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MULTIPLICITY RESULTS FOR THE WAVE SYSTEM USING THE LINKING THEOREM

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Abstract. We investigate the existence of solutions of the one-dimensional wave system

$$u_{tt} - u_{xx} + \mu g(u+v) = f(u+v) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R,$$
$$v_{tt} - v_{xx} + \nu g(u+v) = f(u+v) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R,$$

with Dirichlet boundary condition. We find them by applying linking inequlaities.

1. Introduction

In [1] and [2], the authors investigate multiplicity of solutions for a piecewise linear perturbation of the one-dimensional wave operator under Dirichlet boundary condition on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and periodic condition on the variable t. The wave system with Dirichlet boundary condition,

$$u_{tt} - u_{xx} + \mu g(u+v) = f(u+v) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R,$$
$$v_{tt} - v_{xx} + \nu g(u+v) = f(u+v) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R.$$

we have extended. We applied the linking inequalities to studying multiple nontrivial solutions for the system.

In section 2, we have a concern with the wave equation

$$u_{tt} - u_{xx} + bu^+ - au^- = f(u) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R,$$

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with Dirichlet boundary condition. We find a suitable functional I on a Hilbert space H and prove the suitable version of the Palais-Smale condition for the topological method. And we find the two linking type inequalities, relative to two diferent decompositions of the space H. In section 3, we applied the results in order to study the wave system.

2. The single wave equation

We consider the following one-dimensional nonlinear wave equation

$$
\begin{aligned}
&u_{tt} - u_{xx} + bu^+ - au^- = f(u) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R, \\
&u(\pm\tfrac{\pi}{2}, x) = 0, \\
&\text{u is } \pi\text{-periodic in t and even in x and t,}
\end{aligned} \tag{1}
$$

where f is defined by

$$
f(s) = \begin{cases} |s|^{p-2}s, & s \geq 0 \\ |s|^{q-2}s, & s < 0 \end{cases} \tag{2}
$$

where $p, q > 2$ and $p \neq q$.

2.1. The Palais Smale condition. To begin with, we consider the associated eigenvalue problem

$$
\begin{aligned}
&u_{tt} - u_{xx} = \lambda u \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R \\
&u(\pm\tfrac{\pi}{2}, x) = 0 \\
&u(t,x) = u(-t,x) = u(t,-x) = u(t+\pi, x).
\end{aligned} \tag{3}
$$

A simple computation shows that equation (3) has infinitely many eigenvalues λ_{mn} and the corresponding eigenfunctions ϕ_{mn} given by

$$\lambda_{mn} = (2n+1)^2 - 4m^2,$$

$$\phi_{mn}(t,x) = \cos 2mt \cos(2n+1)x \qquad (m, n = 0, 1, 2, \cdots).$$

Let Q be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H the Hilbert space defined by

$$H = \{u \in L^2(Q) | u \text{ is even in } x \text{ and } t\}.$$

Then the set $\{\phi_{mn} | m, n = 0, 1, 2, \cdots\}$ is an orthogonal base of H and H consists of the functions

$$u(x,t) = \sum_{m,n=0}^{\infty} a_{mn}\phi_{mn}(t,x)$$

with the norm given by

$$\|u\|^2 = \sum_{m,n=0}^{\infty} a_{mn}^2.$$

We denote by $(\Lambda_i^-)_{i\geq 1}$ the sequence of the negative eigenvalues of equation (3), by $(\Lambda_i^+)_{i\geq 1}$ the sequence of the positive ones, so that

$$\cdots < \Lambda_1^- = -3 < \Lambda_1^+ = 1 < \Lambda_2^+ = 5 < \cdots .$$

We consider an orthonormal system of eigenfunctions $\{e_i^-, e_i^+, i \geq 1\}$ associated with the eigenvalues $\{\Lambda_i^-, \Lambda_i^+, i \geq 1\}$. We set

$$H^+ = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \geq 0\},$$

$$H^- = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \leq 0\}.$$

We define the linear projections $P^- : H \to H^-, P^+ : H \to H^+$.

We also introduce two linear operators $R : H \to H^+, S : H \to H^-$ by

$$S(u) = \sum_{i=1}^{\infty} \frac{a_i^- e_i^-}{\sqrt{-\Lambda_i^-}}, R(u) = \sum_{i=1}^{\infty} \frac{a_i^+ e_i^+}{\sqrt{\Lambda_i^+}}$$

if

$$u = \sum_{i=1}^{\infty} a_i^- e_i^- + \sum_{i=1}^{\infty} a_i^+ e_i^+.$$

It is clear that S and R are compact and self adjoint on H.

Definition 2.1. *Let $I_b : H \to R$ be defined by*

$$\begin{aligned} I_b(u) = \frac{1}{2}\|P^+u\|^2 \quad &- \quad \frac{1}{2}\|P^-u\|^2 \\ &+ \quad \frac{b}{2}\|[Au]^+\|^2 - \frac{a}{2}\|[Au]^-\|^2 - \int_\Omega F(Au)dtdx \end{aligned}$$

where $A = R + S$ and $F(s) = \int_0^s f(\tau)d\tau$.

It is straightforward that

$$\nabla I_b(u) = P^+u - P^-u + bA(Au)^+ - aA(Au)^- - Af(Au).$$

Following the idea of Hofer (see [3]) one can show that

Proposition 2.1. *$I_b \in C^{1,1}(H, R)$. Moreover $\nabla I_b(u) = 0$ if and only if $w = (R + S)(u)$ is a weak solution of (P), that is,*

$$\int_\Omega ((w_{tt} - w_{xx})v + b[w]^+v - a[w]^-v)dtdx = \int_\Omega f(w)vdtdx$$

for all smooth $v \in H$*.*

In this section, we suppose $b > 0$. Under this assumption, we have a concern with multiplicity of solutions of equation (1). Here we suppose that f is defined by equation (2).

In the following, we consider the following sequence of subspaces of $L^2(Q)$:

$$H_n = (\oplus_{i=1}^n H_{\Lambda_i^-}) \oplus (\oplus_{i=1}^n H_{\Lambda_i^+})$$

where H_Λ is the eigenspace associated to Λ.

Lemma 2.1. *The functional* I_b *satisfies* $(P.S.)_\gamma^*$ *condition, with respect to* (H_n)*, for all* γ*.*

Proof. Let (k_n) be any sequence in N with $k_n \to \infty$. And let (u_n) be any sequence in H such that $u_n \in H_n$ for all n, $I_b(u_n) \to \gamma$ and $\nabla(I_b)\mid_{H_{k_n}}(u_n) \to 0$.

First, we prove that (u_n) is bounded. By contradiction let $t_n = \|u_n\| \to \infty$ and set $\hat{u_n} = u_n/t_n$. Up to a subsequence $\hat{u_n} \rightharpoonup \hat{u}$ in H for some $\hat{u}$ in H. Moreover

$$\begin{aligned} 0 \quad &\leftarrow \quad < \nabla(I_b)_{H_{k_n}}(u_n), \hat{u_n} > -\frac{2}{t_n} I_b(u_n) \\ &= \quad \frac{2}{t_n}\int_\Omega F(Au_n)dtdx - \frac{1}{t_n}\int_\Omega f(Au_n)Au_n dtdx \\ &= \quad \int_\Omega -\frac{p-2}{p}(t_n)^{p-1}[(A\hat{u_n})^+]^p + \frac{q+2}{q}(t_n)^{q-1}[(A\hat{u_n})^-]^q dtdx. \end{aligned}$$

Since $t_n \to \infty$, $(A\hat{u_n})^+ \to 0$ and $(A\hat{u_n})^- \to 0$. This implies $A\hat{u} = 0$ and $\hat{u} = 0$, a contradiction.

So (u_n) is bounded and we can suppose $u_n \rightharpoonup u$ for some $u \in H$. We know that

$$\nabla(I_b)_{H_{k_n}}(u_n) = P^+u_n - P^-u_n + bA(Au_n)^+ - aA(Au_n)^- - Af(Au_n).$$

Since A is the compact operator, $P^+u_n - P^-u_n$ converges strongly, hence $u_n \to u$ strongly and $\nabla I_b(u) = 0$. □

2.2. The first result applying the linking theorem. Fixed Λ_i^-. We prove the Theorem via a linking argument.

First of all, we introduce a suitable splitting of the space H. Let

$$Z_1 = \oplus_{j=i+1}^\infty H_{\Lambda_j^-}, Z_2 = H_{\Lambda_i^-}, Z_3 = \oplus_{j=1}^{i-1} H_{\Lambda_j^-} \oplus H^+$$

Lemma 2.2. *There exists* R *such that* $\sup_{v \in Z_1 \oplus Z_2, \|v\| = R} I_b(v) \le 0$.

Proof. If $v \in Z_1 \oplus Z_2$ then

$$I_b(v) = -\frac{1}{2}\|v\|^2 + \frac{b}{2}\|[Sv]^+\|^2 - \frac{a}{2}\|[Sv]^-\|^2 - \int_\Omega F(Sv)dtdx.$$

Since

$$\begin{aligned}\frac{b}{2}\|[Sv]^+\|^2 &- \frac{a}{2}\|[Sv]^-\|^2 - \int_\Omega F(Sv)dtdx \\ &= \int_\Omega \frac{b}{2}([Sv]^+)^2 - \frac{1}{p}([Sv]^+)^p - \frac{a}{2}([Sv]^-)^2 - \frac{1}{q}([Sv]^-)^q dtdx,\end{aligned}$$

there exists R such that $\frac{b}{2}\|[Sv]^+\|^2 - \frac{a}{2}\|[Sv]^-\|^2 - \int_\Omega F(Sv)dtdx \leq 0$ for all $\|v\| = R$. Hence

$$I_b(v) \leq -\frac{1}{2}\|v\|^2 \leq 0.$$

□

LEMMA 2.3. *There exists ρ such that* $\inf_{u\in Z_2\oplus Z_3, \|u\|=\rho} I_b(u) > 0$.

Proof. Let $\sigma \in [0,1]$. We consider the functional $I_{b,\sigma} : Z_2 \oplus Z_3 \to R$ defined by

$$\begin{aligned}I_{b,\sigma}(u) &= \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 \\ &+ \frac{b}{2}\|[Au]^+\|^2 - \frac{a}{2}\|[Au]^-\|^2 - \sigma\int_\Omega F(Au)dtdx.\end{aligned}$$

We claim that there exists a ball $B_\rho = \{u \in Z_2 \oplus Z_3 | \|u\| < \rho\}$ such that

1. $I_{b,\sigma}$ are continuous with respect to σ,
2. $I_{b,\sigma}$ satisfies $(P.S)$ condition,
3. 0 is a minimum for $I_{b,0}$ in B_ρ,
4. 0 is the unique critical point of $I_{b,\sigma}$ in B_ρ.

Then by a continuation argument of Li-Szulkin's (see[4]), it can be shown that 0 is a local minimum for $I_b|_{Z_2\oplus Z_3} = I_{b,1}$ and Lemma is proved.

The continuity in σ and the fact that 0 is a local minimum for $I_{b,0}$ are straightforward. To prove $(P.S.)$ condition one can argue as in the previous Lemma, when dealing with I_b.

To prove that 0 is isolated we argue by contradiction and suppose that there exists a sequence (σ_n) in $[0,1]$ and sequence (u_n) in $Z_2 \oplus Z_3$ such that $\nabla I_{b,\sigma_n}(u_n) = 0$ for all $n, u_n \neq 0$, and$u_n \to 0$. Set $t_n = \|u_n\|$ and $\hat{u_n} = u_n/t_n$ then $t_n \to 0$. Let $\hat{v_n} = P^-\hat{u_n}$ and $\hat{w_n} = P^+\hat{u_n}$. Since

$\hat{v_n}$ varies in a finite dimensional space, we can suppose that $\hat{v_n} \to \hat{v}$ for some $\hat{v}$. We get

$$(4) \quad \begin{aligned} \frac{1}{t_n}\nabla I_{b,\sigma}(u_n) &= \hat{w_n} - \hat{v_n} \\ &+ \frac{b}{t_n}A(Au_n)^+ - \frac{a}{t_n}A(Au_n)^- - \frac{\sigma_n}{t_n}Af(Au_n) = 0. \end{aligned}$$

Multiplying by $\hat{w_n}$ yields

$$\|\hat{w_n}\|^2 = \frac{\sigma_n}{t_n}\int_\Omega f(Au_n)A\hat{w_n}dtdx - \frac{b}{t_n}\int_\Omega (Au_n)^+ A\hat{w_n}dtdx.$$

We know that

$$\begin{aligned} \int_\Omega (Au_n)^+ A\hat{w_n}dtdx &= \int_\Omega P^+(Au_n)^+ A\hat{u_n}dtdx \\ &= \int_\Omega P^+(Au_n)^+(A\hat{u_n})^+dtdx. \end{aligned}$$

Since $b > 0$, there exists a sequence (ϵ_n) such that $\epsilon_n \to 0$ and $0 < \epsilon_n < b$ for all n. That is

$$\frac{b}{t_n}\int_\Omega (Au_n)^+ A\hat{w_n}dtdx \geq \frac{\epsilon_n}{t_n}\int_\Omega P^+(Au_n)^+(A\hat{u_n})^+dtdx.$$

Then

$$\begin{aligned} \|\hat{w_n}\|^2 &\leq \frac{1}{t_n}\int_\Omega f(Au_n)A\hat{w_n}dtdx - \frac{\epsilon_n}{t_n}\int_\Omega P^+(Au_n)^+(A\hat{u_n})^+dtdx \\ &\leq \int_\Omega \frac{|f(Au_n)|}{t_n}|A\hat{w_n}|dtdx + \epsilon_n\int_\Omega |P^+(A\hat{u_n})^+||(A\hat{u_n})^+|dtdx. \end{aligned}$$

Since A is a compact operator

$$\begin{aligned} |f(Au_n)| &= |\{([t_nA\hat{u_n}]^+)^{p-1} - ([t_nA\hat{u_n}]^-)^{q-1}\}| \\ &\leq {t_n}^{p-1}|[A\hat{u_n}]^+|^{p-1} + {t_n}^{q-1}|[A\hat{u_n}]^-|^{q-1} \\ &\leq {t_n}^m(M_1 + {t_n}^{M-m}M_2) \end{aligned}$$

for some M_1 and M_2 where $m = \min\{p-1, q-1\}$ and $M = \max\{p-1, q-1\}$. We get that

$$\int_\Omega \frac{|f(Au_n)|}{t_n}|A\hat{w_n}|dxdt \leq {t_n}^m(M_1 + {t_n}^{M-m}M_2)\int_\Omega |A\hat{w_n}|dtdx \leq o(1).$$

Hence

$$\|\hat{w_n}\|^2 \leq o(1) + \epsilon_n \int_\Omega |P^+(A\hat{u_n})^+||(A\hat{u_n})^+|dtdx. \tag{5}$$

Since $\int_\Omega |P^+(A\hat{u_n})^+||(A\hat{u_n})^+|dxdt$ is bounded and equation (5) holds for every ϵ_n, $\hat{w_n} \to 0$ and so $(\hat{u_n})$ converges. Since $|f(Au_n)| \leq {t_n}^m(M_1 + {t_n}^{M-m}M_2)$, we get

$$\frac{\sigma_n}{t_n}|f(Au_n)| \leq \frac{1}{t_n}|f(Au_n)| \leq {t_n}^{m-1}(|M_1 + {t_n}^{M-m}M_2) \leq o(1).$$

Then $\frac{\sigma_n}{t_n}Af(Au_n) \to 0$. From equation (4), $(\hat{v_n})$ converges to zero, but this is impossible if $\|\hat{u_n}\| = 1$. □

Definition 2.2. *Let H be an Hilbert space, $Y \subset H$, $\rho > 0$ and $e \in H \setminus Y$, $e \neq 0$. Set:*

$$\begin{aligned} B_\rho(Y) &= \{x \in Y \mid \|x\| \leq \rho\}, \\ S_\rho(Y) &= \{x \in Y \mid \|x\| = \rho\}, \\ \triangle_\rho(e, Y) &= \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\| \leq \rho\}, \\ \Sigma_\rho(e, Y) &= \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\| = \rho\} \cup \{v \mid v \in Y, \|v\| \leq \rho\}. \end{aligned}$$

Theorem 2.1. *If $b > 0$, then the problem (1) has at least one nontrivial solution.*

Proof. Let $e \in Z_2$. By Lemma 3.1 and Lemma 3.2, for a suitable large R and a suitable small ρ, we have the linking inequality

$$\sup I_b(\Sigma_R(e, Z_1)) < \inf I_b(S_\rho(Z_2 \oplus Z_3)).$$

Moreover $(P.S.)^*_\gamma$ holds. By standard linking arguments, it follows that there exists a critical point u for I_b with $\alpha \leq I_b(u) \leq \beta$, where $\alpha = \inf I_b(S_\rho(Z_2 \oplus Z_3))$ and $\beta = \sup I_b(\Delta_R(e, Z_1))$. Since $\alpha > 0$, then $u \neq 0$. □

2.3. The second result applying the linking theorem. We assume in this section that $i \geq 2$ and we set

$$W_1 = \oplus_{j=i}^{\infty} H_{\Lambda_j^-}, W_2 = \oplus_{j=1}^{i-1} H_{\Lambda_j^-}, W_3 = H^+.$$

Notice that $W_1 = Z_1 \oplus Z_2$ and $W_2 \oplus W_3 = Z_3$.

Lemma 2.4. $\liminf_{\|u\| \to +\infty, u \in W_1 \oplus W_2} I_b(u) \leq 0.$

Proof. Let $(u_n)_n$ be a sequence in $W_1 \oplus W_2$ such that $\|u_n\| \to \infty$. We set $t_n = \|u_n\|$ and $\hat{u_n} = u_n/t_n$. Since S is a compact operator,

$$\frac{b}{2}\frac{\|[Su_n]^+\|^2}{t_n^2} - \frac{a}{2}\frac{\|[Su_n]^-\|^2}{t_n^2} - \int_\Omega \frac{F(Su_n)}{t_n^2} dtdx$$
$$= \int_\Omega \frac{b}{2}([S\hat{u_n}]^+)^2 - \frac{{t_n}^{p-2}}{p}([S\hat{u_n}]^+)^p - \frac{a}{2}([S\hat{u_n}]^-)^2 - \frac{{t_n}^{q-2}}{q}([S\hat{u_n}]^-)^q dtdx$$
$$\to -\infty.$$

Then

$$\frac{I_b(u_n)}{\|u_n\|^2} = -\frac{1}{2} + \frac{b}{2}\frac{\|[Su_n]^+\|^2}{t_n^2} - \frac{a}{2}\frac{\|[Su_n]^-\|^2}{t_n^2} - \int_\Omega \frac{F(Su_n)}{t_n^2} dtdx \to -\infty.$$

Hence

$$\liminf_{\|u\| \to +\infty, u \in W_1 \oplus W_2} I_b(u) \leq 0.$$

□

LEMMA 2.5. *There exists* $\hat{\rho}$ *such that* $\inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)) > 0$.

Proof. Repeating the same arguments used in 2.3, we get the conclusion. □

THEOREM 2.2. *Let* $i \geq 2$. *If* $b > 0$, *then the problem* (1) *has at least two nontrivial solution.*

Proof. Using the conclusion of 2.1, we have that there exist a nontrivial critical point u with

$$I_b(u) \leq \sup I_b(\Delta_R(e, Z_1))$$

where e, R were given in Lemma 3.1 and 3.2. We can choose that $\hat{R} \geq R$. Take any $\hat{e}$ in W_2, then we have a second linking inequality,

$$\sup I_b(\Sigma_{\hat{R}}(\hat{e}, W_1)) \leq \inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)).$$

Since $(P.S.)^*_\gamma$ holds, there exists a critical point $\hat{u}$ such that

$$\inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)) \leq I_b(\hat{u}) \leq \sup I_b(\Delta_{\hat{R}}(\hat{e}, W_1)).$$

Since $\hat{R} \geq R$ and $Z_1 \oplus Z_2 = W_1$,

$$\Delta_R(e, Z_1) \subset B_{\hat{R}}(W_1) \subset \Sigma_{\hat{R}}(\hat{e}, W_1).$$

Then

$$\begin{aligned} I_b(u) &\le \sup I_b(\Delta_R(e, Z_1)) \\ &\le \sup I_b(\Sigma_{\hat{R}}(\hat{e}, W_1)) < \inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)) \le I_b(\hat{u}). \end{aligned}$$

Hence $u \neq \hat{u}$. □

3. Solutons of the wave system

In this section we investigate the existence of solutions $(u(t,x), v(t,x))$ of wave system with Dirichlet boundary condition

$$\begin{aligned} &u_{tt} - u_{xx} + \mu g(u+v) = f(u+v) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R, \\ &v_{tt} - v_{xx} + \nu g(u+v) = f(u+v) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R, \\ &u(\pm\tfrac{\pi}{2}, x) = 0, v(\pm\tfrac{\pi}{2}, x) = 0, \\ &\text{u and v is } \pi\text{-periodic in t and even in x and t,} \end{aligned} \tag{6}$$

where $g(u) = bu^+ - au^-$ and f is defined by (2).

THEOREM 3.1. *Let μ, ν be positive constants and let $i \ge 2$. If $b > 0$, then the problem* (6) *has at least two nontrivial solutions.*

Proof. We get that

$$(u - \frac{\mu}{\nu}v)_{tt} - (u - \frac{\mu}{\nu}v)_{xx} = (1 - \frac{\mu}{\nu})f(u+v)$$

By contraction mapping principle, the problem

$$\begin{aligned} u_{tt} - u_{xx} &= F \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R, \\ u(\pm\tfrac{\pi}{2}, x) &= 0 \end{aligned}$$

has a unique solution. If u_1 is a solution of $u_{tt} - u_{xx} = (1 - \frac{\mu}{\nu})f$, then the solution (u, v) of problem (6) satisfies

$$u - \frac{\mu}{\nu}v = u_1.$$

On the other hand, from problem (6) we get the equation

$$\begin{aligned} &(u+v)_{tt} - (u+v)_{xx} + (\mu+\nu)g(u+v) = 2f(u+v) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R, \\ &u(\pm\tfrac{\pi}{2}, x) = 0, v(\pm\tfrac{\pi}{2}, x) = 0, \\ &\text{u and v is } \pi\text{-periodic in t and even in x and t.} \end{aligned}$$

Put $w = u + v$. Then the above equation is equivalent to

$$\begin{aligned} & w_{tt} - w_{xx} + (\mu + \nu) g(w) = 2f(w) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R, \\ & w(\pm\tfrac{\pi}{2}, x) = 0, \\ & \text{w is } \pi\text{-periodic in t and even in x and t.} \end{aligned} \tag{7}$$

By Theorem 2.2, equation (7) has at least two nontrivial solution. If w_1 is a solution of problem (7), then the solution (u, v) of problem (6) satisfies

$$u + v = w_1.$$

Hence we get the solution (u, v) of problem (6) from the following systems:

$$\begin{aligned} u - \frac{\mu}{\nu} v &= u_1, \\ u + v &= w_1. \end{aligned} \tag{8}$$

Since $1 + \frac{\mu}{\nu} > 0$, system (8) has a unique solution (u, v). Therefore system (6) has at least two nontrivial solutions. □

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