Korean J. Math. **33** (2025), No. 3, pp. 193–203
https://dx.doi.org/10.11568/kjm.2025.33.3.193

UNIVARIATE POLYNOMIALS OF CONSECUTIVE DEGREES THAT FORM A SAGBI BASIS

Berehanu Belayneh*, Dawit Solomon, and Mohammed Tesemma†

Abstract. In this paper, we provide necessary and sufficient conditions for polynomials of consecutive degrees that form a SAGBI basis in the univariate polynomial ring. The special case of three polynomials with consecutive degrees is also considered.

1. Introduction

One of the algorithmic approaches to studying ideals in commutative algebra and algebraic geometry is the introduction of Gröbner basis by Bruno Buchberger in his 1965 PhD thesis. A similar set of special generators for subalgebras of multivariable polynomial rings, known as SAGBI basis, was introduced in the late 1980's by Robbiano & Sweedler [9], and independently by Kapur & Madlener [6]. The term SAGBI is an acronym for Subalgebra Analogue to Gröbner Bases for Ideals. However, unlike ideals, not every subalgebra of a multivariable polynomial ring is finitely generated, hence has no finite SAGBI basis. Furthermore, there are subalgebras that are finitely generated but have no finite SAGBI basis, see [9, Example 1.20]. It is indeed an open problem to determine which subalgebras have finite SAGBI basis; see [11, p. 100]. For further reference on SAGBI theory, its connection and application to other fields we refer to [2], [3], [4], [5], [7, Chapter 6.6], [9], [11, Chapter 11], and [12]. Let's consider a univariate polynomial ring, $\Bbbk[x]$, with indeterminate x over a base field $\Bbbk$. Torstensson et al. [13], set a sufficient condition when two polynomials in $\Bbbk[x]$ form a SAGBI basis for the subalgebra they generate. There is no obvious generalization of their results to three or more polynomials. Our interest in this paper is inspired by their work, where we set necessary and sufficient conditions for three or more polynomials of consecutive degrees to form a SAGBI basis. In Section 2 we state the definition of SAGBI basis suitable for the univariate setting as well as some known results. Section 3 contains our main results. Several examples and counter examples will also be given along the way.

Received August 21, 2024. Revised June 29, 2025. Accepted August 19, 2025.
2010 Mathematics Subject Classification: 13P10.
Key words and phrases: initial algebra, subduction algorithm, SAGBI basis, T-polynomial.

† Supported by Simons Fellow Award #SFA-23-3.
* Corresponding author.
© The Kangwon-Kyungki Mathematical Society, 2024.

2. Definition and Examples

Throughout this paper, $\Bbbk[x]$ denotes the polynomial ring in one variable x with coefficients in a field $\Bbbk$. Let $f = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial in $\Bbbk[x]$ where $a_i \in \Bbbk, n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, and $a_n \neq 0$.

DEFINITION 1. Given f as above;

(i) The *initial of* f is the lead monomial given by $\mathrm{in}(f) = x^n$ where $n = \deg(f)$ is the degree of f.

(ii) The *lead term* of f is $\mathrm{lt}(f) = a_n x^n$.

A subalgebra R of $\Bbbk[x]$ is a subring that contains the base field $\Bbbk$, hence is of the form $R = \Bbbk[G]$ for some $G \subseteq \Bbbk[x]\backslash\Bbbk$. By a *G-monomial* we mean a power product of elements of G. The set of all such monomials is given by

$$G_{\mathrm{mon}} = \{\Pi_{i=1}^{t} f_i^{\lambda_i} : f_i \in G;\ \lambda_i, t \in \mathbb{N}_0\}$$

The set G is called a generator of the subalgebra, and is not unique. If G can be chosen to be finite, say $G = \{f_1, \ldots, f_s\}$, we say R is finitely generated and write

$$R = \Bbbk[f_1, \ldots, f_s] = \{p(f_1, \ldots, f_s) : p \in \Bbbk[y_1, \ldots, y_s]\}$$

DEFINITION 2 (*Initial algebra)*. Given a subalgebra R of $\Bbbk[x]$, the *initial algebra of* R, denoted $\mathrm{in}(R)$, is the monomial algebra generated by the lead monomial of each nonconstant polynomial in R. That is;

$$\mathrm{in}(R) = \Bbbk[\mathrm{in}(f) : f \in R\backslash\Bbbk]$$

DEFINITION 3 (*SAGBI basis)*. Let R be a subalgebra of $\Bbbk[x]$ and $S \subseteq R$. We say S is a SAGBI basis for R if

$$\mathrm{in}(R) = \Bbbk[\mathrm{in}(f) : f \in S]$$

If S can be chosen to be finite, we say R has finite SAGBI basis. As noted in the introduction, there are subalgebras of a multivariable polynomial ring that have no finite SAGBI basis even if the subalgebra itself is finitely generated. However this is not the case in the univariate polynomial ring, as all subalgebras are finitely generated. We include an outline of the proof for completeness.

THEOREM 2.1. *Every subalgebra of a polynomial ring in one variable has finite SAGBI basis.*

Proof. Let R be a subalgebra of $\Bbbk[x]$, the set $A = \{\deg(f) : f \in R\backslash\Bbbk\}$ is a subsemigroup of the natural numbers $(\mathbb{N}, +)$. By [10, Corollary 1] such semigroups are finitely generated, say $A = \langle \deg(f_1), \ldots, \deg(f_\ell)\rangle_{\mathrm{mon}}$ for some $\ell \in \mathbb{N}$ and $f_i \in R$. It follows that

$$\mathrm{in}(R) = \Bbbk[\mathrm{in}(f) : f \in R\backslash\Bbbk] = \Bbbk[\mathrm{in}(f_1), \ldots, \mathrm{in}(f_\ell)]$$

Hence by definition the set $S = \{f_1, \ldots, f_\ell\}$ is a finite SAGBI basis for R. □

There is an algorithm in SAGBI theory called *subalgebra reduction (subduction) algorithm*. It is similar to division algorithm that is suitable for subalgebras. This algorithm allows polynomial rewriting in terms of the basis and other computational applications in SAGBI basis theory. For further reference we refer to [7, 6.6B] [9, 1.5].

Algorithm. Subduction Algorithm

INPUT: A SAGBI basis $S = \{f_1, \dots f_\ell\}$ for $R \leq \mathbb{k}[x]$, and $f \in \mathbb{k}[x]$.
OUTPUT: An expression of f as a polynomial in S, provided $f \in R$.
WHILE: f is not a constant in $\mathbb{k}$ do

1. Find exponents $i_1, ..., i_\ell \in \mathbb{N}_0$ and $k \in \mathbb{k}^*$, such that $\mathrm{lt}(f) = k\,\mathrm{lt}(f_1)^{i_1} \cdots \mathrm{lt}(f_\ell)^{i_\ell}$
2. If no representation of step 1 exists, then output f not in R. STOP
3. Otherwise $p = kf_1^{i_1} \cdots f_\ell^{i_\ell}$ and replace $f = f - p$

Output the polynomial $f = \sum p + k_1$ where $k_1 \in \mathbb{k}$.

An interesting problem raised in the work of Torstensson et al. [13] is when a priori a given set of polynomials, say $S = \{f_1, \dots, f_n\}$ form a SAGBI basis of the subalgebra $\mathbb{k}[S]$ they generate? In such case we simply say that S *is a SAGBI basis.* Among other things they showed that:

> A set of two polynomials $S = \{f, g\} \subseteq \mathbb{k}[x]$ form a SAGBI basis if $\deg(f)$ and $\deg(g)$ are relatively prime.

However, a set of three or more polynomials, whose degrees are pairwise relatively prime, may not necessarily form a SAGBI basis of the subalgebra they generate. Here is an example.

Example 1. Let $S = \{f = x^3 + 1, g = x^4 - 2x + 1, h = x^7 + x^5 + 3\}$. The degrees of the polynomials are pairwise relatively prime however S is not a SAGBI basis since the polynomial $T = h - fg \in \mathbb{k}[S]$ but $\mathrm{in}(T) = x^5$ is not in $\mathbb{k}[\mathrm{in}(f), \mathrm{in}(g), \mathrm{in}(h)] = \mathbb{k}[x^3, x^4, x^7]$.

Proposition 2.2. *Let $S = \{f_1, \dots, f_s\} \subseteq \mathbb{k}[x]$. The following are equivalent:*

(i). S is a SAGBI basis.
(ii). For each $0 \neq g \in \mathbb{k}[S]$ there exists $\lambda_i \in \mathbb{N}_0$ such that $\mathrm{in}(g) = \mathrm{in}(\prod_{i=1}^s f_i^{\lambda_i})$
(iii). For each $0 \neq g \in \mathbb{k}[S]$ there exists $\lambda_i \in \mathbb{N}_0$ such that $\deg(g) = \sum_{i=1}^s \lambda_i \deg(f_i)$

Proof. $\underline{(i) \Leftrightarrow (ii).}$ By definition, S is a SAGBI basis if and only if each nonzero polynomial $g \in \mathbb{k}[S]$ satisfies $\mathrm{in}(g) \in \mathbb{k}[\mathrm{in}(f_1), \dots, \mathrm{in}(f_s)]$. But as $\mathrm{in}(g)$ is a monomial it can only be expressed as power product of monomials in the generators of $\mathbb{k}[\mathrm{in}(f_1), \dots, \mathrm{in}(f_s)]$. Hence

$$\mathrm{in}(g) = \prod_{i=1}^{s} \mathrm{in}(f_i)^{\lambda_i} = \prod_{i=1}^{s} \mathrm{in}(f_i^{\lambda_i}) = \mathrm{in}(\prod_{i=1}^{s} f_i^{\lambda_i}),$$

for some $\lambda_i \in \mathbb{N}_0$.
$\underline{(ii) \Leftrightarrow (iii).}$ First observe that

$$\deg(\prod_{i=1}^{s} f_i^{\lambda_i}) = \sum_{i=1}^{s} \deg(f_i^{\lambda_i}) = \sum_{i=1}^{s} \lambda_i \deg(f_i).$$

Hence,

$$\mathrm{in}(g) = \mathrm{in}(\prod_{i=1}^{s} f_i^{\lambda_i}) = x^{\deg(\prod_{i=1}^{s} f_i^{\lambda_i})} \Leftrightarrow \deg(g) = \sum_{i=1}^{s} \lambda_i \deg(f_i).$$

□

Example 2. Let $\mathbb{Q}$ denote the field of rationals. The set of two polynomials $S = \{f = 2x^4 + x^3 - x - 5, g = x^2 - 1\}$ is not a SAGBI basis, since $h := f - 2g^2 = x^3 + 4x^2 - x - 6 \in \Bbbk[S]$ but $\deg(h) = 3$ is not an $\mathbb{N}_0$-linear combination of $\{\deg(f) = 4, \deg(g) = 2\}$.

3. Main Results

Let $S = \{f_1, \ldots, f_\ell\} \subseteq \Bbbk[x]$, by a T-polynomial in $\Bbbk[S]$ we mean a difference $T = P_1 - kP_2$ for some $k \in \Bbbk$ and $P_1, P_2 \in S_{\text{mon}}$ satisfying $\operatorname{lt}(P_1) = k \operatorname{lt}(P_2)$. It is the analogue of S-polynomials in Gröbner basis theory. Observe that $\deg(P_1) = \deg(P_2) > \deg(T)$. Thus $\deg(T)$ depends on the lower terms of the polynomials P_1 and P_2, which is one of the challenges in determining SAGBI basis.

Lemma 3.1. *Let S be a SAGBI basis of a subalgebra R and $m = \min\{\deg(f) : f \in S\}$, then there is no nonconstant polynomial in R of degree less than m.*

Proof. Let g be a non constant polynomial in R, then by Proposition 2.2(iii)

$$\deg(g) = \sum \lambda_i \deg(g_i), \quad \text{where } \lambda_i \in \mathbb{N}_0$$

Since $g \neq 0$ there is some j where $\lambda_j \neq 0$. It follows that $\deg(g) \geq \deg(g_j) \geq m = \min\{\deg(f) : f \in S\}$. □

Theorem 3.2. *Let $\ell \geq m$ be positive integers, a set of ℓ polynomials $B = \{f_1, f_2, ..., f_\ell\} \subseteq \Bbbk[x]$ with consecutive degrees $m, m+1, \ldots, m+\ell-1$ is a SAGBI basis if and only if the subalgebra $\Bbbk[B]$ does not contain a nonconstant polynomial of degree less than m.*

Proof. Suppose $\{f_1, f_2, ..., f_\ell\}$ is a SAGBI basis. By Lemma 3.1, the degree of each nonconstant polynomial in $\Bbbk[B]$ is at least $m = \min\{\deg(f_i) : f_i \in B\}$. Conversely, assume $\Bbbk[B]$ does not contain a nonconstant polynomials of degree less than m. Hence for any $h \in \Bbbk[B] \backslash \Bbbk$ we have $\deg(h) \geq m$. Applying division algorithm on $\deg(h)$ and m we have,

$$\begin{aligned}
\deg(h) &= qm + r \text{ for some } q \geq 1 \text{ and}, 0 \leq r < m \leq \ell \\
&= (q-1)m + (m+r) \\
&= (q-1)\deg(f_1) + \deg(f_{r+1})
\end{aligned}$$

Hence by Proposition 2.2(iii), the set $B = \{f_1, f_2, ..., f_m\}$ is a SAGBI basis. □

Example 3. Let

$$B = \{f_1 = x^4 + x^2, f_2 = x^5, f_3 = x^6 - 5x^4, f_4 = x^7 - 3x^5\}$$

be four polynomials of consecutive degrees $4, 5, 6, 7$ in $\mathbb{R}[x]$, thus $\ell = m = 4$. Observe that there are no monomials of degree 1 or 3 in each of the four polynomials f_1, f_2, f_3, f_4, hence $\mathbb{R}[S]$ does not contain any polynomial of degree 1 and 3. On the other hand if there is a polynomial of degree 2 in $\mathbb{R}[B]$, it must be of the form $f_1 + h$ where $h = -x^4 \in \mathbb{R}[S]$. But this is also impossible by observing the terms of the generators. Therefore R has no polynomial of degree $1, 2$ or 3. Hence by Theorem 3.2 the set $B = \{f_1, f_2, f_3, f_4\}$ forms a SAGBI basis.

The next two examples demonstrate we can't drop any of the conditions in theorem 3.2.

Example 4. Let

$$B = \{f_1 = x^3 + 1,\ f_2 = x^4 - x,\ f_3 = x^5 + x^2\}$$

where $\ell = 3 \geq m = 3$ are 3-polynomials of consecutive degrees $3, 4, 5$. Here we drop the condition $\Bbbk[B]$ does not contain nonconstant polynomial of degree less than $m = 3$. Consider the polynomial

$$g = f_2^2 - f_1 f_3 + 4f_3 = 4x^2 \in \mathbb{R}[B].$$

But $\deg(g) = 2$ can't be expressed as linear combination $\lambda_1 3 + \lambda_2 4 + \lambda_3 5$ where $\lambda_i \in \mathbb{N}_0$. Hence B is not a SAGBI basis.

Example 5. The theorem is also false if $\ell < m$. For this consider the three polynomials For this consider

$$B = \{f_1 = x^4 + 2x^3,\ f_2 = x^5,\ f_3 = 2x^6 - 3x^5\}$$

where $\ell = 3 < 4 = m$. First, a quick observation to the monomials in each polynomial f_1, f_2, f_3 we can conclude that $\Bbbk[f_1, f_2, f_3]$ doesn't contain any polynomials of degree *one* and *two*. On the other hand if there is a polynomial h of degree *three* in $\Bbbk[f_1, f_2, f_3]$, then $h = k_1 x^3 + k_2$ for some $k_1, k_2 \in \Bbbk$. Moreover $h = k_1 f_1 + g$ for some $k \in \Bbbk$ and $g \in \Bbbk[f_1, f_2, f_3]$. This is again impossible with a careful observation of the degrees of each term in the polynomials f_1, f_2, f_3. But the polynomial

$$g = f_1 f_3 - 2f_2^2 - f_1 f_2 + 8f_1^2 = 32x^7 + 32x^6 \in \mathbb{R}[f_1, f_2, f_3]$$

Note that $\deg(g) = 7$ can not be expressed as $\mathbb{N}_0 - \mathit{linear}$ combination of the set $\{4, 5, 6\}$. Hence $\{f_1, f_2, f_3\}$ is not a SAGBI basis.

Lemma 3.3. *Let $S = \{f_1, f_2, ..., f_\ell\} \subseteq \Bbbk[x]$ be a set of ℓ-polynomials, (with $\ell < m$) of consecutive degrees $m, m+1, \ldots, m+\ell-1$. If S is a SAGBI basis then the subalgebra $\Bbbk[S]$ does not contain polynomials of degree d where $m + \ell \leq d < 2m$.*

Proof. Suppose $h \in \Bbbk[f_1, ..., f_\ell]$ with $m + \ell \leq \deg(h) < 2m$. Since $\{f_1, f_2, ..., f_\ell\}$ is a SAGBI basis then $\deg(h) = \lambda_1 \deg(f_1) + \lambda_2 \deg(f_2) + ... + \lambda_\ell \deg(f_\ell)$ for some $\lambda_i \in \mathbb{N}$. Since $\deg(h) > \deg(f_\ell) = m + \ell - 1$, at least two of λ_i should be nonzero (or at least one λ_i should be greater than or equal to 2). Hence,

$$\begin{aligned}
\deg(h) &= \lambda_1 \deg(f_1) + \lambda_2 \deg(f_2) + ... + \lambda_\ell \deg(f_\ell). \\
&\geq \lambda_i \deg(f_i) + \lambda_j \deg(f_j). \\
&= \lambda_i (m + i - 1) + \lambda_j (m + j - 1). \\
&= \lambda_i m + \lambda_j m + \lambda_i (i - 1) + \lambda_j (j - 1). \\
&\geq 2m.
\end{aligned}$$

This contradicts the assumption $\deg(h) \leq 2m - 1$. Therefore, $\Bbbk[f_1, ..., f_\ell]$ does not contain polynomials of degree $m + \ell, ..., 2m - 1$. □

Remark 1. Given a set polynomials $B = \{f_1, \ldots, f_\ell\}$ of consecutive degrees $\deg(f_i) = m + (i - 1),\ \ i \in \{1, \ldots, \ell\}$ for some $m \in \mathbb{N}$. In Theorem 3.2 we establish a necessary and sufficient condition when B is a SAGBI basis provided that $\ell \geq m$. In the next result we cut the number of such polynomials by half and determine when they form a SAGBI basis.

Theorem 3.4. *Let $m \in \mathbb{N}$ and $\lfloor \frac{m}{2} \rfloor$ denote the greatest integer less than or equal to $\frac{m}{2}$. A set of $\lfloor \frac{m}{2} \rfloor + 1$ polynomials $S = \{f_1, f_2, ..., f_{\lfloor \frac{m}{2} \rfloor+1}\}$ with consecutive degrees $m, m+1, \ldots, m + \lfloor \frac{m}{2} \rfloor$ is a SAGBI basis if and only if $\Bbbk[S]$ does not contain any nonconstant polynomial of degree d, where $d < m$ or $m + \lfloor \frac{m}{2} \rfloor < d < 2m$.*

Proof. $\underline{(\Rightarrow):}$ Assume the set of polynomials $S = \{f_1, f_2, ..., f_{\lfloor \frac{m}{2} \rfloor+1}\}$ with consecutive degrees $m, m+1, \ldots, m + \lfloor \frac{m}{2} \rfloor$ form a SAGBI basis. Applying Lemma 3.1 and Lemma 3.3, the subalgebra $\Bbbk[S]$ does not contain nonconstant polynomials of degree $d < m$ or $m + \lfloor \frac{m}{2} \rfloor < d < 2m$.

$\underline{(\Leftarrow):}$ Let $\Bbbk[S]$ doesn't contain any non constant polynomial of degree d where $d < m$ or $m + \lfloor \frac{m}{2} \rfloor < d < 2m$, we show S forms a SAGBI basis. This will be achieved by considering two cases on the parity of m and additional subcases.
Case 1. m is even, (hence $\lfloor \frac{m}{2} \rfloor = \frac{m}{2}$.)

Let h be a nonconstant polynomial in $\Bbbk[S]$. By hypothesis $\Bbbk[S]$ does not contain polynomials of degree d where $0 < d < m$ or $m + \frac{m}{2} < d < 2m$. Hence either $m \leq \deg(h) \leq m + \frac{m}{2}$ or $\deg(h) \geq 2m$.
If $m \leq \deg(h) \leq m + \frac{m}{2}$, then $\deg(h) = \deg(f_i)$ for some $i \in \{1, 2, ..., \frac{m}{2} + 1\}$. On the other hand if $\deg(h) \geq 2m$, by division algorithm, $\deg(h) = qm + r$ where $q \geq 2$ and $0 \leq r < m$. We divide this into two subcases on the remainder r.

1.1. $0 \leq r \leq \frac{m}{2}$

$$\begin{aligned} \deg(h) &= qm + r \\ &= (q-1)m + m + r \\ &= (q-1)\deg(f_1) + \deg(f_{r+1}) \end{aligned}$$

1.2. $\frac{m}{2} < r \leq m - 1$
In this case r can be expressed as $r = \frac{m}{2} + i - 1$ for some $2 \leq i \leq \frac{m}{2}$. Hence,

$$\begin{aligned} \deg(h) &= qm + r \\ &= (q-2)m + 2m + \frac{m}{2} + i - 1 \\ &= (q-2)m + (m + \frac{m}{2} - 1) + (m + i) \\ &= (q-2)\deg(f_1) + \deg(f_{\frac{m}{2}}) + \deg(f_{i+1}) \end{aligned}$$

In both cases we express $\deg(h)$ as $\mathbb{N}_0$-linear combination of $\deg(f_i)$, $i = 1, \ldots, \frac{m}{2} + 1$. Hence, $S = \{f_1, ..., f_{\frac{m}{2}+1}\}$ is a SAGBI basis.

Case 2. m is odd, (hence $\lfloor \frac{m}{2} \rfloor = \frac{m-1}{2}$.)

Here again let h be a nonconstant polynomial in $\Bbbk[S]$. By hypothesis, either $m \leq \deg(h) \leq m + \frac{m-1}{2}$ or $\deg(h) \geq 2m$
If $m \leq \deg(h) \leq m + \frac{m-1}{2}$, then $\deg(h) = \deg(f_i)$, for some $i \in \{1, 2, ..., \frac{m-1}{2} + 1\}$. Next assume $\deg(h) \geq 2m$. By division algorithm, $\deg(h) = qm + r$, where $q \geq 2$, and $0 \leq r < m$. Here we take three subcases on r.

2.1. $0 \le r \le \frac{m-1}{2}$

$$\begin{aligned}\deg(h) &= qm + r\\ &= (q-1)m + m + r\\ &= (q-1)\deg(f_1) + \deg(f_{r+1})\end{aligned}$$

2.2. $\frac{m-1}{2} < r \le m-2$.

In this case r can be expressed as $r = \frac{m-1}{2} + (i-1)$ where $2 \le i \le \frac{m-1}{2}$. Hence,

$$\begin{aligned}\deg(h) &= qm + r\\ &= (q-2)m + 2m + \frac{m-1}{2} + (i-1)\\ &= (q-2)m + (m + \frac{m-1}{2} - 1) + (m+i)\\ &= (q-2)\deg(f_1) + \deg(f_{\frac{m-1}{2}}) + \deg(f_{i+1}).\end{aligned}$$

2.3. $r = m-1$

$$\begin{aligned}\deg(h) &= qm + r\\ &= qm + m - 1\\ &= (q-2)m + 2m + 2(\frac{m-1}{2})\\ &= (q-2)m + 2(m + \frac{m-1}{2})\\ &= (q-2)\deg(f_1) + 2\deg(f_{\frac{m-1}{2}+1})\end{aligned}$$

In all cases $\deg(h)$ is linear combination

$$\lambda_1 \deg(f_1) + \ldots + \lambda_{\frac{m-1}{2}+1} \deg(f_{\frac{m-1}{2}+1})$$

where $\lambda_i \in \mathbb{N}_0$, thus proving S is a SAGBI basis. □

EXAMPLE 6. In this example we choose $m = 5$ and hence $\lfloor \frac{5}{2} \rfloor = \frac{5-1}{2} = 2$. Consider the following $\lfloor \frac{m}{2} \rfloor + 1 = 3$ polynomials with degrees $m = 5, m+1 = 6, m + \lfloor \frac{m}{2} \rfloor = 7$ given by

$$f_1 = x^5 + x, f_2 = x^6 + x^2, f_3 = x^7 + x + 1.$$

Consider the polynomial

$$h = f_2^2 - f_1 f_3 = x^8 - x^6 - x^5 + x^4 - x^2 - x \in \mathbb{R}[f_1, f_2, f_3]$$

Observe that $\deg(h) = 8$ satisfies $m + \lfloor \frac{m}{2} \rfloor = 7 < \deg(h) < 10 = 2m$. By Theorem 3.4 the above set is not a SAGBI basis. This can easily be confirmed by the fact that $\deg(h) = 8$ is not $\mathbb{N}_0$-linear combination of $\deg(f_1) = 5, \deg(f_2) = 6, \deg(f_3) = 7$.

We next state and prove a lemma on a submonoid of natural numbers generated by three consecutive integers. It will have a immediate application to our last result on SAGBI basis.

LEMMA 3.5. *Let $m \in \mathbb{N}_0$ and $\mathcal{H} := \langle m, m+1, m+2 \rangle_{\text{mon}} = \{\lambda_1 m + \lambda_2(m+1) + \lambda_3(m+2) : \lambda_i \in \mathbb{N}_0\}$ be the additive monoid generated by $\{m, m+1, m+2\}$, then*

$$a \in \mathcal{H} \Leftrightarrow a \notin \bigcup_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} \{t \in \mathbb{N} : i(m+2) < t < (i+1)m\}.$$

Proof. Let's start with $m = 0$ or 1. In this case $\mathcal{H}$ contains 1, hence $\mathcal{H} = \mathbb{N}_0$. On the other hand $\lfloor \frac{m}{2} \rfloor - 1 = -1$, and hence $\bigcup_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} = \emptyset$ proving the Lemma. Next let's show for $m \geq 2$ and m even.
$(\Rightarrow)$: Assume $a \in \mathcal{H}$ and m is even. Hence $\lfloor \frac{m}{2} \rfloor = \frac{m}{2}$.

$$\begin{aligned} a \in \mathcal{H} \ \Rightarrow \ a &= \lambda_1 m + \lambda_2(m+1) + \lambda_3(m+2) \text{ for some } \lambda_i \in \mathbb{N} \\ &= (\lambda_1 + \lambda_2 + \lambda_3)m + (\lambda_2 + 2\lambda_3) \end{aligned}$$

We want to show $a \notin \bigcup_{i=0}^{\frac{m}{2}-1} \{t \in \mathbb{N} : i(m+2) < t < (i+1)m\}$. Assume to the contrary $i(m+2) < a < (i+1)m$, for some $i \in \mathbb{N}$, where $0 \leq i < \frac{m}{2}$. It follows that;

$$i(m+2) < a = (\lambda_1 + \lambda_2 + \lambda_3)m + (\lambda_2 + 2\lambda_3) < (i+1)m \qquad (*)$$

Let's take two subcases on $\lambda_1 + \lambda_2 + \lambda_3$

Case 1.1. $\lambda_1 + \lambda_2 + \lambda_3 > i$

$$\begin{aligned} &\Rightarrow \ \lambda_1 + \lambda_2 + \lambda_3 \geq i+1 \\ &\Rightarrow \ (\lambda_1 + \lambda_2 + \lambda_3)m \geq (i+1)m \\ &\Rightarrow \ a = (\lambda_1 + \lambda_2 + \lambda_3)m + (\lambda_2 + 2\lambda_3) \geq (\lambda_1 + \lambda_2 + \lambda_3)m \geq (i+1)m \\ &\Rightarrow \ a \geq (i+1)m. \end{aligned}$$

But this contradicts equation $(*)$ above where $a < (i+1)m$.

Case 1.2. $\lambda_1 + \lambda_2 + \lambda_3 \leq i$. From equation $(*)$ we have

$$\begin{aligned} & i(m+2) < (\lambda_1 + \lambda_2 + \lambda_3)m + (\lambda_2 + 2\lambda_3) < (i+1)m \\ \Rightarrow \ & im + 2i < (\lambda_1 + \lambda_2 + \lambda_3)m + (\lambda_2 + 2\lambda_3) \\ \Rightarrow \ & im + 2i < im + (\lambda_2 + 2\lambda_3) \ \text{ by assumption } \lambda_1 + \lambda_2 + \lambda_3 \leq i \\ \Rightarrow \ & 2i < \lambda_2 + 2\lambda_3 \\ \Rightarrow \ & \lambda_2 + 2\lambda_3 > 2i \geq 2(\lambda_1 + \lambda_2 + \lambda_3) = 2\lambda_1 + 2\lambda_2 + 2\lambda_3 \\ \Rightarrow \ & 0 > 2\lambda_1 + \lambda_2. \ \text{(this contradicts the fact that both } \lambda_1, \lambda_2 \in \mathbb{N}.) \end{aligned}$$

In both subcases the contradictions prove our claim.
$(\Leftarrow)$: Now given $a \notin B := \bigcup_{i=0}^{\frac{m}{2}-1} \{t \in \mathbb{N} : i(m+2) < t < (i+1)m\}$.

$$\begin{aligned} \Rightarrow a \in B' &= \bigcup_{i=0}^{\frac{m}{2}-2} [(i+1)m, (i+1)(m+2)] \cup [m \cdot \frac{m}{2}, \ \infty) \\ &= C \cup D \end{aligned}$$

where $C = \bigcup_{i=0}^{\frac{m}{2}-2} [(i+1)m, (i+1)(m+2)]$ and $D = [m \cdot \frac{m}{2}, \ \infty)$.

Let a $\in$ C.

$$\begin{aligned} &\Rightarrow \ a \in [(i+1)m, (i+1)(m+2)], \text{ for some } i \in \{0, 1, ..., \frac{m}{2} - 2\} \\ &\Rightarrow \ (i+1)m \leq a \leq (i+1)m + 2(i+1) \\ &\Rightarrow \ a = (i+1)m + r, \quad \text{where} \quad 0 \leq r \leq 2(i+1). \end{aligned}$$

We further take two subcases on the parity of r.

C1. r is even. From $0 \le r \le 2(i+1)$ it follows that $0 \le \frac{r-2}{2} \le i$, and $i - \frac{r-2}{2} \in \mathbb{N}$. Hence from above;

$$\begin{aligned} a &= (i+1)m + r \\ &= (i - \frac{r-2}{2})m + \frac{r}{2}(m+2) \in \mathcal{H}. \end{aligned}$$

C2. r is odd. In this case $\frac{r-1}{2}$ is a nonnegative integer and $\frac{r-1}{2} < i$. It follows that

$$\begin{aligned} a &= (i+1)m + r \\ &= (i - \frac{r-1}{2})m + \frac{r-1}{2}(m+2) + (m+1) \in \mathcal{H}. \end{aligned}$$

Let $\mathbf{a} \in \mathbf{D}$. By division algorithm $a = qm + r$, where $q \ge \frac{m}{2}$ and $r \in \{0, 1, ..., m-1\}$. We again take two subcases on r

D1. r is even. Hence $\frac{r}{2}$ is a nonnegative integer which is less than or equal to p, that is, $q - \frac{r}{2} \in \mathbb{N}$, and

$$\begin{aligned} a &= qm + r \\ &= (q - \frac{r}{2})m + \frac{r}{2}(m+2) \in \mathcal{H} \end{aligned}$$

D2. r is odd. In this case $\frac{r+1}{2}$ is a nonnegative integer which is between 1 and p, that is, $p - \frac{r+1}{2}$, and $\frac{r+1}{2} - 1 \in \mathbb{N}$, and

$$\begin{aligned} a &= qm + r \\ &= (q - \frac{r+1}{2})m + (\frac{r+1}{2} - 1)(m+2) + (m+1) \in \mathcal{H}. \end{aligned}$$

In all cases we have shown $a \in \mathcal{H}$ as desired.
Finally, the case m is odd can be verified with similar steps as in the even case. We leave the details to avoid repetition. □

This lemma leads us to our last observation to determine when three polynomials of consecutive degrees form a SAGBI basis.

THEOREM 3.6. *Let $f_1, f_2, f_3 \in \Bbbk[x]$ be a set of three polynomials with consecutive degrees $m, m+1$ and $m+2$ respectively, for some $m \in \mathbb{N}$. Then $\{f_1, f_2, f_3\}$ is a SAGBI basis if and only if $\Bbbk[f_1, f_2, f_3]$ does not contain a polynomial h with*

$$\deg(h) \in \bigcup_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} \{t : i(m+2) < t < (i+1)m\}.$$

Proof. Let $m \in \mathbb{N}$ and $S = \{f_1, f_2, f_3\} \subseteq \Bbbk[x]$ with degrees $m, m+1, m+2$ respectively be a SAGBI basis. Let $h \in \Bbbk[S]$ then by Proposition 2.2 we have

$$\begin{aligned} \deg(h) &= \lambda_1 \deg(f_1) + \lambda_2 \deg(f_2) + \lambda_3 \deg(f_3) \\ &= \lambda_1 m + \lambda_2(m+1) + \lambda_3(m+2) \end{aligned}$$

Now applying Lemma 3.5 we have, $\deg(h) \notin \bigcup_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} \{t : i(m+2) < t < (i+1)m\}$. □

COROLLARY 3.7. *Let f_1, f_2, f_3 be three polynomials in $\Bbbk[x]$ of consecutive degree $m, m+1, m+2$ that form a SAGBI basis. The cardinality of the set $\Lambda = \{\deg(g) : g \notin \Bbbk[f_1, f_2, f_3]\}$ is $\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1}(m - 1 - 2i)$.*

Proof. In view of Theorem 3.6 if $g \notin \Bbbk[f_1, f_2, f_3]$ then $\deg(g) \in \bigcup_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1}\{t : i(m+2) < t < (i+1)m\}$. Hence

$$\begin{aligned} |\Lambda| &= |\bigcup_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} \{t : i(m+2) < t < (i+1)m\}| \\ &= \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} [(i+1)m - i(m+2) - 1] \\ &= \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} (im + m - im - 2i - 1) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} (m - 1 - 2i). \end{aligned}$$

□

Example 7. Consider three polynomials $f_1 = x^6 - 3x^2 + 1,\ f_2 = x^7 + 4x^6 - 2,\ f_3 = x^8 - 2x^7 + x^4 + 2x^2 - 5$ with respective degrees $6, 7$, and 8. In view of Theorem 3.6 such a set of polynomials forms a SAGBI basis if and only if the subalgebra $\mathbb{Q}[f_1, f_2, f_3]$ avoids polynomials of degree $\underline{1, 2, 3, 4, 5},\ \underline{9, 10, 11}, and\ \underline{17}$. Let's construct successive T-polynomials $T_1 = f_1 f_3 - f_2^2,\ T_2 = T_1 - 10 f_1 f_2,\ T_3 = T_2 - 24 f_1^2$ in $\mathbb{Q}[f, f_2, f_3]$. But the last polynomial

$$T_3 = -2x^{10} - 24x^9 + 27x^8 + 12x^7 - 20x^6 - 221x^4 + 221x^2 - 53$$

is of degree 10 hence $\{f_1, f_2, f_3\}$ is not a SAGBI basis.

Example 8. Consider the three polynomials $g_1 = x^6 - 1, g_2 = x^7 + 4x^6 - 2, g_3 = x^8 - 2x^7 + x^6 - 5$ of consecutive degrees $6, 7$ and 8. The degree of each monomial in the polynomials g_i are $0, 6, 7, 8$. Hence the degree of polynomial in $\mathbb{Q}[g_1, g_2, g_3]$ can not be $\underline{1, 2, 3, 4, 5},\ \underline{9, 10, 11}, and\ \underline{17}$. Therefore by Theorem 3.6 the set $\{g_1, g_2, g_3\}$ forms a SAGBI basis.

Concluding remark. In section 3 the main results deals with the connection of SAGBI basis with semigroups generated by degrees of the polynomials in the SAGBI basis. In the theory of numerical semigroups, such problems are also known as *gaps*. There is a general work in this direction by Alfonsín [1] that also addresses the *Frobenius number*, and *genius* of such semigroups. Our proof however gave a simplified description of the gaps as a union of intervals since we were only interested in special cases of those semigroups. Thanks to Anna Torstensson who brought the work of Alfonsín to our attention. We believe there is even great potential to learn from the tools developed in the subject of numerical semigroups and to address additional SAGBI basis problems.

Acknowledgment. This paper is based on a portion of the second author's PhD thesis at Addis Ababa University, conducted under the supervision of the first and third authors. We would like to thank Anna Torstensson and Haohao Wang for reading the final draft of the thesis and providing several valuable comments and suggestions.

References

[1] J. Ramírez Alfonsín, *Gaps in semigroups*, Discrete Math. **308** (18) (2008), 4177–4184. https://www.sciencedirect.com/science/article/pii/S0012365X07006292

[2] S. Anderson, A. Smith, P. Stewart, M. Tesemma, J. Usatine, *A topological structure on certain initial algebras*, Topology Appl. **180** (2015), 199–208.
https://www.sciencedirect.com/science/article/pii/S0166864114004428

[3] A. Conca, J. Herzog, G. Valla, *Sagbi bases with applications to blow-up algebras*, J. reine angew. Math. **474** (1996), 113–138.
https://www.degruyterbrill.com/document/doi/10.1515/crll.1996.474.113/html

[4] K. Gatermann, *Applications of SAGBI-bases in dynamics*, J. Symbolic Comput. **35** (5) (2003), 543–575.
https://www.sciencedirect.com/science/article/pii/S074771710300018X

[5] M. Göbel, *A constructive description of SAGBI basis for polynomial invariants of permutation groups*, J. Symbolic Comput. **26** (3) (1998), 261–272.
https://www.sciencedirect.com/science/article/pii/S0747717198902103

[6] D. Kapur, K. Madlener, *A completion procedure for computing a canonical bases for a k-subalgebra*, Computers and Mathematics **89**, MIT, Cambridge, Mass. (1989), 1–11.

[7] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra 2*, Springer, Heidelberg, 2005, x+586pp.
https://link.springer.com/book/10.1007/3-540-28296-3

[8] Z. Reichstein, *SAGBI bases in rings of multiplicative invariants*, Comment. Math. Helv. **78** (1) (2003), 185–202.
https://link.springer.com/article/10.1007/s000140300008

[9] L. Robbiano, M. Sweedler, *Subalgebra bases*, Lecture Notes in Math. **1430** (1990), 61–87.
https://link.springer.com/chapter/10.1007/BFb0085537

[10] W. Sit, M. Siu, *On the subsemigroups of* $\mathbb{N}$, Math. Mag. **48** (4) (1975), 225–227.
https://www.tandfonline.com/doi/abs/10.1080/0025570X.1975.11976495

[11] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, Univ. Lecture Ser. **8** (1996), xii+162pp.
https://bookstore.ams.org/view?ProductCode=ULECT/8

[12] M. Tesemma, *On initial algebra of multiplicative invariants*, J. Algebra **320** (2008), 3851–3865.
https://www.sciencedirect.com/science/article/pii/S0021869308003451

[13] A. Torstensson, V. Ufnarovski, H. Öfverbeck, *Using resultant for SAGBI basis verification in the univariate polynomial ring*, J. Symbolic Comput. **40** (2005), 1087–1105.
https://www.sciencedirect.com/science/article/pii/S0747717105000775

Berehanu Belayneh
Department of Mathematics, Addis Ababa University, Ethiopia
E-mail: berhanu.bekele@aau.edu.et

Dawit Solomon
Department of Mathematics, Kotebe University of Education, Addis Ababa, Ethiopia
E-mail: dawit.solomon@kue.edu.et

Mohammed Tesemma
Department of Mathematics, Spelman College, Atlanta, GA USA
E-mail: mtesemma@spelman.edu