 4.1, we can point out the following theorem.

THEOREM 4.2: *Let $(z_{lmn}) \in \ell_\infty^3(\mathbb{R})$ and $(p_l), (q_m) \in SVA_{\mathrm{reg}(\alpha)}$. If a sequence (z_{lmn}) is $(\bar{N}, p, q, r)$ summable to a number ℓ and conditions*

$$\frac{P_l}{p_l}\Delta_{100}V_{lmn}^{111^{(0)}}(\Delta_{111}z) = O_L(1), \frac{Q_m}{q_m}\Delta_{010}V_{mn}^{111^{(0)}}(\Delta_{111}z) = O_L(1)$$

$$\text{and } \frac{R_n}{r_n}\Delta_{001}V_{lmn}^{111^{(0)}}(\Delta_{111}z) = O_L(1) \tag{4.8}$$

are satisfied, then (z_{lmn}) is P-convergent to ℓ.

Proof. Assume that (z_{lmn}) is $(\bar{N}, p, q, r)$ summable to ℓ and conditions (4.8) are satisfied. If we indicate that conditions (4.8) imply one of the conditions (4.1), shall we say, $\left(V_{lmn}^{111^{(0)}}(\Delta_{111}z)\right) \in \mathscr{S}_{\mathscr{D}_{(P)})} \cap \mathscr{S}\mathscr{D}_{(Q)} \cap \mathscr{S}\mathscr{D}_{(R)} \cap \mathscr{S}\mathscr{D}\mathscr{S}\mathscr{T}_{(Q)}$, then we prove this theorem with the help of Theorem 4.1. Put $\mu = \operatorname{argmin}\{P_i \geq \lambda P_l\}$, $\eta = \operatorname{argmin}\{Q_j \geq \kappa Q_m\}$ and $\chi = \operatorname{argmin}\{R_k \geq \delta R_n\}$ with $\lambda, \kappa, \delta > 1$. Then, we have for $n_0 \leq m \leq i \leq \mu$ and $n_0 \leq n$

$$V_{lmn}^{111^{(0)}} - V_{lmn}^{111^{(0)}} = \sum_{j=m+1}^{i} \Delta_{10}V_{kn}^{11^{(0)}} \geq -M_1 \sum_{k=m+1}^{i} \frac{p_k}{P_k}$$
$$\geq -M_1\left(\frac{P_\mu}{P_m} - 1\right) \geq -M_1(\lambda - 1 + \lambda o(1))$$

for any constant $M_1 > 0$. If we get lim inf and limit of both sides of last inequality as $m, n \to \infty$ and $\lambda \to 1^+$respectively, then we reach

$$\lim_{\lambda\to 1^+} \liminf_{m,n\to\infty} \min_{P_m \leq P_i \leq \lambda P_m} \left(V_{imn}^{111^{(0)}} - V_{lmn}^{111^{(0)}}\right) \geq 0$$

which means that $\left(V_{lmn}^{111^{(0)}}(\Delta_{111}u)\right) \in \mathscr{S}\mathscr{D}_{(P)}$. On the other hand, we obtain for $n_0 \leq m, n$ and $n_0 \leq m, n \leq j, k \leq \eta, \chi$

$$V_{ijn}^{111^{(0)}} - V_{lmn}^{111^{(0)}} = \sum_{r=n+1}^{j} \Delta_{01}V_{mr}^{111^{(0)}} \geq -M_2 \sum_{r=n+1}^{j} \frac{q_r}{Q_r}$$
$$\geq -M_2\left(\frac{Q_\eta}{Q_n} - 1\right) \geq -M_2(\kappa - 1 + \lambda o(1))$$

for any constant $M_2 > 0$. If we get lim inf and limit of both sides of last inequality as $l, m, n \to \infty$ and $\kappa \to 1^+$respectively, then we reach

$$\lim_{\kappa\to 1^+} \liminf_{l,m,n\to\infty} \min_{Q_m \leq Q_j \leq \kappa Q_m} \left(V_{ljn}^{111^{(0)}} - V_{lmn}^{111^{(0)}}\right) \geq 0$$

which means that $\left(V_{lmn}^{111^{(0)}}(\Delta_{111}z)\right) \in \mathscr{S}\mathscr{D}_{(Q)}$. Similarly, we can easily observe that $\left(V_{lmn}^{111^{(0)}}(\Delta_{111}z)\right) \in \mathscr{S}\mathscr{D}\mathscr{S}\mathscr{T}(Q)$ is verified. Therefore, we conclude with the help of Theorem 4.1 that (z_{lmn}) is P-convergent to ℓ. □

Analogous results for triple sequences of complex numbers can be formulated as follows.

Theorem 4.3: *Let* $(z_{lmn}) \in \ell_\infty^3(\mathbb{R})$ *and* $(p_i), (q_m), (r_n) \in SVA_{\mathrm{reg}(\alpha)}$. *If a sequence* (z_{lmn}) *is* $(\bar{N}, p, q, r)$ *summable to a number* ℓ *and*

$$\left(V_{lmn}^{111^{(0)}}(\Delta_{111} z)\right) \in \mathscr{SO}_{(P)} \cap \mathscr{SO}_{(Q)} \cap \mathscr{SO}_{(R)} \cap \mathscr{SOST}(Q)$$

or

$$\left(V_{lmn}^{111^{(0)}}(\Delta_{111} z)\right) \in \mathscr{SO}_{(P)} \cap \mathscr{SO}_{(Q)} \cap \mathscr{SO}_{(R)} \cap \mathscr{SOST}(P)$$

or

$$\left(V_{lmn}^{111^{(0)}}(\Delta_{111} z)\right) \in \mathscr{SO}_{(P)} \cap \mathscr{SO}_{(Q)} \cap \mathscr{SO}_{(R)} \cap \mathscr{SOST}(R)$$

then (z_{lmn}) *is* P*-convergent to* ℓ.

In regard to Theorem 4.3 , we can point out the following theorem.

Theorem 4.4: *Let* $(z_{lmn}) \in \ell_\infty^3(\mathbb{R})$ *and* $(p_l), (q_m), (r_n) \in SVA_{\mathrm{reg}(\alpha)}$. *If a sequence* (z_{lmn}) *is* $(\bar{N}, p, q, r)$ *summable to a number* ℓ *and conditions*

$$\frac{P_l}{p_l}\Delta_{100} V_{lmn}^{111^{(0)}}(\Delta_{111} z) = O(1), \frac{Q_m}{q_m}\Delta_{010} V_{lmn}^{111^{(0)}}(\Delta_{111} z) = O(1)$$

and

$$\frac{R_n}{R_n}\Delta_{001} V_{lmn}^{111^{(0)}}(\Delta_{111} z) = O(1)$$

are satisfied, then (z_{lmn}) *is* P*-convergent to* ℓ. *Before finishing this section, we discuss some conditions needed for* $(\bar{N}, p, q, r)$ *summable triple sequences to be convergent.*

Theorem 4.5: *Let* $(z_{lmn}) \in \ell_\infty^3(\mathbb{R})$ *and* $(p_i), (q_m), (r_n) \in SVA_{\mathrm{reg}(\alpha)}$. *If a sequence* (z_{lmn}) *is* $(\bar{N}, p, q, r)$ *summable to a number* ℓ *and conditions*

$$\lim_{\lambda,\kappa,\delta \to 1^+} \liminf_{l,m,n\to\infty} \left(\tau_{lmn}^{\mu\eta\chi} - z_{lmn}\right) \geq 0 \tag{4.9}$$

and

$$\lim_{\lambda,\kappa,\delta \to 1^-} \liminf_{l,m,n\to\infty} \left(z_{lmn} - \tau_{\mu\eta\chi}^{lmn}\right) \geq 0 \tag{4.10}$$

are satisfied, then (z_{lmn}) *is* P*-convergent to* ℓ.

Proof. Assume that (z_{lmn}) is $(\bar{N}, p, q, r)$ summable to ℓ and conditions (4.9) and (4.10) are satisfied. To prove that (z_{lmn}) is P-convergent to the same number, it is enough to prove that conditions in (3.5) and (3.6) are verified. For $\mu > l$, $\eta > m$ ans $\chi > n$, we have from Lemma 3.3(i)

$$\sigma^{111}_{\mu mn}(z) - \tau^{\mu\eta\chi}_{lmn}(z) = \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\eta - R_n)}\left(\sigma^{111}_{\mu mn} - \sigma^{111}_{\mu\eta\chi} - \sigma^{111}_{lmn} + \sigma^{111}_{l\eta}\right)$$

(4.11)

$$+ \frac{P_l}{P_\mu - P_l}\left(\sigma^{111}_{lmn} - \sigma^{111}_{\mu mn}\right) + \frac{Q_\eta}{Q_\eta - Q_m}\left(\sigma^{111}_{lmn} - \sigma^{111}_{l\eta n}\right) + \frac{R_\chi}{R_\chi - R_m}\left(\sigma^{111}_{lmn} - \sigma^{111}_{lm\chi}\right).$$

Putting $\mu = \text{argmin}\{P_i \geq \lambda P_l\}$, $\eta = \text{argmin}\{Q_j \geq \kappa Q_n\}$ and $\chi = \text{argmin}\{R_k \geq \delta R_n\}$ with $\lambda, \kappa, \delta > 1$, we can observe inequalities (3.9), (3.10) and, accordingly, (3.11). Grounding that $\sigma^{11}_{mn}(u) \to \ell$ as $l, m, n \to \infty$, we could remark $\sigma^{111}_{lmn} - \sigma^{111}_{ijk} \to 0$ for $i = l$ or μ, $j = m$ or η and $k = n$ or χ as $l, m, n \to \infty$. If we get lim inf of both sides of equality (4.11) as $l, m, n \to \infty$, then we obtain

$$\begin{aligned}
&\liminf_{l,m,n\to\infty}\left(\sigma^{111}_{\mu mn} - \tau^{\mu\eta}_{lmn}\right)\\
&\geq \left(\frac{\lambda}{\lambda-1}\right)\left(\frac{\kappa}{\kappa-1}\right)\left(\frac{\delta}{\delta-1}\right)\liminf_{l,m,n\to\infty}\left(\sigma^{11}_{\mu mn} - \sigma^{111}_{\mu\eta\chi} - \sigma^{111}_{lmn} + \sigma^{111}_{l\eta n} + \sigma^{111}_{lm\chi}\right)\\
&\quad + \frac{1}{\lambda-1}\liminf_{l,m,n\to\infty}\left(\sigma^{111}_{lmn} - \sigma^{111}_{\mu mn}\right) + \frac{1}{\kappa-1}\liminf_{l,m,n\to\infty}\left(\sigma^{11}_{lmn} - \sigma^{11}_{l\eta n}\right)\\
&+ \frac{1}{\delta-1}\liminf_{l,m,n\to\infty}\left(\sigma^{11}_{lmn} - \sigma^{11}_{lm\chi}\right)
\end{aligned}$$

where

$$\frac{P_\mu}{P_\mu - P_l} = \frac{P_\mu/P_l}{P_\mu/P_m - 1} \leq \frac{\lambda}{(\lambda-1)}(1 + o(1)),$$

$$\frac{Q_\eta}{Q_\eta - Q_m} = \frac{Q_\eta/Q_m}{Q_\eta/Q_m - 1} \leq \frac{\kappa}{(\kappa-1)}(1 + o(1))$$

and

$$\frac{R_\chi}{R_\chi - R_n} = \frac{R_\chi/R_n}{R_\chi/R_n - 1} \leq \frac{\delta}{(\delta-1)}(1 + o(1)).$$

If we get limit of both sides of last inequality as $\lambda, \kappa, \delta \to 1^+$, then we arrive

$$\lim_{\lambda,\kappa,\delta\to 1^+}\liminf_{l,m,n\to\infty}\left(\sigma^{111}_{\mu mn} - \tau^{\mu\eta\chi}_{lmn}\right) \geq 0.$$

In the same vein, for $\mu < l$, $\eta < m$ and $\chi < n$, we have from Lemma 3.3(i)

$$\sigma^{111}_{l\eta n}(z) - \tau^{\mu\eta\chi}_{lmn}(z) = \frac{P_\mu Q_\eta}{(P_\mu - P_l)(Q_\eta - Q_n)}\left(\sigma^{111}_{\mu mn} - \sigma^{111}_{\mu\eta\chi} - \sigma^{111}_{lmn} + \sigma^{111}_{l\eta n}\right)$$

(4.12)

$$+ \frac{P_\mu}{P_\mu - P_l}\left(\sigma^{111}_{lmn} - \sigma^{111}_{\mu mn}\right) + \frac{Q_m}{Q_\eta - Q_m}\left(\sigma^{111}_{lmn} - \sigma^{111}_{l\eta n}\right) + \frac{R_n}{R_\eta - R_n}\left(\sigma^{111}_{lmn} - \sigma^{111}_{lm\chi}\right).$$

If we get lim inf of both sides of equality (4.12) as $l, m, n \to \infty$, then we obtain

$$\begin{aligned}
&\liminf_{l,m,n\to\infty} \left(\sigma^{111}_{l\eta n}(z) - \tau^{\mu\eta\chi}_{lmn}(z)\right) \\
&\geq \left(\frac{\lambda}{\lambda-1}\right)\left(\frac{\kappa}{\kappa-1}\right)\left(\frac{\delta}{\delta-1}\right)\left(\sigma^{111}_{\mu mn} - \sigma^{111}_{\mu\eta\chi} - \sigma^{111}_{lmn} + \sigma^{111}_{l\eta n} + \sigma^{111}_{lm\chi}\right) \\
&\quad + \frac{\lambda}{\lambda-1}\liminf_{l,m,n\to\infty}\left(\sigma^{111}_{lmn} - \sigma^{111}_{\mu mn}\right) + \frac{\kappa}{\kappa-1}\liminf_{l,m,n\to\infty}\left(\sigma^{111}_{lmn} - \sigma^{111}_{l\eta n}\right) \\
&+ \frac{\delta}{\delta-1}\liminf_{l,m,n\to\infty}\left(\sigma^{111}_{lmn} - \sigma^{111}_{lm\chi}\right).
\end{aligned}$$

If we get limit of both sides of last inequality as $\lambda, \kappa, \delta \to 1^+$, then we arrive

$$\lim_{\lambda,\kappa,\delta\to 1^+}\liminf_{l,m,n\to\infty}\left(\sigma^{111}_{l\eta n}(z) - \tau^{\mu\eta\chi}_{lmn}(z)\right) \geq 0.$$

In addition to what is attained above, for $\mu < l$, $\eta < m$ and $\chi < n$, we have from Lemma 3.3(i)

$$\begin{aligned}
&\sigma^{111}_{\mu\eta\chi} - \tau^{\mu\eta\chi}_{lmn} \\
&= \frac{P_l Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)}\left(\sigma^{111}_{l\eta n} - \sigma^{111}_{\mu\eta\chi}\right) \\
&+ \frac{P_\mu Q_m R_n}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)}\left(\sigma^{111}_{\mu mn} - \sigma^{111}_{lmn}\right) + \frac{Q_m}{Q_\eta - Q_m}\left(\sigma^{111}_{lmn} - \sigma^{111}_{\mu\eta\chi}\right) \\
&+ \frac{R_n}{R_\chi - R_n}\left(\sigma^{111}_{lmn} - \sigma^{11}_{\mu\eta\chi}\right). \qquad (4.13)
\end{aligned}$$

If we get lim sup of both sides of equality (4.13) as $l, m, n \to \infty$, then we obtain

$$\begin{aligned}
&\limsup_{l,m,n\to\infty}\left(\sigma^{111}_{\mu\eta\chi} - \tau^{\mu\eta\chi}_{lmn}\right) \\
&\leq \frac{\kappa}{(\lambda-1)(\kappa-1)(\delta-1)}\limsup_{l,m,n\to\infty}\left(\sigma^{111}_{l\eta n} - \sigma^{111}_{\mu\eta\chi}\right) + \frac{\lambda}{(\lambda-1)(\kappa-1)(\delta-1)}\limsup_{l,m,n\to\infty}\left(\sigma^{111}_{\mu mn} - \sigma^{111}_{lmn}\right) \\
&+ \frac{\delta}{(\lambda-1)(\kappa-1)(\delta-1)}\limsup_{l,m,n\to\infty}\left(\sigma^{111}_{\mu mn} - \sigma^{111}_{lmn}\right) + \frac{1}{\kappa-1}\limsup_{l,m,n\to\infty}\left(\sigma^{111}_{lmn} - \sigma^{111}_{\mu\eta\chi}\right).
\end{aligned}$$

If we get limit of both sides of last inequality as $\lambda, \kappa, \delta \to 1^+$, then we arrive

$$\lim_{\lambda,\kappa,\delta\to 1^+}\limsup_{l,m,n\to\infty}\left(\sigma^{111}_{\mu\eta\chi} - \tau^{\mu\eta\chi}_{lmn}\right) \leq 0.$$

Hence, we can state that conditions in (3.5) are verified. Following a similar procedure to above for $\mu < l$, $\eta < m$ and $\chi < n$, we can behold that conditions in (3.6) are also verified. In that case, we reach from Lemma 3.4 that $\left(V^{111^{(0)}}_{lmn}(\Delta_{111} z)\right)$ is P-convergent to 0 . Therefore, we conclude from the triple weighted Kronecker identity that (z_{lmn}) is P-convergent to ℓ. □

References

[1] S. Baron and U. Stadtmüller, *Tauberian theorems for power series methods applied to double sequences*, J. Math. Anal. Appl., **211** (2) (1997), 574–589.

[2] C. Belen, *Some Tauberian theorems for weighted means of bounded double sequences*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), **63** (1) (2017), 115–122.

[3] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, 1989.
[4] R. Bojanic and E. Seneta, *A unified theory of regularly varying sequences*, Math. Z., **134** (1973), 91–106.
[5] T. J. I. A. Bromwich, *An Introduction to the Theory of Infinite Series*, MacMillan & Co., Ltd., London, 1908.
[6] C. P. Chen and J. M. Hsu, *Tauberian theorems for weighted means of double sequences*, Anal. Math., **26** (4) (2000), 243–262.
[7] İ. Çanak, *A theorem for convergence of generator sequences*, Comput. Math. Appl., **61** (2) (2011), 408–411.
[8] A. Esi, *Statistical convergence of triple sequences in topological groups*, Ann. Univ. Craiova Math. Comput. Sci. Ser., **40** (1) (2013), 29–33.
[9] Á. Fekete, *Tauberian conditions for double sequences that are statistically summable by weighted means*, Sarajevo J. Math., **1** (14) (2) (2005), 197–210.
[10] G. H. Hardy, *On the convergence of certain multiple series*, Proc. Lond. Math. Soc., **2** (1) (1904), 124–128.
[11] T. Jalal and I. A. Malik, *I-convergent triple sequence spaces over n-normed space*, Asia Pac. J. Math., **5** (2) (2018), 233–242.
http://dx.doi.org/10.22199/issn.0717-6279-4867
[12] T. Jalal and I. A. Malik, *Some new triple sequence spaces over n-normed space*, Proyecciones (Antofagasta), **37** (3) (2018), 547–564.
http://dx.doi.org/10.4067/S0716-09172018000300547
[13] T. Jalal and I. A. Malik, *I-Convergence of triple difference sequence spaces over n-normed space*, Tbilisi Math. J., **11** (4) (2018), 93–102.
https://doi.org/10.32513/tbilisi/1546570888
[14] T. Jalal and I. A. Malik, *Topological and algebraic properties of triple n-normed spaces*, Proc. 5th Int. Conf. Math. Comput., Springer Singapore, (2021), 187–196.
[15] A. H. Jan and T. Jalal, *On lacunary Δ^m-statistical convergence of triple sequence in intuitionistic fuzzy n-normed space*, Korean J. Math., **31** (3) (2023), 349–361.
https://doi.org/10.11568/kjm.2023.31.3.349
[16] A. H. Jan and T. Jalal, *On lacunary Δ^m-statistical convergence of triple sequence in intuitionistic fuzzy normed space*, Bull. Transilv. Univ. Brasov Ser. III Math. Comput. Sci., **66** (4) (2024), 101–116.
https://doi.org/10.31926/but.mif.2024.4.66.1.7
[17] A. H. Jan, Y. A. Bhat, T. Jalal, and N. A. Sheikh, *On generation and properties of triple sequence-induced frames in Hilbert spaces*, J. Numer. Anal. Approx. Theory, To appear, (2024).
[18] J. Karamata, *Sur un mode de croissance régulière. Théorèmes fondamentaux*, Bull. Soc. Math. Fr., **61** (1933), 55–62.
[19] V. N. Mishra, K. Khatri, and L. N. Mishra, *Strong Cesàro summability of triple Fourier integrals*, Fasciculi Math., **53** (2014), 95–112.
[20] L. N. Mishra, M. Raiz, L. Rathour, and V. N. Mishra, *Tauberian theorems for weighted means of double sequences in intuitionistic fuzzy normed spaces*, Yugosl. J. Oper. Res., **32** (3) (2022), 377–388.
https://doi.org/10.2298/YJOR210915005M
[21] V. N. Mishra and L. N. Mishra, *Trigonometric approximation of signals (functions) in $L^p (p \geq 1)$-norm*, Int. J. Contemp. Math. Sci., **7** (19) (2012), 909–918.
[22] V. N. Mishra, L. N. Mishra, N. Subramanian, and S. A. A. Abdulla, *Analytic weighted rough statistical convergence with rate of rough convergence and Voronovskaya theorem of triple difference sequences*, Appl. Sci., **22** (2020), 157–168.
[23] F. Móricz, *Tauberian theorems for Cesàro summable double sequences*, Studia Math., **110** (1) (1994), 83–96.
[24] F. Móricz and U. Stadtmüller, *Summability of double sequences by weighted mean methods and Tauberian conditions for convergence in Pringsheim's sense*, Int. J. Math. Math. Sci., (2004), 65–68, 3499–3511.

[25] Z. Önder, E. Savaş, and İ. Çanak, *On the weighted generator of double sequences and its Tauberian conditions*, Adv. Oper. Theory, **8** (3) (2023), 38.
[26] A. Pringsheim, *Zur Theorie der zweifach unendlichen Zahlenfolgen*, Anal. Math., **53** (3) (1900), 289–321.
[27] D. Rai, N. Subramanian, and V. N. Mishra, *The generalized difference of $\int \chi^{2I}$ of fuzzy real numbers over p-metric spaces defined by Musielak Orlicz function*, New Trends Math. Sci., **4** (3) (2016), 296–306.
https://doi.org/10.20852/ntmsci.2016320385
[28] A. Sahiner, M. Gurdal, and F. K. Duden, *Triple sequences and their statistical convergence*, Turk. Klin. Psychiatr. J., **8** (2) (2007), 49–55.
[29] U. Stadtmüller, *Tauberian theorems for weighted means of double sequences*, Anal. Math., **25** (1) (1999), 57–68.
[30] Vandana, Deepmala, N. Subramanian, and V. N. Mishra, *Riesz triple probabilistic of almost lacunary Cesàro C_{111} statistical convergence of χ^3 defined by a Musielak Orlicz function*, Bol. Soc. Paran. Mat., **36** (4) (2018), 23–32.
https://doi.org/10.5269/bspm.v36i4.32870
[31] Vandana, Deepmala, N. Subramanian, and V. N. Mishra, *The intuitionistic triple χ of ideal fuzzy real numbers over p-metric spaces defined by Musielak Orlicz function*, Asia Pac. J. Math., **5** (1) (2018), 1–13.

Asif Hussain Jan
Department of Mathematics, National Institute of Technology,
Hazratbal, Srinagar -190006, Jammu and Kashmir, India.
E-mail: asif_06phd20@nitsri.net

Tanweer Jalal
Department of Mathematics, National Institute of Technology,
Hazratbal, Srinagar -190006, Jammu and Kashmir, India.
E-mail: tjalal@nitsri.ac.in