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A NOTE ON BEST PROXIMITY POINTS FOR F-CONTRACTIVE NON-SELF MAPPINGS

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Abstract. In the year 2012, Wardowski [Wardowski, D., Fixed Point Theory Appl., 94 (2012), 6pp] introduced the notion of F-contraction mapping and presented a fixed point result on complete metric space which generalized the Banach contraction principle. Then, in the year 2014, Omidvari et al. [Omidvari, M., Vaezpour, S.M., Saadati, R., Miskolc Math Notes, 15 (2014), 615-623] considered the concept of F-contraction non-self mappings and presented a best proximity point theorem for this class of mappings to generalize the fixed point theorem of Wardowski. In this note, we show that the existence of best proximity point for F-contraction non-self mappings follow from the Wardowski's fixed point theorem. Also, in this note, we provide a new version of [15, Theorem 2] where instead of considering the continuity of F-proximal contraction of the first kind, we use the concept of p-property. We apply Wardowski's fixed point theorem to prove [15, Theorem 2]. In the last part, we also prove a best proximity point result regarding F-proximal contraction of the second kind where we drop some conditions.

1. Introduction

Let (X, d) be a metric space and $f : M \to N$ be a mapping where M, N are non-empty subsets of the metric space X. If $f(M) \cap M \neq \phi$ then one search for a necessary and sufficient condition under which the mapping f has a fixed point. Here, $f(M)$ denotes the range of f. One of the fundamental results in metric fixed point theory is the Banach contraction principle. In the year 1922, Banach [3] proved that if X is a complete metric space and $f : X \to X$ is a contraction mapping then f has a unique fixed point. Banach contraction principle has a lot of applications in differential equations, integral equations for the existence of solutions. To generalize Banach contraction principle, in the year 2012, Wardowski [19] introduced the notion of F-contraction mapping which includes the class of contraction mappings and presented a fixed point result for such class of mappings in a complete metric space. In 2023, Rossafi et al. [17] introduced the notion of $\theta - \phi$-contraction mappings in b-metric spaces and proved several fixed point theorems for θ-type, $\theta - \phi$ type of mappings in a complete b-metric space. As an application they established the existence and

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uniqueness of solution of a class of integral equation of Fredholm type. In [1], Abdullayev et al. established a class of functions in Hardy space H^1 for which the best trigonometric approximations donot coincide with the best algebraic approximations. In [12], authors introduced the notion of α-ψ contractive mappings with respect to w-distance and obtained a fixed point result for this class of mappings. Also, they give an application of their obtained results to nonlinear fractional differential equations. For a mapping $f : M \to N$ if $f(M) \cap M = \phi$ then the mapping f has no fixed points. In this case, one interesting problem is to search for an element $x \in M$ such that $d(x, f(x)) = d(M, N)$ where $d(M, N) = \inf\{d(x, y) : x \in M, y \in N\}$. Best proximity point problems deal with this situation. Authors often prove best proximity point results to generalize fixed point results for self mappings. In 2011, Basha [4] established best proximity point result for proximal contraction which extends Banach contraction principle for non-self mappings. In the article [4], Basha established the existence of best proximity point for proximal contraction of first kind and second kind respectively with respect to an isometry mapping $g : A \to A$ where A is a nonempty closed subset of a metric space X. In [5], Basha et al. proved some best proximity point results for non-self Kannan type mappings, Chatterjea type mappings. They defined the notion of weak K-cyclic contraction and K-cyclic contraction with respect to mappings $T : A \to B$ and $S : B \to A$. Then they established best proximity pair results for these two classes of mappings and established the corresponding Kannan fixed point theorem and Chatterjea fixed point theorem as a corollary. In 2021, Mishra et al. [13] introduced the notion of almost generalized proximal $(\alpha - \psi - \phi - \theta)$-weakly contractive mappings with rational expressions and proved a best proximity point results for this class of mappings from which several fixed point theorems can be deduced as a corollary. In 2022, Gabeleh and Patle [9] considered a new class of condensing operators in reflexive Busemann convex space and studied best proximity points (pairs) for this class of mappings. In [9], they defined the notion of cyclic (noncyclic) $\alpha - \psi$ and $\beta - \psi$ condensing operators with respect to α and β admissible mappings and studied the best proximity point (pair) for these classes of mappings. As an appication, they established optimal solutions of a syatem of second order differential equations with two initial conditions. In [16], authors has given an application of best proximity point (pair) for existence of optimal solution of a system of differential equations involving ψ-Hilfer fractional derivative. In [11] authors have obtained best proximity point results for $\mathcal{Z}$ contraction and Suzuki type $\mathcal{Z}$ contractions and provided an application of their results in fractional order functional differential equations. For more recent best proximity points (pairs) and applications, readers can see [9, 14, 16] and the references therein. In [10], Jayapriya et al. used Sawi transform to derive a generalized Hyers-Ulam stability results for linear homogeneous and non-homogeneous differential equations. In [18], authors proved a best proximity point theorem in the context of probabilistic metric space and extend the Banach contraction principle for such spaces. In [8], Gabeleh showed that, the best proximity point result obtained in [18], become straightforward consequence of the corresponding fixed point result. In [2], authors proved best proximity point results for interpolative proximal contractions and provided examples for the results. In 2021, Gabeleh and Markin [7] showed that the best proximity point result obtained in [2] follows from the corresponding fixed point theorem for interpolative contractions. In the year 2014, Omidvari et al. [15] considered F-contraction non-self mappings and presented a best proximity point result to generalize Wardowski's fixed point theorem. In this note, we show that the existence of best proximity point for F-contraction non-self mappings follow from the Wardowski's fixed point theorem. The main advantage of our result is that we have to find fixed point of the mapping $S : A_0 \to A_0$ to get the best proximity point of the non-self mapping. So, our result provides a relation between best proximity point of F-contraction non-self mappings and fixed point of a self mapping from A_0 into A_0. Also, In our proof, the main challenge is to contruct the self mapping which satisfy all the conditions of the fixed point theorem. In [15], Omidvari et al. introduced the concept of F-proximal contraction of the first kind and proved Theorem 3 in [15] by taking the mapping to be continuous. For more details on F-contractions, readers can see [6] and the references therein. In this short note, by applying Wardowski's fixed point theorem, we provide an alternative proof of the Theorem without considering the continuity assumption of the mapping. Next, in the last part of the paper we consider F-proximal contraction of the second kind. We prove a best proximity point result regarding F-proximal contraction of the second kind where we drop some conditions from [15, Theorem 3]. We donot consider isometry mapping $g : A \to A$ in the theorem and instead, we use the continuity of the mapping $F : \mathbb{R}^+ \to \mathbb{R}$.

2. Preliminaries and main results

Throughout this article $\mathbb{R}^+$ denotes the set of all positive real numbers and $\mathbb{R}$ denotes the set of all real numbers. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying the following conditions:

1. f is strictly increasing;
2. for every sequence $\{\alpha_n\} \subset \mathbb{R}^+$ we have $\lim_{n\to\infty} \alpha_n = 0 \iff \lim_{n\to\infty} f(\alpha_n) = -\infty$;
3. there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0+} \alpha^k f(\alpha) = 0$.

In this paper the set of all functions $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying the three conditions above will be denoted by Ω. We first recall the definition of F-contraction mapping from [19] as follows.

Definition 2.1. [19] Let (X, d) be a metric space and $F \in \Omega$. A mapping $T : X \to X$ is said to be an F-contraction mapping if there exists $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

By using Definition 2.1, Wardowski proved the following fixed point theorem in [19].

Theorem 2.2. [19, Theorem 2.1] *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction mapping where $F \in \Omega$. Then T has a unique fixed point and for any $x_0 \in X$, the sequence $\{T^n(x_0)\}$ will converge to the fixed point of T.*

In this paper, the following notations will be needed from [15]. Let (X, d) be a metric space and A, B be nonempty subsets of X. Then,

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\};$$

$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

Definition 2.3. [15] Let (X,d) be a metric space and A, B be two non-empty subsets of X with $A_0 \neq \phi$. If for every $u_1, u_2 \in A$ and $v_1, v_2 \in B$

$$\begin{cases} d(u_1, v_1) = d(A,B) \\ d(u_2, v_2) = d(A,B) \end{cases} \Longrightarrow d(u_1, u_2) = d(v_1, v_2)$$

then the pair (A, B) is said to have the p-property.

Definition 2.4. [15] Let (X,d) be a metric space and A, B be two non-empty subsets of X. The set A is said to be approximatively compact with respect to B if for every sequence $\{x_n\}$ in A satisfying the condition that $d(z, x_n) \to d(z, A)$ as $n \to \infty$ for some $z \in B$, has a convergent subsequence. Here $d(z, A) = \inf\{d(z, a) : a \in A\}$.

In [15], Omidvari et al. considered F-contraction non-self mappings and proved the following best proximity point theorem to generalize the Wardowski's fixed point theorem.

Theorem 2.5. [15, Theorem 1] *Let A, B be non-void closed subsets of a complete metric space (X,d) such that $A_0 \neq \phi$. Let $T : A \to B$ be an F-contraction non-self mapping such that $T(A_0) \subseteq B_0$. Assume that the pair (A,B) has the p-property. Then there exists unique $x* \in A$ such that $d(x^*, Tx^*) = d(A,B)$.*

Theorem 2.6. *Theorem 2.5 is a direct consequence of Theorem 2.2.*

Proof. Let $x \in A_0$. As $T(A_0) \subseteq B_0$, so, $T(x) \in B_0$. So, there exists $y \in A_0$ such that $d(y, T(x)) = d(A,B)$. Now, we will show that $y \in A_0$ is unique. Suppose there exists $y_1, y_2 \in A_0$ such that $d(y_1, T(x)) = d(A,B)$ and $d(y_2, T(x)) = d(A,B)$. Since the pair (A,B) have the p-property, so, we have

$$\begin{aligned} d(y_1, y_2) &= d(Tx, Tx) \\ \Rightarrow d(y_1, y_2) &= 0 \\ \Rightarrow y_1 &= y_2. \end{aligned}$$

Define a mapping $S : A_0 \to A_0$ by $Sx = y$ having the property that $d(Sx, Tx) = d(A,B)$. Now, we show that A_0 is closed. Let $x \in \bar{A}_0$. Then there exists $\{x_n\} \subset A_0$ such that $x_n \to x$ as $n \to \infty$. Since $x_n \in A_0$ so, there exists $y_n \in B_0$ such that $d(x_n, y_n) = d(A,B)$. Similarly, for $x_m \in A_0$, there exists $y_m \in B_0$ such that $d(x_m, y_m) = d(A,B)$. Since, the pair (A,B) have the p-property, so we have $d(x_n, x_m) = d(y_n, y_m)$. This shows that the sequence $\{y_n\}$ is a Cauchy sequence in B. Since B is closed, so, there exists $y \in B$ such that $y_n \to y$ as $n \to \infty$. Now, since $d(x_n, y_n) \to d(x,y)$, so, we have $d(x,y) = d(A,B)$. This shows that $x \in A_0$. So, A_0 is a closed subset of X and hence it is a complete metric space. Now, we show that $S : A_0 \to A_0$ is an F-contraction mapping. Let $x_1, x_2 \in A_0$ with $Sx_1 \neq Sx_2$. Since $d(Sx_1, Tx_1) = d(A,B)$ and $d(Sx_2, Tx_2) = d(A,B)$ and (A,B) has the p-property so we have, $d(Sx_1, Sx_2) = d(Tx_1, Tx_2)$. As, $T : A \to B$ is an F-contraction non-self mapping, so, we have

$$\begin{aligned} \tau + F(d(Tx_1, Tx_2)) &\leq F(d(x_1, x_2)) \\ \Longrightarrow \tau + F(d(Sx_1, Sx_2)) &\leq F(d(x_1, x_2)). \end{aligned}$$

This shows that $S : A_0 \to A_0$ is an F-contraction mapping. So, from Theorem 2.2 due to Wardowski [19], we can conclude that the mapping S has a fixed point in A_0. So, there exists $z \in A_0$ such that $S(z) = z$. Also, $d(z, T(z)) = d(Sz, Tz) = d(A,B)$. This shows that z is a best proximity point for the mapping $T : A \to B$. □

EXAMPLE 2.7. Consider $X = [0,2]\times\mathbb{R}$ with the usual metric. Let $A = \{0\}\times\mathbb{R}$ and $B = \{2\}\times\mathbb{R}$. Let us define a mapping $T : A \to B$ by $T(0,x) = (2, \frac{x}{e^2})$. Here, it can be seen that $A_0 = A, B_0 = B$ and $d(A,B) = 2$. We consider $F(\alpha) = ln\alpha$ for $\alpha \in \mathbb{R}^+$. It can be seen that T is an F-contraction non-self mapping with $\tau = 2$ and the pair (A,B) have the p-property. Now, we have to construct our mapping $S : A_0 \to A_0$ with the property $d(S(x),T(x)) = d(A,B)$. Let $(0,x) \in A = A_0$. Now,

$$\begin{aligned} d(S(0,x),T(0,x)) &= 2 \\ \Longrightarrow d((0,y),(2,\frac{x}{e^2})) &= 2 \\ \Longrightarrow \sqrt{4+(y-\frac{x}{e^2})^2} &= 2 \\ \Longrightarrow y &= \frac{x}{e^2}. \end{aligned}$$

So, the mapping $S : A_0 \to A_0$ be defined by $S(0,x) = (0, \frac{x}{e^2})$ for $(0,x) \in A_0$. The mapping S has a unique fixed point $(0,0)$. So, by the construction in our Theorem 2.6, $(0,0)$ will be the best proximity point of T. It can also be checked that $d((0,0),T(0,0)) = d(A,B)$, i.e $(0,0)$ is the best proximity point of the mapping T.

In [15], Omidvari et al. introduced the concept of F-proximal contraction of the first kind and presented a best proximity point theorem for this class of mappings in [15, Corollary 3]. First of all, we recall the concept of F-proximal contraction of the first kind from [15] as follows.

DEFINITION 2.8. [15] Let (X,d) be a metric space and (A,B) be a pair of nonempty subsets of X. Let $F \in \Omega$. A mapping $T : A \to B$ is said to be a F-proximal contraction of the first kind if there exists $\tau > 0$ such that

$$\begin{cases} d(u_1,T(x_1)) = d(A,B) \\ d(u_2,T(x_2)) = d(A,B) \end{cases} \Longrightarrow \tau + F(d(u_1,u_2)) \le F(d(x_1,x_2))$$

for all $u_1,u_2,x_1,x_2 \in A$ with $u_1 \neq u_2$ and $x_1 \neq x_2$.

By using definition 2.8, Omidvari et al. presented the following result.

THEOREM 2.9. [15, Theorem 2] *Let A,B be non-void, closed subsets of a complete metric space (X,d) such that $A_0 \neq \phi$. Let $T : A \to B$ and $g : A \to A$ satisfies the following conditions:*
i) T is a continuous F-proximal contraction of the first kind;
ii) $T(A_0) \subseteq B_0$;
iii) g is an isometry;
iv) $A_0 \subseteq g(A_0)$;
then there exists a unique element $x^* \in A$ such that $d(g(x^*),T(x^*)) = d(A,B)$.

In Theorem 2.9, Omidvari et al. used the continuity of F-proximal contraction of the first kind. In our next result, we prove Theorem 2.9 without considering the continuity of the mapping and instead, we use p-property.

THEOREM 2.10. *Let A,B be non-void, closed subsets of a complete metric space (X,d) such that $A_0 \neq \phi$. Let the pair (A,B) have the p-property. Let $T : A \to B$ and $g : A \to A$ satisfies the following conditions:*
i) T is a F-proximal contraction of the first kind;

ii) $T(A_0) \subseteq B_0$;
iii) g *is an isometry;*
iv) $A_0 \subseteq g(A_0)$;
then there exists a unique element $x^* \in A$ *such that* $d(g(x^*), T(x^*)) = d(A, B)$.

Proof. Let $x \in A_0$. As $T(A_0) \subseteq B_0$, so, $T(x) \in B_0$. So, there exists $y \in A_0$ such that $d(y, T(x)) = d(A, B)$. Now, we will show that $y \in A_0$ is unique. Suppose there exists $y_1, y_2 \in A_0$ such that $d(y_1, T(x)) = d(A, B)$ and $d(y_2, T(x)) = d(A, B)$. Since the pair (A, B) have the p-property, so, we have

$$\begin{aligned} d(y_1, y_2) &= d(Tx, Tx) \\ \Rightarrow d(y_1, y_2) &= 0 \\ \Rightarrow y_1 &= y_2. \end{aligned}$$

So, corresponding to $x \in A_0$ there exists unique $y \in A_0$ such that $d(y, Tx) = d(A, B)$. As $A_0 \subseteq g(A_0)$, so, for $y \in A_0$ for which $d(y, T(x)) = d(A, B)$, there exists $y' \in A_0$ such that $g(y') = y$. As g is an isometry, so, y' is unique. Let $S : A_0 \to A_0$ be defined by $S(x) = y'$. So, the mapping S has the property that for all $x \in A_0$, $d(g(S(x)), T(x)) = d(A, B)$. We will show that the mapping $S : A_0 \to A_0$ is an F-contraction. Let $x_1, x_2 \in A_0$ with $Sx_1 \neq Sx_2$. So, $x_1 \neq x_2$. As $d(g(Sx_1), Tx_1) = d(A, B)$ and $d(g(Sx_2), Tx_2) = d(A, B)$ and the pair (A, B) has the p-property so, $d(g(Sx_1), g(Sx_2)) = d(Tx_1, Tx_2)$. Also, since T is a F-proximal contraction of the first kind, so we have

$$\begin{aligned} &\tau + F(d(g(Sx_1), g(Sx_2))) \leq F(d(x_1, x_2)) \\ \Longrightarrow\ &\tau + F(d(Sx_1, Sx_2)) \leq F(d(x_1, x_2)) \text{ as } g \text{ is an isometry.} \end{aligned}$$

This shows that $S : A_0 \to A_0$ is an F-contraction mapping. Also, by proceeding the same proof as in Theorem 2.6, we can show that A_0 is closed and hence A_0 is a complete metric space. By Wardowski's fixed point theorem S has a fixed point $z \in A_0$. Now, $d(g(z), T(z)) = d(g(Sz), T(z)) = d(A, B)$. Also, it can be easily seen that this point $z \in A$ is unique. □

After proving Theorem 2.9, Omidvari et al. presented the following best proximity point result for the class of F-proximal contraction of the first kind. First of all, we recall the result from [15] as follows.

Corollary 2.11. [15, Corollary 3] *Let* A, B *be non-void, closed subsets of a complete metric space* (X, d) *such that* $A_0 \neq \phi$ *and* A *is approximatively compact with respect to* B*. Let* $T : A \to B$ *satisfies the following conditions:*
i) T *is a continuous* F*-proximal contraction of the first kind.*
ii) $T(A_0) \subseteq B_0$.
Then T *has a unique best proximity point in* A.

With the help of our Theorem 2.10, we will present a new version of corollary 2.11 where we will not consider the mapping to be continuous and also, we will not take the condition that A is approximatively compact with respect to B. But, instead, we use the p-property.

Corollary 2.12. *Let* A, B *be non-void, closed subsets of a complete metric space* (X, d) *such that* $A_0 \neq \phi$ *and the pair* (A, B) *have the* p*-property. Let* $T : A \to B$ *satisfies the following conditions:*

i) T is a F-proximal contraction of the first kind.
ii) $T(A_0) \subseteq B_0$.
Then T has a unique best proximity point in A.

Proof. The proof of corollary 2.12 is same as Theroem 2.10 by taking g as the identity mapping, so omitted. □

Now, we recall the definition of F-proximal contraction of the second from [15] as follows.

Definition 2.13. [15] Let (X, d) be a metric space and (A, B) be a pair of non-empty subsets of X. Let $F \in \Omega$. A mapping $T : A \to B$ is said to be a F-proximal contraction of the second kind if there exists $\tau > 0$ such that

$$\begin{cases} d(u_1, T(x_1)) = d(A, B) \\ d(u_2, T(x_2)) = d(A, B) \end{cases} \implies \tau + F(d(T(u_1), T(u_2))) \leq F(d(T(x_1), T(x_2)))$$

for all $u_1, u_2, x_1, x_2 \in A$ with $T(u_1) \neq T(u_2)$ and $T(x_1) \neq T(x_2)$.

We recall the best proximity point result regarding F-proximal contraction of the second kind from [15] as follows.

Theorem 2.14. [15, Theorem 3] *Let A, B be non-void, closed subsets of a complete metric space (X, d) such that A is approximatively compact with respect to B $A_0 \neq \phi$. Let $T : A \to B$ and $g : A \to A$ satisfies the following conditions:*
i) T is a continuous F-proximal contraction of the second kind;
ii) $T(A_0) \subseteq B_0$;
iii) g is an isometry;
iv) $A_0 \subseteq g(A_0)$;
v) T preserves isometric distance with respect to g;
then there exists an element $x^* \in A$ such that $d(g(x^*), T(x^*)) = d(A, B)$. Moreover, if $x^{**} \in A$ is another element with $d(g(x^{**}), T(x^{**})) = d(A, B)$ then $T(x^*) = T(x^{**})$.

Next, we prove a best proximity point result regarding F-proximal contraction of the second kind where we will not consider the isometry $g : A \to A$ and instead, we use the continuity of the mapping F.

Theorem 2.15. *Let A, B be non-void, closed subsets of a complete metric space (X, d) such that A is approximatively compact with respect to B $A_0 \neq \phi$ and the pair (A, B) satisfies the p-property. Let $T : A \to B$ satisfies the following conditions:*
i) T is a continuous F-proximal contraction of the second kind where $F : \mathbb{R}^+ \to \mathbb{R}$ is continuous;
ii) $T(A_0) \subseteq B_0$;
then there exists an element $x^* \in A$ such that $d(x^*, T(x^*)) = d(A, B)$.

Proof. Let $q_0 \in A_0$. Then $T(q_0) \in B_0$. As the pair (A, B) satisfies the p-property, So, there exists unique $v_{q_0} \in A_0$ such that $d(v_{q_0}, T(q_0)) = d(A, B)$. Now, we define a mapping $G : T(A_0) \to T(A_0)$ by $G(T(q_0)) = T(v_{q_0})$ with the property that, $d(v_{q_0}, T(q_0)) = d(A, B)$. We show that $G : T(A_0) \to T(A_0)$ is a F-contraction mapping. Let, $x_1, x_2 \in A_0$ with $G(T(x_1)) \neq G(T(x_2))$. Let $u_1, u_2 \in A_0$ be such that $d(u_1, T(x_1)) = d(A, B)$ and $d(u_2, T(x_2)) = d(A, B)$. As the mapping T is a F-proximal contraction of the second kind, so, there exists $\tau > 0$ such that

$$\tau + F(d(T(u_1), T(u_2))) \leq F(d(T(x_1), T(x_2)))$$

$$\implies \tau + F(d(G(T(x_1)), G(T(x_2)))) \leq F(d(T(x_1), T(x_2))).$$

So, G is a F-contraction mapping. Let $(x_n) \subset A_0$ be such that $T(x_n) \to s$ as $n \to \infty$. Let $v_n \in A_0$ be such that $d(v_n, T(x_n)) = d(A, B)$. Now, we can show that $d(v_n, s) \to d(A, B)$ as $n \to \infty$. Since A is approximatively compact with respect to B so, there exists a subsequence (v_{n_k}) such that $(v_{n_k}) \to v_0$ and $T((v_{n_k})) \to T(v_0)$. If possible, let there be another subsequence (v_{m_k}) such that $(v_{m_k}) \to w_0$ and $T((v_{m_k})) \to T(w_0)$. Since $d(v_{n_k}, T(x_{n_k})) = d(A, B)$ and $d(v_{m_k}, T(x_{m_k})) = d(A, B)$ so, we have

$$\tau + F(d(T(v_{n_k}), T(v_{m_k})) \leq F(d(T(x_{n_k}), T(x_{m_k}))).$$

As $k \to \infty$ we have $T(v_0) = T(w_0)$.

Now, let $p \in \overline{T(A_0)}$. So, there exists a sequence $(x_n) \subset A_0$ such that $T(x_n) \to p$ as $n \to \infty$. Let $(v_n) \subset A_0$ with $d(v_n, T(x_n)) = d(A, B)$ and the subsequence $T(v_{n_k}) \to T(v_0)$ as $n \to \infty$. Let us define $H : \overline{T(A_0)} \to \overline{T(A_0)}$ by $H(p) = T(v_0)$ and this is well defined with $H|_{T(A_0)} = G$. let $q \in \overline{T(A_0)}$. So, there exists a sequence $(y_n) \subset A_0$ such that $T(y_n) \to q$ as $n \to \infty$. Let $(w_n) \subset A_0$ with $d(w_n, T(y_n)) = d(A, B)$ and the subsequence $T(w_{m_k}) \to T(w_0)$ as $n \to \infty$. Here $H(q) = T(w_0)$. Now, let $d(v_{n_k}, T(x_{n_k})) = d(A, B)$ and $d(w_{m_k}, T(y_{m_k})) = d(A, B)$ be such that $d(T(v_{n_k}), T(w_{m_k})) > 0$ and $d(T(x_{n_k}), T(y_{m_k})) > 0$. Since G is a F-contraction mapping so, we have

$$\tau + F(d(G(T(x_{n_k})), G(T(y_{m_k})))) \leq F(d(T(x_{n_k}), T(y_{m_k})))$$
$$\implies \tau + F(d(T(v_{n_k}), T(w_{m_k}))) \leq F(d(T(x_{n_k}), T(y_{m_k}))).$$

Now, as $k \to \infty$ we have,

$$\tau + F(d(H(p), H(q))) \leq F(d(p, q)).$$

So, the mapping $H : \overline{T(A_0)} \to \overline{T(A_0)}$ is a F-contraction mapping and by Wardowski's fixed point theorem H has a fixed point in $\overline{T(A_0)}$. First of all, suppose that there exists $x^* \in A_0$ with $H(T(x^*)) = T(x^*)$. Let $d(v, T(x^*)) = d(A, B)$. This implies $d(v, H(T(x^*))) = d(A, B)$. So, $d(v, T(v)) = d(A, B)$. In this case, v is a best proximity point of T. Now, let $t \in \overline{T(A_0)} \setminus T(A_0)$ with $H(t) = t$. Then there exists $(t_n) \subset A_0$ with $T(t_n) \to t$ as $n \to \infty$. Also, let $h_n \in A_0$ with $d(h_n, T(t_n)) = d(A, B)$ and $d(h_n, t) \to d(A, B)$ as $n \to \infty$. As A is approximatively compact with respect to B so, there exists a subsequence (h_{n_k}) with $h_{n_k} \to h_0$ and $T(h_{n_k}) \to T(h_0)$. So, by the definition of the mapping H we have $H(t) = T(h_0)$. Now,

$$d(h_0, t) = d(A, B)$$
$$\implies d(h_0, H(t)) = d(A, B)$$
$$\implies d(h_0, T(h_0)) = d(A, B).$$

This shows that $h_0 \in A_0$ is a best proximity point of T. □

3. Conclusion

The main motivation of the current note is to show that the best proximity point result proved by Omidvari et al. [15, Theorem 1] follows from the fixed point result of Wardowski [19, Theorem 2.1]. So, the best proximity point result for non-self mappings is not a real generalization of the corresponding fixed point theory. Also, we show that the best proximity point theorem for F-proximal contraction of the first kind proved by Omidvari et al. [15, Theorem 2] follows from the Wardowski's fixed point theorem without the assumption of continuity and approximatively compactness. Finally, we prove a best proximity point result regarding F-proximal contraction of the second kind mapping where we donot take the isometry mapping $g : A \to A$ and the conditions related to mapping g. Instead, we take the p-property of the pair (A, B) and the continuity of the mapping $F : \mathbb{R}^+ \to \mathbb{R}$.

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