Korean J. Math. **32** (2024), No. 4, pp. 769–782
https://dx.doi.org/10.11568/kjm.2024.32.4.769

EXISTENCE THEOREMS AND EVALUATION FORMULAS FOR SEQUENTIAL YEH-FEYNMAN INTEGRALS

Byoung Soo Kim† and Young-Hee Kim

Abstract. We establish the existence of the sequential Yeh-Feynman integral for functionals of the form $F(x) = G(x)\Psi(x(S,T))$, where G belongs to a Banach algebra of sequential Yeh-Feynman integrable functionals and Ψ need not be bounded or continuous. We also give formulas evaluating the integrals of these functionals. Note that these functionals are often employed in the application of the Feynman integral to quantum theory, and Ψ corresponds to the initial condition associated with Schrödinger equation.

1. Introduction

Cameron and Storvick [4] gave a simple definition of the sequential Feynman integral on Wiener space which is applicable to a rather large class of functionals. In particular, they showed that the sequential Feynman integral exists and equals the analytic Feynman integral for all elements of a Banach algebra $\mathcal{S}^*$ of functionals expressible as Fourier-transform of measures of finite variation on $L_2[a,b]$. Moreover in [6], they established the existence of the sequential Feynman integral and gave explicit formulas for evaluating the integrals for larger classes of functionals containing the Banach algebras $\hat{\mathcal{S}}$ studied in [4].

On the other hand, Yeh [16,17] extended Wiener space to Yeh-Wiener space, that is, a space of functions of two variables. Much varied work on the integrals (analytic Yeh-Feynman integral and sequential Yeh-Feynman integral) on Yeh-Wiener space has been done in [1,9–12,14].

We turn now to reviewing the basic definitions on sequential Yeh-Feynman integral after which we will describe more precisely the results of this paper.

Let $C_2(Q)$ be the Yeh-Wiener space, that is, the space of real valued continuous functions $x(s,t)$ on $Q = [0,S] \times [0,T]$ such that $x(s,0) = x(0,t) = 0$ for all $(s,t) \in Q$. Let a subdivision σ of Q be given:

$$0 = s_0 < s_1 < \cdots < s_l = S, \quad 0 = t_0 < t_1 < \cdots < t_m = T.$$

Received September 12, 2024. Revised October 8, 2024. Accepted October 14, 2024.
2010 Mathematics Subject Classification: 28C20, 46G12.
Key words and phrases: Yeh-Wiener space, sequential Feynman integral, sequential Yeh-Feynman integral, Banach algebra $\hat{\mathcal{S}}(L_2(Q))$.
† This research was supported by Seoul National University of Science and Technology.

Let $X = X(s,t)$ be a quadratic surface in $C_2(Q)$ based on a subdivision σ and the $l \times m$ matrix of real numbers $\Xi = \{\xi_{j,k}\}$ and defined by

$$\begin{aligned} X(s,t) &= X((s,t),\sigma,\Xi) \\ &= \frac{\xi_{j,k} - \xi_{j-1,k} - \xi_{j,k-1} + \xi_{j-1,k-1}}{(s_j - s_{j-1})(t_k - t_{k-1})}(s - s_{j-1})(t - t_{k-1}) \\ &\quad + \frac{\xi_{j,k-1} - \xi_{j-1,k-1}}{s_j - s_{j-1}}(s - s_{j-1}) + \frac{\xi_{j-1,k} - \xi_{j-1,k-1}}{t_k - t_{k-1}}(t - t_{k-1}) + \xi_{j-1,k-1}, \end{aligned}$$

when $(s,t) \in [s_{j-1}, s_j] \times [t_{k-1}, t_k]$, $\xi_{0,0} = \xi_{0,k} = \xi_{j,0} = 0$ for $j = 1, 2, \ldots, l$ and $k = 1, 2, \cdots, m$. Where there is a sequence of subdivisions $\{\sigma_n\}$, then σ, l, m, s_j, t_k and Ξ will be replaced by $\sigma_n, l_n, m_n, s_{n;j}, t_{n;k}$ and Ξ_n.

Let q be a given nonzero real number and let $F(x)$ be a functional defined on a subset of $C_2(Q)$ containing all the quadratic surfaces in $C_2(Q)$. Let $\{\sigma_n\}$ be a sequence of subdivisions such that $\|\sigma_n\| \to 0$ as $n \to \infty$, and let $\{\lambda_n\}$ be a sequence of complex numbers with $\operatorname{Re}\lambda_n > 0$ such that $\lambda_n \to -iq$ as $n \to \infty$. Then if the integral in the right hand side of (1.1) exists for all n and if the following limit exists and is independent of the choice of the sequences $\{\sigma_n\}$ and $\{\lambda_n\}$, we say that the sequential Yeh-Feynman integral with parameter q exists and it is denoted by

$$\begin{aligned} \int^{\mathrm{syf}_q} F(x)\,dx = \lim_{n\to\infty} \gamma_{\sigma_n,\lambda_n} \int_{\mathbb{R}^{l_n m_n}} \exp\Big\{-\frac{\lambda_n}{2}\int_Q \Big[\frac{\partial^2 X}{\partial s\,\partial t}((s,t),\sigma_n,\Xi_n)\Big]^2 ds\,dt\Big\} \\ \times F(X((\cdot,\cdot),\sigma_n,\Xi_n))\,d\Xi_n, \end{aligned} \tag{1.1}$$

where

$$\gamma_{\sigma,\lambda} = \Big(\frac{\lambda}{2\pi}\Big)^{lm/2} \prod_{j=1}^{l}\prod_{k=1}^{m} [(s_j - s_{j-1})(t_k - t_{k-1})]^{-1/2}.$$

Let

$$\begin{aligned} H_\lambda(\sigma,\Xi) &\equiv \gamma_{\sigma,\lambda} \exp\Big\{-\frac{\lambda}{2}\int_Q \Big[\frac{\partial^2 X}{\partial s\,\partial t}((s,t),\sigma,\Xi)\Big]^2 ds\,dt\Big\} \\ &= \Big(\frac{\lambda}{2\pi}\Big)^{lm/2} \prod_{j=1}^{l}\prod_{k=1}^{m} [(s_j - s_{j-1}(t_k - t_{k-1})]^{-1/2} \\ &\quad \times \exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{l}\sum_{k=1}^{m} \frac{(\xi_{j,k} - \xi_{j-1,k} - \xi_{j,k-1} + \xi_{j-1,k-1})^2}{(s_j - s_{j-1})(t_k - t_{k-1})}\Big\}. \end{aligned} \tag{1.2}$$

Thus in terms of $H_\lambda(\sigma,\Xi)$, the sequential Yeh-Feynman integral can be written

$$\int^{\mathrm{syf}_q} F(x)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^{l_n m_n}} H_{\lambda_n}(\sigma_n,\Xi_n) F(X((\cdot,\cdot),\sigma_n,\Xi_n))\,d\Xi_n. \tag{1.3}$$

To describe the class of functionals that we work with in this paper, we need the concept of absolute continuouity for functions of two variables. For any subrectangle $R = [a,b] \times [c,d]$ of Q and a real valued function $x(s,t)$ on Q, let $\Delta_R(x) = x(b,d) - x(a,d) - x(b,c) + x(a,c)$. A function $x(s,t)$ is absolutely continuous on Q if the following two conditions are satisfied [2].

(i) Given $\epsilon > 0$, there exists $\delta > 0$ such that $\sum_{R\in\mathcal{R}} |\Delta_R(x)| < \epsilon$ whenever $\mathcal{R}$ is a finite collection of pairwise non-overlapping subrectangles of R with $\sum_{R\in\mathcal{R}} m(R) < \delta$, where m denotes the Lebesgue measure on $\mathbb{R}^2$.

(ii) The functions $x(\cdot, T)$ and $x(S, \cdot)$ are absolutely continuous of a single variable on $[0, S]$ and $[0, T]$, respectively.

Let $D_2(Q)$ be the class of elements of $C_2(Q)$ such that x is absolutely continuous on Q and $\frac{\partial^2 x}{\partial s\, \partial t}(s,t) \in L_2(Q)$.

For $u, v \in L_2(Q)$ and $x \in C_2(Q)$, we let

$$\langle u, v\rangle = \int_Q u(s,t)v(s,t)\, ds\, dt, \tag{1.4}$$

and

$$\langle u, v\rangle_{j,k} = \int_{t_{k-1}}^{t_k} \int_{s_{j-1}}^{s_j} u(s,t)v(s,t)\, ds\, dt \tag{1.5}$$

for $j = 1, \dots, l$ and $k = 1, \dots, m$. Thus we have

$$\langle u, v\rangle = \sum_{j=1}^{l} \sum_{k=1}^{m} \langle u, v\rangle_{j,k}. \tag{1.6}$$

If there is a sequence of subdivisions $\{\sigma_n\}$, then $\langle u, v\rangle_{j,k}$ will be replaced by $\langle u, v\rangle_{n;j,k}$.

Let $\mathcal{M}(L_2(Q))$ be the class of complex measures of bounded variation defined on $\mathcal{B}(L_2(Q))$, the Borel measurable subsets of $L_2(Q)$. A functional F defined on a subset of $C_2(Q)$ that contains $D_2(Q)$ is said to be an element of $\hat{\mathcal{S}}(L_2(Q))$ if there exists a measure $f \in \mathcal{M}(L_2(Q))$ such that for $x \in D_2(Q)$,

$$F(x) = \int_{L_2(Q)} \exp\Big\{ i \Big\langle v, \frac{\partial^2 x}{\partial s\, \partial t} \Big\rangle \Big\}\, df(v). \tag{1.7}$$

Note that $\hat{\mathcal{S}}(L_2(Q))$ with the norm $\|F\| = \|f\| = \operatorname{var} f$ is a Banach algebra [1].

In the application of the Feynman integral to quantum theory, functionals of the type

$$F(x) = G(x)\Psi(x(S,T)) \tag{1.8}$$

are often employed, with $G \in \hat{\mathcal{S}}(L_2(Q))$ and Ψ corresponding to the initial condition associated with Schrödinger equation [4, 6].

In Section 2 of the present paper, we prove the existence of the sequential Yeh-Feynman integral of the type (1.8) when Ψ is the Fourier transform of a measure of bounded variation on $\mathbb{R}$. However this condition restricts Ψ to be bounded and continuous.

In Section 3, we shall establish the sequential Yeh-Feynman integrability of the type (1.8), where Ψ need not be bounded or continuous. We also establish formulas for the evaluation of such integrals.

2. Sequential Yeh-Feynman integral of some bounded functionals

Let $v \in L_2(Q)$ and let σ be any subdivision

$$0 = s_0 < s_1 < \cdots < s_l = S, \quad 0 = t_0 < t_1 < \cdots < t_m = T.$$

We define the averaged function $v_\sigma(s,t)$ for v on σ by

$$v_\sigma(s,t) = \frac{1}{(s_j - s_{j-1})(t_k - t_{k-1})} \langle v, 1\rangle_{j,k} \tag{2.1}$$

when $s_{j-1} \le s < s_j$ and $t_{k-1} \le t < t_k$ for $j = 1, \dots, l$ and $k = 1, \dots, m$, and

$$v_\sigma(s,t) = 0 \tag{2.2}$$

when $s = S$ and $t = T$.

LEMMA 2.1 (Proposition 2.3 of [10]). *Let $v \in L_2(Q)$ and let $\{\sigma_n\}$ be a sequence of subdivisions of Q such that $\|\sigma_n\| \to 0$ as $n \to \infty$. Then we have*

$$\lim_{n\to\infty} \|v_{\sigma_n}\|_2^2 = \|v\|_2^2.$$

The following theorem shows the existence of the sequential Yeh-Feynman integrable of functionals in $\hat{\mathcal{S}}(L_2(Q))$ which was established in [10].

THEOREM 2.2. *Let $F \in \hat{\mathcal{S}}(L_2(Q))$ be given by (1.7). Then the sequential Yeh-Feynman integral of F exists and is given by*

$$\int^{\mathrm{syf}_q} F(x)\,dx = \int_{L_2(Q)} \exp\Big\{-\frac{i}{2q}\|v\|_2^2\Big\}\,df(v) \tag{2.3}$$

for each nonzero real number q.

Next we consider two more functionals which are different from but are closely related with the expression (1.7). The functional in Theorem 2.3 below was studied in [6–8], while the functional in Theorem 2.4 was studied in [5]. These functionals are often employed in the application of the Feynman integral to quantum theory.

Let $\mathcal{T}$ be the set of functions Ψ defined on $\mathbb{R}$ by

$$\Psi(r) = \int_{\mathbb{R}} \exp\{ir\xi\}\,d\rho(\xi) \tag{2.4}$$

where ρ is a complex Borel measure of bounded variation on $\mathbb{R}$.

For $\xi \in \mathbb{R}$, let $\phi(\xi)$ be the function $v \in L_2(Q)$ such that $v(s,t) = \xi$ for $0 \le s \le S$ and $0 \le t \le T$; thus $\phi : \mathbb{R} \to L_2(Q)$ is continuous. If E is a Borel measurable subset of $L_2[0,T]$, then $\phi^{-1}(E)$ is a Borel measurable subet of $\mathbb{R}$. Let

$$\psi(E) = \rho(\phi^{-1}(E)). \tag{2.5}$$

Thus $\psi \in \mathcal{M}(L_2(Q))$. Transforming the right hand member of (2.4), we have for $x \in D_2(Q)$,

$$\Psi(x(S,T)) = \int_{L_2(Q)} \exp\Big\{i\Big\langle u, \frac{\partial^2 x}{\partial s\,\partial t}\Big\rangle\Big\}\,d\psi(u), \tag{2.6}$$

and $\Psi(x(S,T))$, considered as a functional of x, is an element of $\hat{\mathcal{S}}(L_2(Q))$.

THEOREM 2.3. *For $x \in D_2(Q)$, let $F(x) = G(x)\Psi(x(S,T))$ where $G \in \hat{\mathcal{S}}(L_2(Q))$ and $\Psi \in \mathcal{T}$ are given by (1.7) with corresponding measure g in $\mathcal{M}(L_2(Q))$ and (2.4), respectively. Then F is sequential Yeh-Feynman integrable and*

$$\int^{\mathrm{syf}_q} F(x)\,dx = \int_{L_2(Q)} \int_{\mathbb{R}} \exp\Big\{-\frac{i}{2q}\|v+\xi\|_2^2\Big\}\,d\rho(\xi)\,dg(v) \tag{2.7}$$

for each nonzero real number q.

Proof. Because $\hat{\mathcal{S}}(L_2(Q))$ is a Banach algebra, and $G(x)$ and $\Psi(x(S,T))$ belong to $\hat{\mathcal{S}}(L_2(Q))$ as functions of x, we know that $F \in \hat{\mathcal{S}}(L_2(Q))$. By (1.7) and (2.6), we have for $x \in D_2(Q)$,

$$F(x) = \int_{L_2^2(Q)} \exp\Big\{i\Big\langle v+u, \frac{\partial^2 x}{\partial s\,\partial t}\Big\rangle\Big\}\,dg(v)\,d\psi(u).$$

Making the subtitution $w = v+u$ on the inner integral, we have

$$F(x) = \int_{L_2^2(Q)} \exp\Big\{i\Big\langle w, \frac{\partial^2 x}{\partial s\,\partial t}\Big\rangle\Big\}\,dg_w(w-u)\,d\psi(u),$$

where the subscript w in the measure g indicates that the integration is being performed with respect to the variable w. By the unsymmetric Fubini theorem(Theorem 6.1 in [3]), we have

$$F(x) = \int_{L_2(Q)} \exp\Big\{i\Big\langle w, \frac{\partial^2 x}{\partial s\,\partial t}\Big\rangle\Big\}\,df_{g,\psi}(w),$$

where $f_{g,\psi}$ is a complex measure on $\mathcal{B}(L_2(Q))$ defined by

$$f_{g,\psi}(E) = \int_{L_2(Q)} g(E-u)\,d\psi(u).$$

Now, applying Theorem 2.2, we have

$$\int^{\mathrm{syf}_q} F(x)\,dx = \int_{L_2(Q)} \exp\Big\{-\frac{i}{2q}\|w\|_2^2\Big\}\,df_{g,\psi}(w).$$

By Theorem 6.1 of [3] and the transformation $v = w-u$, we have

$$\int^{\mathrm{syf}_q} F(x)\,dx = \int_{L_2^2(Q)} \exp\Big\{-\frac{i}{2q}\|v+u\|_2^2\Big\}\,dg(v)\,d\psi(u).$$

Finally by (2.5) and the Fubini theorem, we rewrite the right hand side of the last expression to obtain (2.7). □

In [5], Cameron and Storvick proved the existence of the sequential Fourier-Feynman transform of the functionals

$$H(x) = \int_{L_2[0,T]} \exp\{i\langle v, x'\rangle\}\Phi(v)\,d\mu(v), \tag{2.8}$$

for $x \in D[0,T]$, where $D[0,T]$ is the class of elements x in Wiener space such that x is absolutely continuous on $[0,T]$ and $x' \in L_2[0,T]$, and Φ is a bounded measurable functional defined on $L_2[0,T]$. Moreover, recently Kim and Yoo [13, 15, 18] extended the above result for the generalized sequential Fourier-Feynman transform.

In our next theorem, we establish the sequential Yeh-Feynman integrability of the Yeh-Wiener space version of the functionals of the type (2.8). Our results in Theorem 2.4 below can be applied to establish the sequential Fourier-Yeh-Feynman transform of the functionals of the type (2.8).

THEOREM 2.4. *Let Φ be a bounded measurable functional defined on $L_2(Q)$, and let*

$$F(x) = \int_{L_2(Q)} \exp\Big\{i\Big\langle v, \frac{\partial^2 x}{\partial s\,\partial t}\Big\rangle\Big\}\Phi(v)\,df(v) \tag{2.9}$$

for $x \in D_2(Q)$. *Then* F *is sequential Yeh-Feynman integrable and*

$$\int^{\mathrm{syf}_q} F(x)\,dx = \int_{L_2(Q)} \exp\Big\{-\frac{i}{2q}\|v\|_2^2\Big\}\Phi(v)\,df(v) \tag{2.10}$$

for each nonzero real number q.

Proof. Let a measure f_ϕ be defined by $f_\phi(E) = \int_E \Phi(v)\,df(v)$ for $E \in \mathcal{B}(L_2(Q))$. Clearly $f_\phi \in \mathcal{M}(L_2(Q))$ and for $x \in D_2(Q)$,

$$F(x) = \int_{L_2(Q)} \exp\Big\{i\Big\langle v, \frac{\partial^2 x}{\partial s\,\partial t}\Big\rangle\Big\}\,df_\phi(v)$$

so that $F \in \hat{\mathcal{S}}(L_2(Q))$. Applying Theorem 2.2 and replacing $df_\phi(v)$ by $\Phi(v)\,df(v)$, we completes the proof. □

3. Sequential Yeh-Feynman integral of some unbounded functionals

All the functionals we worked with in Section 2 was actually bounded. In this section, we discuss the sequential Yeh-Feynman integrability of some unbounded functionals, that is, we establish the sequential Yeh-Feynman integrability of functionals of the form

$$F(x) = G(x)\Psi(x(S,T)), \tag{3.1}$$

where $G \in \hat{\mathcal{S}}(L_2(Q))$ and Ψ need not be bounded or continuous. The same type of functionals on Wiener space, but not on Yeh-Wiener space, were studied in [6, 8]. In [6], Cameron and Storvick established the sequential Feynman integrability of those functionals. While in [8], Chang et al. studied the sequential Fourier-Feynman transform and a translation theorem for such functionals.

To assist the proof of Theorem 3.5 below, the main result of this section, we first introduce some lemmas.

Lemma 3.1. *Let* $0 = t_0 < t_1 < \cdots < t_m$, *let* ξ_m, $c_1, \ldots, c_m$ *and* $\beta_1, \ldots, \beta_m$ *be real, let* $\alpha > 0$ *and let* $\operatorname{Re}\lambda > 0$. *Let*

$$\begin{aligned} J \equiv \int_{\mathbb{R}^{m-1}} \exp\Big\{-\frac{\lambda}{2}\sum_{k=1}^m \frac{(\xi_k - \xi_{k-1} - c_k + c_{k-1})^2}{\alpha(t_k - t_{k-1})} \\ + i\sum_{k=1}^m \frac{\beta_k(\xi_k - \xi_{k-1} - c_k + c_{k-1})}{\alpha(t_k - t_{k-1})}\Big\}\,d\xi_1\cdots d\xi_{m-1}. \end{aligned} \tag{3.2}$$

Then we have

$$\begin{aligned} J = t_m^{-1/2}\Big(\frac{2\pi\alpha}{\lambda}\Big)^{(m-1)/2}\Big(\prod_{k=1}^m (t_k - t_{k-1})^{1/2}\Big)\exp\Big\{\frac{1}{2\lambda\alpha t_m}\Big(\sum_{k=1}^m \beta_k\Big)^2 \\ - \frac{1}{2\lambda\alpha}\sum_{k=1}^m \frac{\beta_k^2}{t_k - t_{k-1}} + \frac{1}{2\alpha t_m}\Big(-\lambda(\xi_m - c_m)^2 + 2i(\xi_m - c_m)\sum_{k=1}^m \beta_k\Big)\Big\}, \end{aligned} \tag{3.3}$$

where $\xi_0 = c_0 = 0$.

Proof. Since $\alpha > 0$, we have $0 = \tau_0 < \tau_1 < \cdots < \tau_m$, where $\tau_k = \alpha t_k$ for $k = 0, 1, \ldots, m$. Changing the variables $u_k = \xi_k - c_k$ for $k = 0, 1, \ldots, m$, we apply Lemma 1 in [6] to obtain

$$J = \int_{\mathbb{R}^{m-1}} \exp\Big\{-\frac{\lambda}{2}\sum_{k=1}^{m}\frac{(u_k - u_{k-1})^2}{\tau_k - \tau_{k-1}} + i\sum_{k=1}^{m}\frac{\beta_k(u_k - u_{k-1})}{\tau_k - \tau_{k-1}}\Big\}\, du_1 \cdots du_{m-1}$$
$$= \tau_m^{-1/2}\Big(\frac{2\pi}{\lambda}\Big)^{(m-1)/2}\Big(\prod_{k=1}^{m}(\tau_k - \tau_{k-1})^{1/2}\Big)$$
$$\times \exp\Big\{\frac{1}{2\lambda\tau_m}\Big(\sum_{k=1}^{m}\beta_k\Big)^2 - \frac{1}{2\lambda}\sum_{k=1}^{m}\frac{\beta_k^2}{\tau_k - \tau_{k-1}} + \frac{1}{2\tau_m}\Big(-\lambda u_m^2 + 2iu_m\sum_{k=1}^{m}\beta_k\Big)\Big\},$$

which completes the proof. □

To prove the following lemma, we need a mathematical induction on two variables. That is, a statement $P(l, m)$ is true for all natural numbers l and m if it satisfies the following three conditions.

1. $P(1, 1)$ is true.
2. For all $m \geq 1$, if $P(1, m)$ is true, then $P(1, m + 1)$ is true.
3. For all $l \geq 1$, if $P(l, m)$ is true for all $m \geq 1$, then $P(l + 1, m)$ is true for all $m \geq 1$.

Lemma 3.2. *Let l and m be natural numbers. Let $0 = s_0 < s_1 < \cdots < s_l$ and $0 = t_0 < t_1 < \cdots < t_m$. Let $\xi_{l,m}$ and $\beta_{1,1}, \ldots, \beta_{l,m}$ be real, and let $\operatorname{Re}\lambda > 0$. Let*

$$(3.4)\quad \begin{aligned} P(l, m) \equiv \int_{\mathbb{R}^{lm-1}} \exp\Big\{&-\frac{\lambda}{2}\sum_{j=1}^{l}\sum_{k=1}^{m}\frac{(\xi_{j,k} - \xi_{j-1,k} - \xi_{j,k-1} + \xi_{j-1,k-1})^2}{\Delta_{j,k}} \\ &+ i\sum_{j=1}^{l}\sum_{k=1}^{m}\frac{\beta_{j,k}(\xi_{j,k} - \xi_{j-1,k} - \xi_{j,k-1} + \xi_{j-1,k-1})}{\Delta_{j,k}}\Big\} \\ &\times d\xi_{1,1}\cdots d\xi_{1,m}\cdots d\xi_{l-1,1}\cdots d\xi_{l-1,m}\, d\xi_{l,1}\cdots d\xi_{l,m-1}. \end{aligned}$$

Then we have

$$(3.5)\quad \begin{aligned} P(l, m) = (s_l t_m)^{-1/2}\Big(\frac{2\pi}{\lambda}\Big)^{(lm-1)/2}\Big(\prod_{j=1}^{l}\prod_{k=1}^{m}\Delta_{j,k}^{1/2}\Big) \\ \times \exp\Big\{\frac{1}{2\lambda s_l t_m}\Big(\sum_{j=1}^{l}\sum_{k=1}^{m}\beta_{j,k}\Big)^2 - \frac{1}{2\lambda}\sum_{j=1}^{l}\sum_{k=1}^{m}\frac{\beta_{j,k}^2}{\Delta_{j,k}} \\ + \frac{1}{2s_l t_m}\Big(-\lambda\xi_{l,m}^2 + 2i\xi_{l,m}\sum_{j=1}^{l}\sum_{k=1}^{m}\beta_{j,k}\Big)\Big\}, \end{aligned}$$

where $\xi_{0,k} = \xi_{j,0} = 0$ for $j = 0, 1, \ldots, l$ and $k = 0, 1, \ldots, m$, and $\Delta_{j,k} = (s_j - s_{j-1})(t_k - t_{k-1})$ for $j = 1, \ldots, l$ and $k = 1, \ldots, m$.

Proof. We shall use induction on l and m. Let us first assume that $l = m = 1$. Then there is no integrals in the right-hand member of (3.4) and, both expressions in

(3.4) and (3.5) equal to

$$\exp\Big\{\frac{1}{2s_l t_m}(-\lambda\xi_{1,1}^2+2i\xi_{1,1}\beta_{1,1})\Big\},$$

and the lemma is true when $l=m=1$.

Let us next assume that $l=1$ and m be arbitrary number. Then

$$P(1,m)=\int_{\mathbb{R}^{m-1}}\exp\Big\{-\frac{\lambda}{2}\sum_{k=1}^{m}\frac{(\xi_{1,k}-\xi_{1,k-1})^2}{\Delta_{1,k}}+i\sum_{k=1}^{m}\frac{\beta_{1,k}(\xi_{1,k}-\xi_{1,k-1})}{\Delta_{1,k}}\Big\}\,d\xi_{1,1}\cdots d\xi_{1,m-1}.$$

Now we apply Lemma 3.1 with $c_k=0$ for $k=0,1,\ldots,m$ and $\alpha=s_1$ to obtain

$$\begin{aligned}P(1,m)={}&t_m^{-1/2}\Big(\frac{2\pi s_1}{\lambda}\Big)^{(m-1)/2}\Big(\prod_{k=1}^{m}(t_k-t_{k-1})^{1/2}\Big)\exp\Big\{\frac{1}{2\lambda s_1 t_m}\Big(\sum_{k=1}^{m}\beta_{1,k}\Big)^2\\&-\frac{1}{2\lambda s_1}\sum_{k=1}^{m}\frac{\beta_{1,k}^2}{\Delta_k}+\frac{1}{2s_1t_m}\Big(-\lambda\xi_{1,m}^2+2i\xi_{1,m}\sum_{k=1}^{m}\beta_{1,k}\Big)\Big\},\end{aligned}$$

which is equal to the right-hand side of (3.5) when $l=1$, and the lemma is true when $l=1$ and m is an arbitrary natural number.

Let us next assume that the lemma is true when $l\ge 1$ and $m\ge 1$, and proceed to establish that the lemma is valid when $l+1$ and $m\ge 1$. Multiplying (3.4) with l by

$$\begin{aligned}\exp\Big\{&-\frac{\lambda}{2}\sum_{k=1}^{m}\frac{(\xi_{l+1,k}-\xi_{l,k}-\xi_{l+1,k-1}+\xi_{l,k-1})^2}{\Delta_{l+1,k}}\\&+i\sum_{k=1}^{m}\frac{\beta_{l+1,k}(\xi_{l+1,k}-\xi_{l,k}-\xi_{l+1,k-1}+\xi_{l,k-1})}{\Delta_{l+1,k}}\Big\},\end{aligned}$$

and integrating with respect to $\xi_{l,m},\xi_{l+1,1},\ldots,\xi_{l+1,m-1}$ on $\mathbb{R}^m$ we obtain

$$\begin{aligned}P(l+1,m)=\int_{\mathbb{R}^{(l+1)m-1}}&\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{l+1}\sum_{k=1}^{m}\frac{(\xi_{j,k}-\xi_{j-1,k}-\xi_{j,k-1}+\xi_{j-1,k-1})^2}{\Delta_{j,k}}\\&+i\sum_{j=1}^{l+1}\sum_{k=1}^{m}\frac{\beta_{j,k}(\xi_{j,k}-\xi_{j-1,k}-\xi_{j,k-1}+\xi_{j-1,k-1})}{\Delta_{j,k}}\Big\}\\&\times d\xi_{1,1}\cdots d\xi_{1,m}\cdots d\xi_{l,1}\cdots d\xi_{l,m}\,d\xi_{l+1,1}\cdots d\xi_{l+1,m-1}.\end{aligned}$$

By the induction hypothesis we obtain

$$\begin{aligned}P(l+1,m)={}&(s_lt_m)^{-1/2}\Big(\frac{2\pi}{\lambda}\Big)^{(lm-1)/2}\Big(\prod_{j=1}^{l}\prod_{k=1}^{m}\Delta_{j,k}^{1/2}\Big)\\&\times\exp\Big\{\frac{1}{2\lambda s_lt_m}\Big(\sum_{j=1}^{l}\sum_{k=1}^{m}\beta_{j,k}\Big)^2-\frac{1}{2\lambda}\sum_{j=1}^{l}\sum_{k=1}^{m}\frac{\beta_{j,k}^2}{\Delta_{j,k}}\Big\}K(l,m),\end{aligned}\tag{3.6}$$

where

$$K(l,m) \equiv \int_{\mathbb{R}^m} \exp\Big\{-\frac{\lambda}{2}\sum_{k=1}^{m}\frac{(\xi_{l+1,k}-\xi_{l,k}-\xi_{l+1,k-1}+\xi_{l,k-1})^2}{\Delta_{j,k}} + i\sum_{k=1}^{m}\frac{\beta_{l+1,k}(\xi_{l+1,k}-\xi_{l,k}-\xi_{l+1,k-1}+\xi_{l,k-1})}{\Delta_{l+1,k}} + \frac{1}{2s_l t_m}\Big(-\lambda\xi_{l,m}^2+2i\xi_{l,m}\sum_{j=1}^{l}\sum_{k=1}^{m}\beta_{j,k}\Big)\Big\}\,d\xi_{l,m}\,d\xi_{l+1,1}\cdots\xi_{l+1,m-1}.$$

Now we apply Lemma 3.1 with $c_k=\xi_{l,k}$ and $\alpha=s_{l+1}-s_l$ to obtain

$$K(l,m) = t_m^{-1/2}\Big(\frac{2\pi(s_{l+1}-s_l)}{\lambda}\Big)^{(m-1)/2}\Big(\prod_{k=1}^{m}(t_k-t_{k-1})^{1/2}\Big) \times \exp\Big\{\frac{1}{2\lambda(s_{l+1}-s_l)t_m}\Big(\sum_{k=1}^{m}\beta_{l+1,k}\Big)^{1/2} - \frac{1}{2\lambda(s_{l+1}-s_l)}\sum_{k=1}^{m}\frac{\beta_{l+1,k}^2}{t_k-t_{k-1}}\Big\}K_1(l,m)$$

where

$$K_1(l,m) \equiv \int_{\mathbb{R}} \exp\Big\{\frac{1}{2(s_{l+1}-s_l)t_m}\Big(-\lambda(\xi_{l+1,m}-\xi_{l,m})^2+2i(\xi_{l+1,m}-\xi_{l,m})\sum_{k=1}^{m}\beta_{l+1,k}\Big) + \frac{1}{2s_l t_m}\Big(-\lambda\xi_{l,m}^2+2i\xi_{l,m}\sum_{j=1}^{l}\sum_{k=1}^{m}\beta_{j,k}\Big)\Big\}\,d\xi_{l,m}$$

$$= \exp\Big\{\frac{1}{2(s_{l+1}-s_l)t_m}\Big(-\lambda\xi_{l+1,m}^2+2i\xi_{l+1,m}\sum_{k=1}^{m}\beta_{l+1,k}\Big)\Big\} \times \int_{\mathbb{R}}\exp\Big\{-\frac{\lambda s_{l+1}}{2(s_{l+1}-s_l)s_l t_m}\xi_{l,m}^2 + \Big(\frac{\lambda\xi_{l+1,m}}{(s_{l+1}-s_l)t_m} - \frac{i}{(s_{l+1}-s_l)t_m}\sum_{k=1}^{m}\beta_{l+1,k} + \frac{i}{s_l t_m}\sum_{j=1}^{l}\sum_{k=1}^{m}\beta_{j,k}\Big)\xi_{l,m}\Big\}\,d\xi_{l,m}.$$

Performing the last integration on $\xi_{l,m}$, we obtain

$$K(l,m) = \Big(\frac{s_l}{s_{l+1}}\Big)^{1/2}\Big(\frac{2\pi(s_{l+1}-s_l)}{\lambda}\Big)^{m/2}\Big(\prod_{k=1}^{m}(t_k-t_{k-1})^{1/2}\Big) \times \exp\Big\{\frac{1}{2\lambda(s_{l+1}-s_l)t_m}\Big(\sum_{k=1}^{m}\beta_{l+1,k}\Big)^{1/2} - \frac{1}{2\lambda(s_{l+1}-s_l)}\sum_{k=1}^{m}\frac{\beta_{l+1,k}^2}{t_k-t_{k-1}}\Big\} \times \exp\Big\{\frac{1}{2(s_{l+1}-s_l)t_m}\Big(-\lambda\xi_{l+1,m}^2+2i\xi_{l+1,m}\sum_{k=1}^{m}\beta_{l+1,k}\Big) + \frac{(s_{l+1}-s_l)s_l t_m}{2\lambda s_{l+1}}\Big(\frac{\lambda}{(s_{l+1}-s_l)t_m}\xi_{l+1,m} - \frac{i}{(s_{l+1}-s_l)t_m}\sum_{k=1}^{m}\beta_{l+1,k} + \frac{i}{s_l t_m}\sum_{j=1}^{l}\sum_{k=1}^{m}\beta_{j,k}\Big)^2\Big\}.$$

Substituting this value of $K(l,m)$ in the last member of (3.6) above we find that

$$\begin{aligned}P(l+1,m) &= (s_{l+1}t_m)^{-1/2}\Big(\frac{2\pi}{\lambda}\Big)^{[(l+1)m-1]/2}\Big(\prod_{j=1}^{l+1}\prod_{k=1}^{m}\Delta_{j,k}^{1/2}\Big)\\ &\times\exp\Big\{\frac{1}{2\lambda s_{l+1}t_m}\Big(\sum_{j=1}^{l+1}\sum_{k=1}^{m}\beta_{j,k}\Big)^2-\frac{1}{2\lambda}\sum_{j=1}^{l+1}\sum_{k=1}^{m}\frac{\beta_{j,k}^2}{\Delta_{j,k}}\\ &+\frac{1}{2s_{l+1}t_m}\Big(-\lambda\xi_{l+1,m}^2+2i\xi_{l+1,m}\sum_{j=1}^{l+1}\sum_{k=1}^{m}\beta_{j,k}\Big)\Big\},\end{aligned}$$

that is, the lemma is true when $l+1$ and $m\geq 1$. Hence by mathematical induction the lemma is valid for all natural numbers l and m. □

Applying Lemma 3.2 with $s_l = S$, $t_m = T$ and $\beta_{j,k} = \langle v,1\rangle_{j,k}$ for $j = 1,\dots,l$ and $k = 1,\dots,m$, and using the relationship (1.6), we obtain the following lemma.

LEMMA 3.3. *Let $v\in L_2(Q)$ and let $\xi_{l,m}$ be a real number. Let σ be a subdivision of Q and let $\operatorname{Re}\lambda>0$. Let*

$$\begin{aligned}J_{\sigma,\lambda}(\xi_{l,m},v) &\equiv \gamma_{\sigma,\lambda}\int_{\mathbb{R}^{lm-1}}\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{l}\sum_{k=1}^{m}\frac{(\xi_{j,k}-\xi_{j-1,k}-\xi_{j,k-1}+\xi_{j-1,k-1})^2}{(s_j-s_{j-1})(t_k-t_{k-1})}\\ &+i\sum_{j=1}^{l}\sum_{k=1}^{m}\langle v,1\rangle_{j,k}\frac{\xi_{j,k}-\xi_{j-1,k}-\xi_{j,k-1}+\xi_{j-1,k-1}}{(s_j-s_{j-1})(t_k-t_{k-1})}\Big\}\\ &\times d\xi_{1,1}\cdots d\xi_{1,m}\cdots d\xi_{l-1,1}\cdots d\xi_{l-1,m}\,d\xi_{l,1}\cdots d\xi_{l,m-1}.\end{aligned}\tag{3.7}$$

Then the above integral exists and its value is given by

$$\begin{aligned}J_{\sigma,\lambda}(\xi_{l,m},v) &= \Big(\frac{\lambda}{2\pi ST}\Big)^{1/2}\exp\Big\{\frac{1}{2\lambda ST}(\lambda i\xi_{l,m}+\langle v,1\rangle)^2\\ &-\frac{1}{2\lambda}\sum_{j=1}^{l}\sum_{k=1}^{m}\frac{\langle v,1\rangle_{j,k}^2}{(s_j-s_{j-1})(t_k-t_{k-1})}\Big\}.\end{aligned}\tag{3.8}$$

Using Lemma 2.1 we easily obtain the following lemma.

LEMMA 3.4. *Let v and $\{\sigma_n\}$ be given as in Lemma 2.1. Let $\{\lambda_n\}$ be a sequence of complex numbers such that $\operatorname{Re}\lambda_n>0$ and $\lambda_n\to -iq$ as $n\to\infty$. Let $J_{\sigma_n,\lambda_n}(\xi,v)$ be given by (3.7) with σ,λ and $\xi_{l,m}$ replaced by σ_n,λ_n and ξ. Then we have*

$$\lim_{n\to\infty}J_{\sigma_n,\lambda_n}(\xi,v)=\Big(\frac{-iq}{2\pi ST}\Big)^{1/2}\exp\{K_q(\xi,v)\},\tag{3.9}$$

where

$$K_q(\xi,v)\equiv\frac{i}{2qST}(q\xi+\langle v,1\rangle)^2-\frac{i}{2q}\|v\|_2^2.\tag{3.10}$$

Now we are ready to establish the existence of the sequential Yeh-Feynman integral of functionals of the form (3.1).

THEOREM 3.5. *For $x \in D_2(Q)$, let $F(x) = G(x)\Psi(x(S,T))$ where $G \in \hat{\mathcal{S}}(L_2(Q))$ is given by (1.7) with corresponding measure g in $\mathcal{M}(L_2(Q))$ and $\Psi \in L_1(\mathbb{R})$. Then for each nonzero real number q, F is sequential Yeh-Feynman integrable and*

$$\int^{\mathrm{syf}_q} F(x)\,dx = \Big(\frac{-iq}{2\pi ST}\Big)^{1/2} \int_{L_2(Q)} \int_{\mathbb{R}} \exp\{K_q(\xi,v)\}\Psi(\xi)\,d\xi\,dg(v), \tag{3.11}$$

where $K_q(\xi,v)$ is given by (3.10).

Proof. Let σ be a subdivision of Q

$$0 = s_0 < s_1 < \cdots < s_l = S, \quad 0 = t_0 < t_1 < \cdots < t_m = T.$$

Let λ be a complex number with $\operatorname{Re}\lambda > 0$, and let

$$\begin{aligned} I_{\sigma,\lambda}(F) &\equiv \int_{\mathbb{R}^{lm}} H_\lambda(\sigma,\Xi)F(X((\cdot,\cdot),\sigma,\Xi))\,d\Xi \\ &= \gamma_{\sigma,\lambda}\int_{\mathbb{R}^{lm}} \exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{l}\sum_{k=1}^{m}\frac{(\xi_{j,k}-\xi_{j-1,k}-\xi_{j,k-1}+\xi_{j-1,k-1})^2}{(s_j-s_{j-1})(t_k-t_{k-1})}\Big\} \\ &\quad \times F(X((\cdot,\cdot),\sigma,\Xi))\,d\Xi. \end{aligned}$$

By (1.7), we have

$$\begin{aligned} F(X((\cdot,\cdot),\sigma,\Xi)) = \int_{L_2[0,T]} &\exp\Big\{i\sum_{j=1}^{l}\sum_{k=1}^{m}\langle v,1\rangle_{j,k}\frac{\xi_{j,k}-\xi_{j-1,k}-\xi_{j,k-1}+\xi_{j-1,k-1}}{(s_j-s_{j-1})(t_k-t_{k-1})}\Big\} \\ &\times \Psi(\xi_{l,m})\,dg(v) \end{aligned}$$

and so

$$\begin{aligned} I_{\sigma,\lambda}(F) = \gamma_{\sigma,\lambda}\int_{\mathbb{R}^{lm}}\int_{L_2(Q)} &\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{l}\sum_{k=1}^{m}\frac{(\xi_{j,k}-\xi_{j-1,k}-\xi_{j,k-1}+\xi_{j-1,k-1})^2}{(s_j-s_{j-1})(t_k-t_{k-1})} \\ &+ i\sum_{j=1}^{l}\sum_{k=1}^{m}\langle v,1\rangle_{j,k}\frac{\xi_{j,k}-\xi_{j-1,k}-\xi_{j,k-1}+\xi_{j-1,k-1}}{(s_j-s_{j-1})(t_k-t_{k-1})}\Big\}\Psi(\xi_{l,m})\,dg(v)\,d\Xi. \end{aligned}$$

By Lemma 3.2 with $\beta_{j,k} = 0$ for $j = 1,\ldots,l$ and $k = 1,\ldots,m$, we have

$$\begin{aligned} &\int_{\mathbb{R}^{lm}}\int_{L_2(Q)} \exp\Big\{-\frac{\operatorname{Re}\lambda}{2}\sum_{j=1}^{l}\sum_{k=1}^{m}\frac{(\xi_{j,k}-\xi_{j-1,k}-\xi_{j,k-1}+\xi_{j-1,k-1})^2}{(s_j-s_{j-1})(t_k-t_{k-1})}\Big\}|\Psi(\xi_{l,m})|\,dg(v)\,d\Xi \\ &\le |\gamma_{\sigma,\lambda}|^{-1}\Big(\frac{\operatorname{Re}\lambda}{2\pi ST}\Big)^{1/2}\int_{L_2(Q)}\int_{\mathbb{R}}\exp\Big\{-\frac{\operatorname{Re}\lambda}{2ST}\xi_{l,m}^2\Big\}|\Psi(\xi_{l,m})|\,d\xi_{l,m}\,dg(v) \end{aligned}$$

which is finite, and so we apply Fubini theorem and Lemma 3.3 to obtain

$$I_{\sigma,\lambda}(F) = \int_{L_2[0,T]}\int_{\mathbb{R}} J_{\sigma,\lambda}(\xi,v)\Psi(\xi)\,d\xi\,dg(v),$$

where $J_{\sigma,\lambda}(\xi,v)$ is given by (3.7) with $\xi_{l,m}$ replaced by ξ.

Now let $\{\sigma_n\}$ be a sequence of subdivisions of Q such that $\|\sigma_n\| \to 0$ as $n \to \infty$, and let $\{\lambda_n\}$ be a sequence of complex numbers such that $\operatorname{Re}\lambda_n > 0$ and $\lambda_n \to -iq$ as $n \to \infty$. Then we have

$$I_{\sigma_n,\lambda_n}(F) = \int_{L_2[0,T]}\int_{\mathbb{R}} J_{\sigma_n,\lambda_n}(\xi,v)\Psi(\xi)\,d\xi\,dg(v).$$

But by the Schwartz inequality,

$$\frac{\langle v,1\rangle^2}{ST}-\sum_{j=1}^{l_n}\sum_{k=1}^{m_n}\frac{\langle v,1\rangle^2_{n;j,k}}{(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\le 0$$

and so we apply dominated convergence theorem and use Lemma 3.4 to conclude that

$$\lim_{n\to\infty} I_{\sigma_n,\lambda_n}(F)=\Big(\frac{-iq}{2\pi ST}\Big)^{1/2}\int_{L_2[0,T]}\int_{\mathbb{R}}\exp\{K_q(\xi,v)\}\Psi(\xi)\,d\xi\,dg(v)$$

and this completes the proof. □

From Theorems 2.3 and 3.5 and the linearity of the sequential Yeh-Feynman integral we have the following theorem.

THEOREM 3.6. *For $x\in D_2(Q)$, let $F(x)=G(x)\Psi(x(S,T))$ where $G\in\hat{\mathcal{S}}(L_2(Q))$ is given by (1.7) with corresponding measure g in $\mathcal{M}(L_2(Q))$ and $\Psi=\Psi_1+\Psi_2$ with $\Psi_1\in L_1(\mathbb{R})$ and $\Psi_2\in\mathcal{T}$ is given by (2.4). Then for each nonzero real number q, F is sequential Yeh-Feynman integrable and*

$$\begin{aligned}\int^{\mathrm{syf}_q}F(x)\,dx&=\Big(\frac{-iq}{2\pi ST}\Big)^{1/2}\int_{L_2(Q)}\int_{\mathbb{R}}\exp\{K_q(\xi,v)\}\Psi_1(\xi)\,d\xi\,dg(v)\\&\quad+\int_{L_2(Q)}\int_{\mathbb{R}}\exp\Big\{-\frac{i}{2q}\|v+\xi\|_2^2\Big\}\,d\rho(\xi)\,dg(v)\end{aligned}\tag{3.12}$$

where $K_q(\xi,v)$ is given by (3.10).

Cameron and Storvick [4] gave an example of sequential Feynman integrable functional which does not lie in the Banach algebra $\hat{\mathcal{S}}$ on Wiener space. As a corollary of our result Theorem 3.5, we obtain an extension of the Cameron and Storvick's example to the sequential Yeh-Feynman integral. That is, we establish the existence and an evaluation formula for sequential Yeh-Feynman integral of L_1 functional in Corollary 3.7 below.

COROLLARY 3.7. *For $x\in D_2(Q)$, let $F(x)=\Psi(x(S,T))$ where $\Psi\in L_1(\mathbb{R})$. Then the sequential Yeh-Feynman integral of F exists and is given by*

$$\int^{\mathrm{syf}_q}F(x)\,dx=\Big(\frac{-iq}{2\pi ST}\Big)^{1/2}\int_{\mathbb{R}}\exp\Big\{\frac{iq\xi^2}{2ST}\Big\}\Psi(\xi)\,d\xi\tag{3.13}$$

for each nonzero real number q.

Proof. Since the constant function $G(x)\equiv 1$ is obtained by taking g the probability measure concentrated at $v=0$ in $L_2(Q)$, $G(x)\equiv 1$ belongs to the Banach algebra $\hat{\mathcal{S}}(L_2(Q))$. Now (3.13) follows immediately from (3.11) □

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Byoung Soo Kim
School of Natural Sciences, Seoul National University of Science and Technology, Seoul 01811, Korea.
E-mail: mathkbs@seoultech.ac.kr

Young-Hee Kim
Ingenium College of Liberal Arts, Kwangwoon University
Seoul 01897, Korea.
E-mail: yhkim@kw.ac.kr