Korean J. Math. **32** (2024), No. 4, pp. 727–731
https://dx.doi.org/10.11568/kjm.2024.32.4.727

ON THE N-SUPERCYCLICITY OF ISOMETRIES ON BANACH SPACES

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Abstract. In this paper, we present a simple and self-contained proof that isometries are not N-supercyclic and m-isometries are not supercyclic, providing an alternative to the proof given by the authors in [4, 5, 11].

1. Introduction

The notion of m-isometry, as an extension of isometry, was introduced by Agler in the eighties, and it was thoroughly studied by Agler and Stankus in a series of three papers [1–3]. A bounded linear operator T on a complex Hilbert space H is called an m-isometry if it satisfies

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^{k}=0. \tag{1}$$

It is easy to see that the condition is equivalent to

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{m}{k}\|T^{k}x\|^{2}=0 \quad (\text{ for all } x\in H) \tag{2}$$

The definition of m-isometric on Banach spaces was provided by Bayart, and some preliminary properties related to them, similar to those existing in Hilbert space, were developed. Bayart [5] used condition (2) as the basis for defining isometries on Banach spaces. In fact if X is a Banach space and $T: X\to X$ is a bounded linear operator the T is an m-isometry if and only if condition (2) holds. For any $x\in X$ and any $j,n,k\geq 0$, let us define the Beta function as follows:

$$\beta_j(x)=\beta_j(T,x):=\frac{1}{j!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\|T^{k}x\|^{2}.$$

In terms of the Beta function the operator T is an m-isometry if and only if $\beta_m(x)=0$. Additionally, T is an m-isometry and not an $(m-1)$-isometry if and only if $\beta_m\equiv 0$ and $\beta_{m-1}(x)\neq 0$ for some $x\in X$. The Beta function helps us to better identify the properties of isometric operators. The following result briefly states the preliminary

Received August 2, 2024. Revised September 29, 2024. Accepted October 10, 2024.
2010 Mathematics Subject Classification: 47A16.
Key words and phrases: Supercyclic operators, m-isometry.
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properties of m-isometries on Banach spaces. For the proof, one can refer to reference [5]. Recall that the symbol $n(j)$ used in the following proposition has the following meaning:

$$n(j) = \begin{cases} 1 & \text{if } n = 0 \text{ or } j = 0, \\ n(n-1)\cdots(n-j+1) & \text{otherwise.} \end{cases}$$

PROPOSITION 1.1. *Let T be any m-isometry. Then*

(i) $\|T^n x\|^2 = \sum_{j=0}^{m-1} n(j)\beta_j(x)$ *for every* $n \geq 1$.

(ii) *If $m > 1$ then $\|T^n x\| \to +\infty$ for every $x \in X$.*

(iv) *If T is invertible then T^{-1} is also an m-isometry.*

(v) *For every $x \in X$. There exists $n_x \in \mathbb{N}$ such that $\|T^{n+1}x\| \geq \|T^n x\|$ for any $n \geq n_x$.*

(vi) *The approximate point spectrum of T lies in the unit circle and so $\sigma(T) \subset \mathbb{T}$ or $\sigma(T) = \mathbb{D}$.*

In the context of isometric operators, Ansari and Bourdon [4] proved that isometries are not supercyclic. However, the proof is based on a result by R. Godement [12] which states that isometries always have non-trivial invariant subspaces. The study of the dynamics of m-isometries began in [8] where it was proved that an m-isometry acting on a Hilbert space H with an injective covariance operator cannot be N-supercyclic. In [11], the authors showed that m-isometries on a Hilbert space are not supercyclic. The result was extended in [5] by showing that, for any $N, m \geq 1$, an m-isometry cannot be N-supercyclic, without any further assumptions on the m-isometry or on the underlying Banach space X.

In this article, our aim is to provide an alternative, simple, and independent proof of the non-supercyclic nature of isometries and m-isometries.

2. Main result

Recall that a bounded linear operator T on a Banach space X is called N-supercyclic, $N \geq 1$, if there exists a subspace M of X with $dim(M) = N$ such that

$$orb(T, M) = \bigcup_{n \geq 0} T^n(M)$$

is dense in X. If $N = 1$ then T is called briefly supercyclic.

Authors in [4] showed that if $\{\|T^n\|\}$ is bounded and T is supercyclic, then at least one orbit of T must tend to zero:

THEOREM 2.1. *Suppose T is a bounded linear operator on a Banach space with the following properties:*

(a) There exists $M > 0$ such that $\|T^n\| \leq M$ for each positive integer n.
(b) For each nonzero $x \in X$, $T^n x \nrightarrow 0$ as $n \to \infty$.

Then T has no supercyclic vectors.

Theorem 2.2. *Let T be an isometry on a separable Banach space X with $dim(X) > N$ then T is not N-supercyclic.*

Proof. By contradiction, suppose that T is an N-supercyclic isometry and there exists an N-dimensional subspace M such that $\text{orb}(T, M)$ is dense in X. Assume that M is generated by the linearly independent vectors $x_1, x_2, \dots, x_N$.

Claim: For any $z \in X$, there exists a sequence $\{n_k\}$ of positive integers and a unique vector $x \in M$ such that $\|x\| = \|z\|$ and $T^{n_k}x \to z$.

To see this, let $z \in X$. Then there exist scalar sequences $\{\lambda_{1k}\}, \{\lambda_{2k}\}, \dots, \{\lambda_{Nk}\}$ and some integer sequence $\{n_k\}$ such that

$$T^{n_k}\big(\lambda_{1k}x_1 + \lambda_{2k}x_2 + \ldots + \lambda_{Nk}x_N\big) \to z.$$

Consider the bounded linear functionals $\{f_1, f_2, \dots, f_N\}$ such that $f_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. Then

$$\big\|\lambda_{1k}x_1 + \lambda_{2k}x_2 + \ldots + \lambda_{Nk}x_N\big\| = \big\|T^{n_k}\big(\lambda_{1k}x_1 + \lambda_{2k}x_2 + \ldots + \lambda_{Nk}x_N\big)\big\| \le M$$

for some scalar M. Hence

$$\begin{aligned} |\lambda_{1k}| &= |f_1(\lambda_{1k}x_1 + \lambda_{2k}x_2 + \ldots + \lambda_{Nk}x_N)| \\ &\le \|f_1\|\|\lambda_{1k}x_1 + \lambda_{2k}x_2 + \ldots + \lambda_{Nk}x_N\| \le \|f\|M. \end{aligned}$$

The above observations lead to the conclusion that the bounded sequence $\{\lambda_{1k}\}$ has a convergent subsequence. By passing to a subsequence, we can assume that $\lambda_{1k} \to \lambda_1$ for some scalar λ_1. By continuing this process for other sequences and passing to successive subsequences, we can assume that $\lambda_{ik} \to \lambda_i$ for some scalar λ_i, $i = 1, 2, \dots, N$. Thus

$$\begin{aligned} \big\|\lambda_1x_1 + \lambda_2x_2 + \ldots + \lambda_Nx_N\big\| &= \lim_k \big\|\lambda_{1k}x_1 + \lambda_{2k}x_2 + \ldots + \lambda_{Nk}x_N\big\| \\ &= \lim_k \big\|T^{n_k}\big(\lambda_{1k}x_1 + \lambda_{2k}x_2 + \ldots + \lambda_{Nk}x_N\big)\big\| \\ &= \|z\|. \end{aligned}$$

Let $x = \lambda_1x_1 + \lambda_2x_2 + \ldots + \lambda_Nx_N$. Then $x \in M$, $\|x\| = \|z\|$, and $T^{n_k}x \to z$. Considering the relation:

$$\|x_1 - x_2\| = \|T^{n_k}(x_1 - x_2)\| \quad (x_1, x_2 \in X),$$

the uniqueness of x is deduced. Now the mapping $\Lambda : X \to M$ by $\Lambda(z) = x$ defines a function satisfying $\|\Lambda(z)\| = \|z\|$. This forces that $\dim(X) \le \dim(M) = N$, which is a contradiction. □

Theorem 2.3. *Let X be a Banach space with $dim(X) > 1$ then for every positive integer m, an m-isometry is never supercyclic.*

Proof. If $m = 1$, then T is an isometry and by the Theorem 2.2, it is not supercyclic. Assume that $m > 1$. Let $T : X \to X$ be an m-isometry, and by contradiction, suppose that x_0 is a supercyclic vector for T. Then for any $x \in X$, there exist sequences $\{\lambda_k\}$ of scalars and $\{n_k\}$ of positive integers such that

$$\lambda_k T^{n_k} x_0 \to x.$$

Applying Proposition 1.1 (ii), $\|T^{n_k}x\| \to +\infty$, hence $\lambda_k \to 0$. This implies that the set $\{\lambda T^n x : |\lambda| \le 1, n \ge 0\}$ must be dense in X. However, the recent set itself is a subset of the balanced convex hull of $\text{orb}(T, x)$. Using Theorem 2.3 in [7],

$\sup_n |f(T^n x)| = +\infty$ which implies that $\sigma_p(T^*) \cap \overline{\mathbb{D}} = \emptyset$ (see Proposition 2.5 in [7]). Considering Proposition 1.1 (vi), T is invertible. Hence T^{-1} is also supercyclic and so is an m-isometry by Proposition 2.1. Let y_0 be a supercyclic vector for T^{-1}. Then using part (v) of Proposition 1.1 for both m-isometries T and T^{-1}, there exists some integer N such that

$$\|\lambda T^n x_0\| \leq \|T(\lambda T^n x_0)\| \quad \text{and} \quad \|\lambda T^{-n} y_0\| \leq \|T^{-1}(\lambda T^{-n} x_0)\|$$

for every integer $n \geq N$ and every scalar λ. Since both $\mathbb{C}\,\mathrm{orb}(T, x_0)$ and $\mathbb{C}\,\mathrm{orb}(T^{-1}, y_0)$ are dense in X, $\|x\| \leq \|Tx\|$ and $\|x\| \leq \|T^{-1}x\|$ for every $x \in X$. This implies that T is an isometry and so by Theorem 2.2, it cannot be supercyclic. □

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