Korean J. Math. **34** (2026), No. 1, pp. 21–40
https://dx.doi.org/10.11568/kjm.2026.34.1.21

ANALYSIS OF ULAM-HYERS STABILITY AND THE EXISTENCE OF SOLUTIONS IN NONLINEAR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING INTEGRAL BOUNDARY CONDITIONS

Muntazeer Ansari and Lakshmi Narayan Mishra*

Abstract. The current study addresses a boundary value problem involving integral boundary conditions with Caputo fractional differential equations and employs the boundary value problem (BVP) framework to establish the existence of solutions via Schaefer's fixed point theorem. Additionally, it leverages contraction mapping principles to prove uniqueness and investigates Ulam-Hyers stability of fractional-order BVPs using Grönwall's inequality. As an illustration, three examples are provided to demonstrate the applicability of our main results.

1. Introduction

FDEs have become increasingly widespread and important due to their wide applications across numerous scientific domains, including biology, biological technology, biophysics, biological processes, population dynamics [22], electrochemistry, chemical technology, physics, mechanics, electrodynamics, aerodynamics, viscoelasticity [9], electrical systems, control systems, porous media [33], and the fitting of experiment data are a few illustrations [28]. Fractional-order models have been preferred to integer-order ones because of their capacity for memory retention. To structure and process the problems with mathematical simulation, nonlinear FDEs with IBCs emerge in various fields, including electrodynamics, chemistry, physics, aerodynamics, polymers, and others [3, 21]. Integrals and derivatives with fractional order are defined in several ways, encompassing the CFD and the Riemann-Liouville (R-L), Hadamard, and Weyl versions, etc. Except in a few special cases, these definitions are usually distinct, since authors wish to preserve particular qualities of such operators. In the recent past, numerous new derivatives, like the Hadamard derivative [12], and additionally, the ψ-Riemann-Liouville fractional integral [26], allow for a more flexible treatment of fractional calculus, accommodating different types of memory effects and nonlocal behaviors, but the Riemann and Caputo derivatives [1], predominate in being extensively utilized. After investigation, delve into the Caputo-Fabrizio

Received September 15, 2024. Revised September 15, 2025. Accepted December 8, 2025.
2010 Mathematics Subject Classification: 26A33, 45G10, 45M10, 47H10.
Key words and phrases: Caputo fractional differential equations, Integral boundary condition, Schaefer's fixed point theorem, Banach contraction principle, Ulam-Hyers stability.
* Corresponding author.

fractional derivative, which offers a non-singular kernel and interesting properties. Afterwards, Baleanu and Atangana fractional derivative [10], which offers a non-singular kernel and interesting properties. In the scientific literature, researchers have proposed various analytical and numerical methods for solving FDEs. Over the past few decades, various numerical techniques have emerged for solving linear fractional integro-DEs, including the application of shifted Chebyshev polynomials combined with least squares approximation [24]. Likewise, it has been suggested to solve fractional-order diffusion equations using the Chebyshev polynomials. In [23], fractional-order integro-DEs were solved utilizing cubic B-splines. Likewise, systems of nonlinear FDEs have been suggested in [32], examining DEs of fractional order. For that, several effective discretization methods have been developed, as referenced in [13, 14]. However, these discretization techniques often result in dense linear systems, which can become computationally prohibitive. For instance, recent works have explored innovative formulations and solution methods in various fields that employ meshless methods based on radial basis functions for solving three-dimensional multi-term time fractional partial differential equations (PDEs) [36], arising in engineering phenomena, as well as studies presenting exact analytical solutions for nonlinear PDEs using Hermite polynomials [15]. Analogously, the use of residual neural networks has been explored for the advancement of fractional differential equations [8]. Let's delve into the fascinating world of FDEs and explore their existence, uniqueness, and stability properties. These factors are essential for comprehending and examining dynamic systems modeled by FDEs. In this context, we provide a concise summary of some recent research findings related to the existence, uniqueness, and stability of solutions for FDEs involving the CFD. These results are based on the application of the Monch's FPT serves as the foundation for the outcomes related to the process of measure of non-compactness in [27, 29]. Along with the cited works, a few more contributions [35], FDEs solutions have been studied analytically and numerically using FPT. Researchers have diligently explored the existence, uniqueness, and stability of solutions with various fractional derivatives and integrals; see [25]. The work of Ulam [34], on stability of FDEs in the Ulam sense, is considered the starting point for Ulam stability research. Many intriguing types of Ulam stabilities have emerged in response to the work of Ulam-stability. Researchers have introduced various stability notions, including the following, which are called the UHS and HURS, the generalized, etc. Thus, for ordinary DEs, functional DEs, IEs, integro-DEs, partial DEs, and so on, the Ulam stability, the UHS, the HURS, the EU of the solutions, and so on were thoroughly examined for integro-DEs and partial DEs. Researchers in this specialized sector are becoming increasingly interested in FDEs. An essential component of FDEs, stability analysis has been discussed here [16], and the UHS and the existence of a solution of FDEs of a nonlinear singular type by the use of p-Laplacian have been discussed; see [4, 19]. In recent years, there has also been an increasing number of publications exploring the UHS as it relates to FDEs. In many different types of FDEs, the theories of EU and UHS, once established, have served as a motivating force for numerous researchers to contribute to this field; see [6, 30].

In their research, Alam et al. [5], explored for significant types of Ulam-stability and also examined the EU of solutions, including the fractional derivative of R-L:

$$ {}^{\mathcal{R}}D^{\ell_1}_{0,\wp}\, F(\wp) = \mathfrak{S}(\wp, F(\wp)), {}^{\mathcal{R}}D^{\ell_1}_{0,\wp}\, F(\wp)), \quad \wp \in I = [0, T], \text{ where } T > 0, $$

employing the BCs of fractional order

$$ {}^{\mathcal{R}}D^{\ell_2}_{0,\wp} F(0^+) = \beta {}^{\mathcal{R}}D^{\ell_2}_{0,\wp} F(T^-), $$
$$ {}^{\mathcal{R}}D^{\ell_3}_{0,\wp} F(0^+) = \alpha {}^{\mathcal{R}}D^{\ell_3}_{0,\wp} F(T^-), $$

where $\beta, \alpha \neq 1$, $\ell_1 \in (1,2)$, $\ell_2, \ell_3 < \ell_1$, $\mathfrak{S} : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the ${}^{\mathcal{R}}D^{\ell}$ operator represents the ℓ order R-L fractional derivative.
The mathematical contributions of Bezgir and Ghazanfari [11] include an in-depth analysis of EU properties for nonlinear fractional integro-DEs.

$$ {}^{c}D^{\ell_1}_{0^+} F(\wp) = \mathfrak{S}(\wp, F(\wp)), {}^{c}D^{\ell_2}_{0^+} F(\wp), {}^{\mathcal{R}}I^{\ell_3}_{0^+} F(\wp)), \quad \wp \in I = [0,T],\ T > 0, $$

with the non-local BCs

$$ {}^{c}D^{\ell_2}_{0^+} F(\wp_1) + {}^{c}D^{\ell_2}_{0^+} F(\wp_2) = \mathcal{K}_1 F'(1), $$
$$ F(0) + F(1) = \mathcal{K}_2 {}^{\mathcal{R}}I^{\ell_3}_{0^+} F(\wp_3). $$

The parameters ℓ_1, ℓ_2, and ℓ_3 are subject to the condition $\ell_1 \in (1,2)$, ℓ_2, $\wp_1, \wp_2, \wp_3 \in (0,1)$, $\mathcal{K}_1, \mathcal{K}_2 \in \mathbb{R}$, $\ell_1 > \ell_2 + 1$, while ℓ_3 is non-negative. The symbols ${}^{c}D^{\ell}_{0^+}$ and ${}^{\mathcal{R}}I^{\ell_3}_{0^+}$ possess as the CFD and R-L fractional integral of various orders, respectively.
Building upon the research mentioned earlier and the ongoing investigations in this domain, researchers are inspired to explore further and contribute to the field. The mathematical framework of this study concerns the investigation of EU results for nonlinear fractional-order BVP with IBCs as

$$ \begin{cases} {}^{c}D^{\ell} F(\wp) = \mathfrak{S}(\wp, F(\wp)), \\ F(0) - \zeta F'(0) = \int_0^1 \mathfrak{T}(\bar{\xi}) d\bar{\xi}, \ \varkappa F(1) + \varrho F'(1) = \int_0^1 \mathfrak{X}(\bar{\xi}) d\bar{\xi}, \end{cases} \tag{1} $$

where $\nu \geq 0$, $\varrho \geq 0$, $\zeta \geq 0$, and $\varkappa \geq 0$, and ${}^{c}D^{\ell}$ denotes the nonlinear CFD of order ℓ such that $1 < \ell \leq 2$ and $\mathfrak{S} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function. The primary objective is to apply IBCs, as stated in `Eq.` (1), to demonstrate the existence of solutions for nonlinear CFD on a continuous and bounded domain.
Furthermore, we find the existence and uniqueness of the solution. The technique used here is the Schaefer's FPT and BCP.
After the introduction, the remaining part of the paper is structured in the following manner:
Section **2** presents the fundamental definitions, auxiliary lemmas, and preliminary results that establish the theoretical framework for the subsequent analysis. Section **3** presents the main results and is devoted to the rigorous development of the principal analytical framework and the derivation of the core theoretical results. In Subsection **3.1**, the fractional differential equation with IBCs corresponding to the proposed problem (1) is formulated and investigated. The existence and uniqueness of solution are rigorously demonstrated through the application of Banach's and Schaefer's FPTs. Moreover, Subsection **3.2** is devoted to a comprehensive stability analysis, wherein the UHS of the proposed problem (1) is established. Section **4** is concerned with three illustrative applications, which serve to validate and exemplify the theoretical findings. Finally, Section **5** provides the concluding remarks, summarizing the principal contributions of the work and outlining potential directions for future research.

2. Notations, definitions and auxiliary facts

Essential notations and background informations are provided to facilitate understanding of our primary findings, as outlined in this section. Consider the interval $\mathscr{J} = [0,1]$. Consider set of real numbers denoted by $\mathbb{R}$. We also have a space $\mathscr{M} = C(\mathscr{J}, \mathbb{R})$, which represents the collection of continuous functions from an interval $\mathscr{J}$ to real numbers $\mathbb{R}$. It can be demonstrated that $\mathscr{M}$ forms a Banach space, equipped with the supremum norm defined as: $\|\mathscr{M}\| = \{\sup|\mathscr{F}(\wp)| : \wp \in [0,1]\}$.

Notation used:

- FDEs: Fractional differential equations;
- DEs: Differential equations;
- CFD: Caputo fractional derivative;
- R-L: Riemann Liouville;
- BVP: Boundary value problem;
- IBCs: Integral boundary conditions;
- UHS: Ulam-Hyers stability;
- HURS: Hyers-Ulam-Rassias stability;
- FPT: Fixed point theory;
- EU: Existence and uniqueness;
- BCP: Banach contraction principle.

To make it easier for the reader to analyze the problem, fundamental principles and outcomes from the theory of fractional calculus are discussed. The most recent literature provides these materials, see [2, 7, 20].

Definition 2.1. ([31]) A generalization of Liouville's fractional integral operator is the modified form proposed by Riemann. The iterated integral formula of Cauchy's

$$\int_k^{\alpha_0} d\alpha_1 \int_k^{\alpha_1} d\alpha_2 \cdots \int_k^{\alpha_{n-1}} f(\alpha_n) d\alpha_n = \frac{1}{\Gamma(n)} \int_k^{\alpha} \frac{f(\bar{\Upsilon})}{(\alpha - \bar{\Upsilon})^{1-n}} d\bar{\Upsilon},$$

where Γ denotes Euler's gamma function. For every positive real value of n, the right-hand side of this equation clearly has results. Consequently, the following is an intuitive definition of the fractional derivative:

Definition 2.2. ([31]) The CFD of order ℓ for a continuous function $f : [0, \infty) \to \mathbb{R}$, is defined as

$${}^{c}\mathcal{D}^{\ell} f(\wp) = \frac{1}{\Gamma(n-\ell)} \int_0^{\alpha} (\wp - \bar{\Upsilon})^{n-\ell-1} f^n(\bar{\Upsilon}) d\bar{\Upsilon}, \ n = [\ell] + 1,$$

and $f^n(\bar{\Upsilon})$ exists, where $n = [\ell] + 1$ shows the integer part of the real number order ℓ.

Remark 2.3. When conditions are natural n on $f(\wp)$, the CFD is transform in to the fixed integer under the $f(\wp)$ function's derivative as $\ell \to$ n.

Remark 2.4. Let ℓ, $\zeta > 0$, $n = [\ell] + 1$ and $\zeta > n$, then the following relation hold

$${}^{c}\mathcal{D}^{\ell} \ \wp^{\zeta-1} = \frac{\Gamma(\zeta)}{\Gamma(\zeta - \ell)} \wp^{\zeta-1}.$$

Definition 2.5. ([31]) For a continuous function $f(\wp)$, the R-L fractional integral of order $\ell > 0$ is described as

$$I^{\ell} f(\wp) = \frac{1}{\Gamma(\ell)} \int_0^{\wp} (\wp - \bar{\Upsilon})^{\ell-1} f(\bar{\Upsilon}) d\bar{\Upsilon}.$$

DEFINITION 2.6. ([33]) The fractional derivative of order $\ell > 0$ in the sense of R-L for a continuous function $f(\wp)$ is defined as follows:

$$\begin{aligned} {}^{RL}D^{\ell} f(\wp) :=& D^n I^{n-\ell} f(\wp) \\ =& \frac{1}{\Gamma(n-\ell)} \left(\frac{d}{dt}\right)^n \int_a^{\wp} (\wp - \bar{\Upsilon})^{n-\ell-1} f(\bar{\Upsilon}) d\bar{\Upsilon}, \quad n-1 < \ell < n, \end{aligned}$$

where, $n = [\ell] + 1$ represents the smallest integer greater than the real number ℓ provided that the right-hand side is defined pointwise on (a, ∞).

DEFINITION 2.7. ([33]) A generalized form of the R-L fractional derivative of order ℓ and parameter φ is defined as follows:

$${}^{H}D^{\ell,\varphi} f(\wp) = I^{\varphi(n-\ell)} D^n I^{(n-\ell)(1-\varphi)} f(\wp),$$

where $n-1 < \ell < n,\ 0 \le \varphi \le 1,\ \wp > a.$

REMARK 2.8. It is evident that when $\varphi = 0$ takes the form of the Riemann-Liouville fractional differential equation expressed as follows:

$${}^{RL}D^{\ell} u(\wp) = f(\wp, u(\wp)), \quad \wp \in [a,b],$$

and if $\varphi = 1$ takes the form of the Caputo fractional differential equation expressed as follows:

$${}^{c}D^{\ell} u(\wp) = f(\wp, u(\wp)), \quad \wp \in [a,b].$$

LEMMA 2.9. *(* [31])
Let $\ell > 0$, be order of FDE for the homogeneous type

$${}^{c}D^{\ell} F(\wp) = 0,$$

has a solution and given that, $c_{\mathfrak{i}} \in \mathbb{R}, \quad \mathfrak{i} = 0, 1, ..., \mathfrak{n}$ and $\mathfrak{n} = [\ell] + 1$,

$$F(\wp) = c_1 + c_2 \wp + c_3 \wp^2 + ... + c_{\mathfrak{n}} \wp^{\mathfrak{n}-1}.$$

LEMMA 2.10. *(* [31]) Suppose that $\mathfrak{n}$ is the least integer such that $\mathfrak{n} \ge \ell > 0$, and $c_{\mathfrak{i}} \in \mathbb{R}$, where $\mathfrak{i} = 0, 1, 2, ..., \mathfrak{n} - 1$, then

$$I^{\ell}\ {}^{c}D^{\ell} F(\wp) = F(\wp) + c_0 + c_1 \wp + c_2 \wp^2 + ... + c_{\mathfrak{n}-1} \wp^{\mathfrak{n}-1}.$$

The further discussion using the common attributes of fractional integral and FDEs are as follows:

LEMMA 2.11. *(* [31]) Consider $\mathcal{X}$ be a Banach space, $\mathcal{J} \subset \mathcal{X}$ be closed, and $\mathcal{F} : \mathcal{J} \to \mathcal{J}$ is a strict contraction,

$$|\mathcal{F}x - \mathcal{F}y| \le \mathcal{K}|x - y|,$$

for some $\mathcal{K} \in (0, 1)$ and all $x, y \in \mathcal{J}$. Then $\mathcal{F}$ has a unique fixed point in $\mathcal{J}$.

LEMMA 2.12. *(* [17]) (Arzela-Ascoli's theorem). In the space C([a, b]), compactness of a family of continuous functions is equivalent to uniform boundedness and equicontinuity.

Lemma 2.13. *([17])* At this point, we are establishing the following concept:
(i) The expression states that $\exists$ a scalar ϖ such that the equation $\digamma = \varpi T\digamma$ admits a solution, specifically, when for $\varpi = 1$.
(ii) The set $\varphi = \digamma \in \mathscr{M};\ \digamma = T\digamma,\ \varpi \in [0,1]$ is unbounded.

Lemma 2.14. *([31])*(Schaefer's Fixed Point Theorem). Assume that $\mathcal{F} : \mathcal{J} \to \mathcal{J}$ be a completely continuous operator in the Banach space $\mathcal{J}$, and let the set $\varphi = u \in \mathcal{J}|u = \mu\mathcal{F}u, 0 < \mu < 1$, be bounded. Then $\mathcal{F}$ has a fixed point in $\mathcal{J}$.

Lemma 2.15. *([18])* (Grönwall's inequality). Assume that $u(\wp) \geq 0$ and $f(\wp \geq 0)$ exist and be a continuous function on $I = [0, \infty)$. For all inequality

$$u(\wp) \leq u_0 + \int_0^{\wp} u(\bar{\Upsilon}) f(\bar{\Upsilon}) d\bar{\Upsilon},\ \wp \in I, \tag{2}$$

holds, where $u_0 \geq 0$, and constant. Then

$$u(\wp) \leq u_0 e^{\int_0^{\wp} f(\bar{\Upsilon})d\bar{\Upsilon}},\ \wp \in I. \tag{3}$$

Definition 2.16. ([34]) The Eq.(1) is Ulam-Hyers stable if there exist a real numbers $C_w > 0$,
such that $\forall\ \epsilon > 0$, and for all solutions $\mathscr{G} \in C([0,1], \mathbb{R})$ of the inequality

$$|\ {}^{c}\mathcal{D}^{\ell}\mathscr{G}(\wp) - \mathfrak{S}(\wp, \mathscr{G}(\wp))| \leq \epsilon,\ \wp \in [0,1], \tag{4}$$

$\exists$ a solution $\digamma \in C([0,1], \mathbb{R})$ of Eq.(1) with

$$|\mathscr{G}(\wp) - \digamma(\wp)| \leq C_w \epsilon, \quad \wp \in [0,1].$$

3. Main results

In this present section, we establish the existence and uniqueness of solutions to the boundary value problem Eq. (1) within a Banach space. The proposed problem Eq. (1) is first transformed into an equivalent integral equation using piecewise continuous function, whose essential properties are subsequently derived and employed to formulate the conditions ensuring uniqueness. The existence and uniqueness of solutions are rigorously demonstrated through the applications of Banach's and Schaefer's fixed point theorems. Furthermore, this subsection encompasses a comprehensive stability analysis, wherein the Ulam-Hyers stability of the proposed problem Eq. (1) is rigorously established.

3.1. Existence and uniqueness results. For the existence of solutions for the problem (1), we need the following auxiliary lemma.

Lemma 3.1. *If any $\digamma(\wp)$, $\mathfrak{T}$, $\mathfrak{X} \in (C[0,1], \mathbb{R})$, $1 < \ell \leq 2$, with that the BVP is there* Eq.(1) *has a solution,*

$$\digamma(\wp) = \int_0^1 \mathbb{G}(\wp, \bar{\Upsilon}) \mathcal{F}(\bar{\Upsilon}) d\bar{\Upsilon} + \mathfrak{T}(\wp), \tag{5}$$

where

$$\mathbb{G}(\wp, \bar{\Upsilon}) = \frac{\wp^2 (3 - \wp)(1 - \bar{\Upsilon})^{\ell-4}}{6\Gamma(\ell - 2)} - \frac{\wp^2 (1 - \bar{\Upsilon})^{\ell-3}}{2\Gamma(\ell - 1)}$$

$$+\begin{cases} \dfrac{(\wp-\bar{\Upsilon})^{\ell-1}}{\Gamma(\ell)}, & if\ 0\leq\bar{\Upsilon}\leq\wp\leq 1\\ 0, & if\ 0\leq\wp\leq\bar{\Upsilon}\leq 1,\end{cases}$$

$$\mathfrak{T}(\wp)=\frac{\varrho+\varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1\mathfrak{T}(\wp,\bar{\Upsilon})d\bar{\Upsilon}+\frac{\zeta(1+\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1\mathfrak{X}(\wp,\bar{\Upsilon})d\bar{\Upsilon}.$$

Proof. Let for BVP `Eq.` (1), F is a solution, then using Lemma (2.10), we can reduce problem

$$F(\wp)=I^{\ell}+c_1+c_2\wp,\quad \text{where } c_1,c_2\in\mathbb{R}, \tag{6}$$

by combining `Eq.` (5) with `Eq.`(6), we get

$$\nu c_1-\zeta c_2=\int_0^1\mathfrak{T}(\bar{\Upsilon})d\bar{\Upsilon}$$
$$\varkappa c_1+(\varrho+\varkappa)c_2=-\varkappa I^{\ell}\mathcal{F}(1)-\varrho I^{\ell-1}\mathcal{F}(1)+\int_0^1\mathfrak{X}(\bar{\Upsilon})d\bar{\Upsilon},$$

which implies that

$$c_1=-\frac{\zeta}{\pounds}(\varkappa^{\ell}\mathcal{F}(1+\varrho I^{\ell-1}\mathcal{F}(1)))+\frac{1}{\pounds}\int_0^1((\varrho+\varkappa)\mathfrak{T}(\bar{\Upsilon})+\zeta\mathfrak{X}(\bar{\Upsilon}))d\bar{\Upsilon},$$
$$c_2=c_1=-\frac{\zeta}{\pounds}(\varkappa^{\ell}\mathcal{F}(1+\varrho I^{\ell-1}\mathcal{F}(1)))+\frac{1}{\pounds}\int_0^1((\varrho+\varkappa)\mathfrak{T}(\bar{\Upsilon})+\zeta\mathfrak{X}(\bar{\Upsilon}))d\bar{\Upsilon}.$$

Now, on putting the values of c_1 and c_2 into `Eq.`(6) yields the result,

$$U(\wp)=I^{\ell}+-\frac{\zeta}{\pounds}(\varkappa^{\ell}\mathcal{F}(1+\varrho I^{\ell-1}\mathcal{F}(1)))+\frac{1}{\pounds}\int_0^1((\varrho+\varkappa)\mathfrak{T}(\wp)+\zeta\mathfrak{X}(\bar{\Upsilon}))d\bar{\Upsilon}$$
$$-\frac{\zeta}{\pounds}(\varkappa^{\ell}\mathcal{F}(1+\varrho I^{\ell-1}\mathcal{F}(1)))+\frac{1}{\pounds}\int_0^1((\varrho+\varkappa)\mathfrak{T}(\bar{\Upsilon})+\zeta\mathfrak{X}(\bar{\Upsilon}))d\bar{\Upsilon},$$

where

$$\pounds=\nu(\varrho+\varkappa)+\zeta\varkappa,$$

that can be written as

$$F(\wp)=\int_0^1\mathbb{G}(\wp,\bar{\Upsilon})\times\mathcal{F}(\bar{\Upsilon})d\bar{\Upsilon}+\mathfrak{T}(\wp),$$

where

$$\mathbb{G}(\wp,\bar{\Upsilon})=\frac{\wp^2(3-\wp)(1-\bar{\Upsilon})^{\ell-4}}{6\Gamma(\ell-2)}-\frac{\wp^2(1-\bar{\Upsilon})^{\ell-3}}{2\Gamma(\ell-1)}+\begin{cases}\dfrac{(\wp-\bar{\Upsilon})^{\ell-1}}{\Gamma(\ell)}, & \text{if } 0\leq\bar{\Upsilon}\leq\wp\leq 1\\ 0, & \text{if } 0\leq\wp\leq\bar{\Upsilon}\leq 1\end{cases}$$

and

$$\mathfrak{T}(\wp)=\frac{\varrho+\varkappa(1-\wp)}{\pounds}\int_0^1\mathfrak{T}(\wp,\bar{\Upsilon})d\bar{\Upsilon}+\frac{\zeta(1+\wp)}{\pounds}\int_0^1\mathfrak{X}(\wp,\bar{\Upsilon})d\bar{\Upsilon},\quad\text{and}\quad \pounds=\nu(\varrho+\varkappa)+\zeta\varkappa.$$

□

Let us now define the space $\mathscr{M} = C(\mathcal{J}, \mathbb{R})$, be a continuous operator from $\mathcal{J}$ to $\mathbb{R}$, equipped, with the supremum norm $\|\mathscr{M}\| = \{\sup|\digamma(\wp)| : \wp \in [0,1]\}$.
It is clear that $(\mathscr{M}, \|.\|_\digamma)$ is a Banach space.
Now, defining the operator $\mathbb{W} : \mathscr{M} \to \mathscr{M}$ as,

$$\begin{aligned} \mathbb{W}(\digamma)(\wp) = & \int_0^1 \mathbb{G}(\wp, \bar{\Upsilon})\mathfrak{S}(\bar{\Upsilon}, \digamma(\bar{\Upsilon}))d\bar{\Upsilon} + \frac{\varrho + \varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1 \mathfrak{T}(\bar{\Upsilon}, \digamma(\bar{\Upsilon}))d\bar{\Upsilon} \\ & + \frac{(\zeta+\nu\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1 \mathfrak{X}(\bar{\Upsilon}, \digamma(\bar{\Upsilon}))d\bar{\Upsilon}. \end{aligned} \tag{7}$$

Observe that the operator $\mathbb{W}$ is the solution of `Eq.` (1).
Using the theorem of the BCP and Schaefer's FPT, considering the EU of the proposed nonlinear Caputo FDEs, we determine two different types of results.

THEOREM 3.2. *Suppose that the positive constants $k, k_1, k_2 > 0$ exist such that, for all $\digamma_1, \digamma_2 \in \mathbb{R}$ and $\wp \in [0,1]$,*

(A_1) $|\mathfrak{S}(\wp, \digamma_1) - \mathfrak{S}(\wp, \digamma_2)| \le k|\digamma_1 - \digamma_2|$,
(A_2) $|\mathfrak{T}(\wp, \digamma_1) - \mathfrak{T}(\wp, \digamma_2)| \le k_1|\digamma_1 - \digamma_2|$,
(A_3) $|\mathfrak{X}(\wp, \digamma_1) - \mathfrak{X}(\wp, \digamma_2)| \le k_2|\digamma_1 - \digamma_2|$,

with

$$\mu = k\left\{\frac{1}{2\Gamma(\ell-1)} + \frac{1}{\Gamma(\ell)} + \frac{1}{\Gamma(\ell+1)}\right\} + \frac{k_1(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa} + \frac{k_2(\zeta+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa} < 1,$$

then Eq. (1)*, defined on* $[0,1]$*, has a unique solution.*

Proof. The operator $\mathbb{W}$, will be demonstrated to be a contraction.

Let $\digamma_1, \digamma_2 \in \mathscr{M}$, then $\forall \wp \in [0,1]$, we have

$$\begin{aligned} & |\mathbb{W}(\wp, \digamma_1) - \mathbb{W}(\wp, \digamma_2)| \\ & \le \int_0^1 |\mathbb{G}(\wp, \bar{\Upsilon})| \times |\mathfrak{S}(\bar{\Upsilon}, \digamma_1(\bar{\Upsilon})) - \mathfrak{S}(\bar{\Upsilon}, \digamma_2(\bar{\Upsilon}))|d\bar{\Upsilon} \\ & \quad + \frac{\varrho + \varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1 |\mathfrak{T}(\bar{\Upsilon}, \digamma_1(\bar{\Upsilon})) - \mathfrak{T}(\bar{\Upsilon}, \digamma_2(\bar{\Upsilon}))|d\bar{\Upsilon} \\ & \quad + \frac{\zeta+\nu\wp}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1 |\mathfrak{X}(\wp, \digamma_1(\bar{\Upsilon})) - \mathfrak{X}(\bar{\Upsilon}, \digamma_2(\bar{\Upsilon}))|d\bar{\Upsilon} \\ & \le k\|\digamma_1 - \digamma_2\| \int_0^1 |\mathbb{G}(\wp, \bar{\Upsilon})|d\bar{\Upsilon} + \frac{\varrho + \varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}k_1\|\digamma_1 - \digamma_2\| \\ & \quad + \frac{\zeta+\nu\wp}{\nu(\varrho+\varkappa)+\zeta\varkappa}k_2\|\digamma_1 - \digamma_2\| \end{aligned}$$

$$
\begin{aligned}
&\leq k\|\digamma_1-\digamma_2\|\Big[\Big|\frac{\wp^2(3-\wp)}{6\Gamma(\ell-2)}\Big|\int_0^1(1-\wp)^{\ell-3}d\bar{\Upsilon}+\Big|\frac{\wp^2}{2\Gamma(\ell-1)}\Big|\int_0^1(1-\bar{\Upsilon})^{\ell-2}d\bar{\Upsilon}\Big]\\
&\quad+\frac{k\|\digamma_1-\digamma_2\|}{\Gamma(\ell)}\int_0^{\wp}(\wp-\bar{\Upsilon})^{\ell-1}d\bar{\Upsilon}+\frac{\varrho+\varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}k_1\|\digamma_1-\digamma_2\|\\
&\quad+\frac{\zeta+\nu\wp}{\nu(\varrho+\varkappa)+\zeta\varkappa}k_2\|\digamma_1-\digamma_2\|\\
&=k\|\digamma_1-\digamma_2\|\Big[\Big|\frac{\wp^2(3-\wp)}{6\Gamma(\ell-1)}\Big|+\Big|\frac{\wp^2}{2\Gamma(\ell)}\Big|\Big]+\frac{k\|\digamma_1-\digamma_2\|}{\Gamma(\ell+1)}\wp^{\ell}\\
&\quad+\frac{\varrho+\varkappa(1-\wp)}{\alpha(\varrho+\varkappa)+\zeta\varkappa}k_1\|\digamma_1-\digamma_2\|+\frac{\zeta+\nu\wp}{\nu(\varrho+\varkappa)+\zeta\varkappa}k_2\|\digamma_1-\digamma_2\|\\
&\leq\|\digamma_1-\digamma_2\|\left[k\left\{\frac{1}{2\Gamma(\ell-1)}+\frac{1}{\Gamma(\ell)}+\frac{1}{\Gamma(\ell+1)}\right\}+k_1\left\{\frac{(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\}\right.\\
&\quad\left.+k_2\left\{\frac{(\zeta+\varkappa)}{\alpha(\varrho+\varkappa)+\zeta\varkappa}\right\}\right],
\end{aligned}
$$

therefore

$$
\begin{aligned}
\mu=k&\left\{\frac{1}{2\Gamma(\ell-1)}+\frac{1}{\Gamma(\ell)}+\frac{1}{\Gamma(\ell+1)}\right\}+k_1\left\{\frac{(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\}\\
&+k_2\left\{\frac{(\zeta+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\},
\end{aligned}
$$

as $\mu<1$, therefore, operator $\mathbb{W}$ is a contraction. As a result, operator $\mathbb{W}$ must have a unique fixed point based on the Banach fixed point theorem, i.e., `Eq.` (1) has a unique solution. □

The fixed point theorem of Schaefer's is used to demonstrate the result that follows:

Theorem 3.3. *Assume that*

(A_4) $\exists$ *a constant* $\chi>0$ *such that* $|\mathfrak{S}(\wp,\digamma)|\leq\chi\ \forall\wp\in[0,1]$, *and* $\digamma\in\mathbb{R}$,

(A_5) $\exists$ *a constant* $\Lambda>0$ *such that* $|\mathfrak{T}(\wp,\digamma)|\leq\Lambda\ \forall\digamma\in\mathscr{M}$,

(A_6) $\exists$ *a constant* $\eta>0$ *such that* $|\mathfrak{X}(\wp,\digamma)|\leq\eta\ \forall\digamma\in\mathscr{M}$,

then `Eq.` *(1) defined on* $[0,1]$, *has at least one solution.*

Proof. We show this result through the use fixed point theorem of Schaefer's.

Step 1: $\mathbb{W}$ is a continuous operator.

Consider the sequence $\{\digamma_n\}$ in $\mathscr{M}$ tending to $\digamma$, i.e., $\digamma_n \to \digamma$, then, $\forall \wp \in [0,1]$.

$$\begin{aligned}
|\mathbb{W}(\wp,\digamma_n)-\mathbb{W}(\wp,\digamma)| \leq & \int_0^1 |\mathbb{G}(\wp,\bar{\Upsilon})| \times |\mathfrak{S}(\wp,\digamma_n(\bar{\Upsilon}))-\mathfrak{S}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))|d\bar{\Upsilon} \\
& + \frac{\varrho+\varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1 |\mathfrak{T}(\bar{\Upsilon},\digamma_n(\bar{\Upsilon}))-\mathfrak{T}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))|d\bar{\Upsilon} \\
& + \frac{\zeta+\nu\wp}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1 |\mathfrak{X}(\bar{\Upsilon},\digamma_n(\bar{\Upsilon}))-\mathfrak{X}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))|d\bar{\Upsilon} \\
\leq & \int_0^1 |\mathbb{G}(\wp,\bar{\Upsilon})| \times \sup_{\bar{\Upsilon}\in[0,1]}|\mathfrak{S}(\bar{\Upsilon},\digamma_n(\bar{\Upsilon}))-\mathfrak{S}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))|d\bar{\Upsilon} \\
& + \frac{\varrho+\varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1 \sup_{\bar{\Upsilon}\in[0,1]}|\mathfrak{T}(\bar{\Upsilon},\digamma_n(\bar{\Upsilon}))-\mathfrak{T}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))|d\bar{\Upsilon} \\
& + \frac{\zeta+\nu\wp}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1 \sup_{\bar{\Upsilon}\in[0,1]}|\mathfrak{X}(\bar{\Upsilon},\digamma_n(\bar{\Upsilon}))-\mathfrak{X}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))|d\bar{\Upsilon}.
\end{aligned}$$

Since $\mathfrak{S}$, $\mathfrak{T}$ and $\mathfrak{X}$ are continuous functions, $\mathbb{W}$ is also continuous.

Step 2: Under continuous map $\mathbb{W}$, the using of bounded set $\mathscr{M}$ is mapped into bounded sets of $\mathscr{M}$.

Now, for $\digamma \in B_\epsilon$ and for each $\digamma \in [0,1]$,

$$\begin{aligned}
|\mathbb{W}(\digamma)(\wp)| \leq & \int_0^1 |\mathbb{G}(\wp,\bar{\Upsilon})| \times |\mathfrak{S}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))|d\bar{\Upsilon} + \Big|\frac{\varrho+\varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\Big|\int_0^1 |\mathfrak{T}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))|d\bar{\Upsilon} \\
& + \Big|\frac{\zeta+\nu\wp}{\nu(\varrho+\varkappa)+\zeta\varkappa}\Big|\int_0^1 |\mathfrak{X}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))|d\bar{\Upsilon} \\
\leq & \Lambda\int_0^1 |\mathbb{G}(\wp,\bar{\Upsilon})|d\bar{\Upsilon} + \varpi\Big|\frac{\varrho+\varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\Big| \\
& + \eta\Big|\frac{\zeta+\nu\wp}{\alpha(\varrho+\varkappa)+\zeta\varkappa}\Big|.
\end{aligned}$$

Thus

$$\|\mathbb{W}(\digamma)\| \leq \Lambda\Big[\frac{1}{2\Gamma(\ell-1)}+\frac{1}{\Gamma(\ell)}+\frac{1}{\Gamma(\ell+1)}\Big] + \frac{\varpi(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa} + \frac{\eta(\zeta+\nu)}{\nu(\varrho+\varkappa)+\zeta\varkappa} < 1,$$

i.e.,

$$\|\mathbb{W}(\digamma)\| < \infty.$$

Step 3: $\mathbb{W}(B_\epsilon)$ is equi-continuous.
Let $\digamma \in \mathbb{W}(B_\epsilon)$ and $\wp_1, \wp_2$ with $\wp_1 < \wp_2$, then

$$\begin{aligned}
&|\mathbb{W}(\digamma)\wp_1 - \mathbb{W}(\digamma)\wp_2| \\
&\leq \Lambda \int_0^1 |\mathbb{G}(\wp_1, \bar{\Upsilon}) - \mathbb{G}(\wp_2, \bar{\Upsilon})|\, d\bar{\Upsilon} + \varpi \left| \frac{\varrho + \varkappa(1-\wp_1)}{\pounds} - \frac{\varrho + \varkappa(1-\wp_2)}{\pounds} \right| \\
&\quad + \eta \left| \frac{\zeta + \nu\wp_1}{\pounds} - \frac{\zeta + \nu\wp_2}{\pounds} \right| \\
&\leq \Lambda \frac{|3(\wp_1^2 - \wp_2^2) - (\wp_1^3 - \wp_2^3)|}{\Gamma(\ell-2)} \int_0^1 (1-\bar{\Upsilon})^{\ell-3}\, d\bar{\Upsilon} \\
&\quad + \Lambda \frac{|\wp_1^2 - \wp_2^2|}{2\Gamma(\ell-1)} \int_0^1 (1-\bar{\Upsilon})^{\ell-2} d\bar{\Upsilon} \\
&\quad + \frac{\Lambda}{\Gamma(\ell)} \left[\int_0^{\wp_1} \left((\wp_2 - \bar{\Upsilon})^{\ell-1} - (\wp_1 - \bar{\Upsilon})^{\ell-1} \right) d\bar{\Upsilon} + \int_{\wp_1}^{\wp_2} (\wp_2 - \bar{\Upsilon})^{\ell-1} d\bar{\Upsilon} \right] \\
&\quad + \varpi \frac{\varkappa|\wp_1 - \wp_2|}{\pounds} + \eta \frac{\nu|\wp_1 - \wp_2|}{\pounds} \\
&\leq \Lambda \frac{|3(\wp_1^2 - \wp_2^2) - (\wp_1^3 - \wp_2^3)|}{\Gamma(\ell-1)} + \Lambda \frac{|\wp_1^2 - \wp_2^2|}{2\Gamma(\ell)} \\
&\quad + \frac{\Lambda}{\Gamma(\ell+1)} \left[(\wp_1 - \wp_2)^\ell + (\wp_1^\ell - \wp_2^\ell) \right] \\
&\quad + \Lambda \frac{(\wp_1 - \wp_2)^\ell}{\Gamma(\ell+1)} + \varpi \frac{\varkappa|\wp_1 - \wp_2|}{\pounds} + \eta \frac{\nu|\wp_1 - \wp_2|}{\pounds},
\end{aligned}$$

when $\wp_1 \to \wp_2$, then the RHS of the above inequality tending to zero.
$\mathbb{W}$ is completely continuous operator resulting by combining Steps 1 to 3, and the consequence of Arzela-Ascoli theorem.
Step 4: Consider $\mathfrak{N} = \{\digamma \in \mathscr{M} : \digamma = \vartheta\mathbb{W}(\digamma) \text{ for some, } 0 < \vartheta < 1\}$.
At present, we have to show that $\mathfrak{N}$ is bounded, for this we proceed as:
For some $0 < \vartheta < 1$, let $\digamma \in \vartheta \implies \digamma(\wp) = \vartheta\mathbb{W}(\digamma)(\wp)$.
Now

$$\begin{aligned}
|\digamma(\wp)| &= |\vartheta\mathbb{W}(\digamma)(\wp)| \\
&\leq \int_0^1 |\mathbb{G}(\wp, \bar{\Upsilon})| \times |\mathfrak{S}(\bar{\Upsilon}, \digamma(\bar{\Upsilon}))| d\bar{\Upsilon} \\
&\quad + \Big| \frac{\varrho + \varkappa(1-\wp)}{\nu(\varrho + \varkappa) + \zeta\varkappa} \Big| \int_0^1 |\mathfrak{T}(\bar{\Upsilon}, \digamma(\bar{\Upsilon}))| d\bar{\Upsilon} \\
&\quad + \Big| \frac{\zeta + \nu\wp}{\nu(\varrho + \varkappa) + \zeta\varkappa} \Big| \int_0^1 |\mathfrak{X}(\bar{\Upsilon}, \digamma(\bar{\Upsilon}))| ds, \\
&\leq \Lambda \int_0^1 |\mathbb{G}(\wp, \bar{\Upsilon})| d\bar{\Upsilon} + \varpi \Big| \frac{\varrho + \varkappa(1-\wp)}{\nu(\varrho + \varkappa) + \zeta\varkappa} \Big| \\
&\quad + \eta \Big| \frac{\zeta + \nu\wp}{\nu(\varrho + \varkappa) + \zeta\varkappa} \Big|, \\
&\leq \Lambda \int_0^1 |\mathfrak{S}(\wp, \bar{\Upsilon})| d\bar{\Upsilon} + \varpi \frac{|\varrho + \varkappa(1-\wp)|}{\pounds} + \eta \frac{|\zeta + \nu\wp|}{\pounds}
\end{aligned}$$

$$\leq \Lambda\Big|\frac{\wp^2(3-\wp)}{6\Gamma(\ell-2)}\Big|\int_0^1(1-\bar{\Upsilon})^{\ell-3}d\bar{\Upsilon}+$$
$$+\Lambda\Big|\frac{\wp^2}{6\Gamma(\ell-1)}\Big|\int_0^1(1-\bar{\Upsilon})^{\ell-2}d\bar{\Upsilon}$$
$$+\Lambda\int_0^{\wp}\frac{(\wp-\bar{\Upsilon})^{\ell-1}}{\Gamma(\ell)}ds+\varpi\frac{(\varrho+\varkappa)}{\pounds}+\eta\frac{(\zeta+\alpha)}{\pounds}.$$

Thus,

$$\|\mathcal{F}\|\leq\Lambda\Big[\frac{1}{2\Gamma(\ell-1)}+\frac{1}{\Gamma(\ell)}+\frac{1}{\Gamma(\ell+1)}\Big]+\varpi\frac{(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}+\eta\frac{(\zeta+\nu)}{\nu(\varrho+\varkappa)+\zeta\varkappa}<\infty,$$

and $\mathfrak{N}$ is a bounded set which proved the result.
Implies that operator $\mathbb{W}$ must contain at least one fixed point, which is the solution of the `Eq.` (1), by Schaefer's fixed point theorem. □

3.2. Stability analysis. In the subsection that follows, we demonstrate UHS of `Eq.` (1) on the interval $[0,1]$.

Remark 3.4. The function $\mathfrak{H}$ is a solution to `Eq.`(4) if and only if there exists a continuous function $\mathfrak{H}\in C([0,1],\mathbb{R})$ that satisfies the following relation

$$|\mathfrak{H}(\wp,\mathscr{G}(\wp))|\leq\epsilon,\quad \wp\in[0,1],$$

and

$$^{c}D^{\ell}\mathscr{G}(\wp)=\mathfrak{S}(\wp,\mathscr{G}(\wp))+\mathfrak{H}(\wp,\mathscr{G}(\wp)),\quad \wp\in[0,1].$$

Theorem 3.5. *Assume that $(A_1)-(A_3)$ hold, then the BVP* `Eq.`(1) *be the stability of Ulam-Hyers with respect to ϵ.*

Proof. For $\epsilon>0$, and every solution in $C([0,1],\mathbb{R})$ of the inequality,

$$|\ ^{c}D^{\ell}\mathscr{G}(\wp)-\mathfrak{S}(\wp,\mathscr{G}(\wp))|\leq\epsilon,\ \wp\in[0,1],$$

Let $\mathcal{F}\in C([0,1],\mathbb{R})$ be unique solution of BVP `Eq.`(1), then $\mathcal{F}(\wp)$ is given by

$$\mathcal{F}(\wp)=\int_0^1\mathbb{G}(\wp,\bar{\Upsilon})\mathcal{F}(\bar{\Upsilon})d\bar{\Upsilon}+\mathfrak{T}(\wp),$$

where

$$\mathbb{G}(\wp,\bar{\Upsilon})=\frac{\wp^2(3-\wp)(1-\bar{\Upsilon})^{\ell-4}}{6\Gamma(\ell-2)}-\frac{\wp^2(1-\bar{\Upsilon})^{\ell-3}}{2\Gamma(\ell-1)}$$
$$+\begin{cases}\dfrac{(\wp-\bar{\Upsilon})^{\ell-1}}{\Gamma(\ell)}, & if\ 0\leq\bar{\Upsilon}\leq\wp\leq1\\ 0, & if\ 0\leq\wp\leq\bar{\Upsilon}\leq1,\end{cases}$$

$$\mathfrak{T}(\wp)=\frac{\varrho+\varkappa(1-\wp)}{\pounds}\int_0^1\mathfrak{T}(\bar{\Upsilon})d\bar{\Upsilon}+\frac{\zeta(1+\wp)}{\pounds}\int_0^1\mathfrak{X}(\bar{\Upsilon})d\bar{\Upsilon},$$

$$\begin{aligned}
&\digamma(\wp)\\
&=\frac{\wp^2(3-\wp)}{6\Gamma(\ell-2)}\int_0^1(1-\bar{\Upsilon})^{\ell-3}\times\mathfrak{S}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))d\bar{\Upsilon}+\frac{\wp^2}{2\Gamma(\ell-1)}\int_0^1(1-\bar{\Upsilon})^{\ell-2}\times\mathfrak{S}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))d\bar{\Upsilon}\\
&\quad+\frac{1}{\Gamma(\ell)}\int_0^{\wp}(\wp-\bar{\Upsilon})^{\ell-1}\times\mathfrak{S}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))d\bar{\Upsilon}+\frac{\varrho+\varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1\mathfrak{T}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))d\bar{\Upsilon}\\
&\quad+\frac{\zeta+\nu\wp}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1\mathfrak{X}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))d\bar{\Upsilon},
\end{aligned}$$

then, we have

$$\begin{aligned}
&\big|\mathscr{G}(\wp)-\digamma(\wp)\big|\\
&\le\Big|\mathscr{G}(\wp)-\frac{\wp^2(3-\wp)}{6\Gamma(\ell-2)}\int_0^1(1-\bar{\Upsilon})^{\ell-3}\times\mathfrak{S}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))d\bar{\Upsilon}\\
&\quad+\frac{\wp^2}{2\Gamma(\ell-1)}\int_0^1(1-\bar{\Upsilon})^{\ell-2}\times\mathfrak{S}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))d\bar{\Upsilon}\\
&\quad+\frac{1}{\Gamma(\ell)}\int_0^{\wp}(\wp-\bar{\Upsilon})^{\ell-1}\times\mathfrak{S}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))d\bar{\Upsilon}+\frac{\varrho+\varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1\mathfrak{T}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))d\bar{\Upsilon}\\
&\quad+\frac{\zeta+\nu\wp}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1\mathfrak{X}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))d\bar{\Upsilon}\Big|\\
&\quad+\Big|\frac{\wp^2(3-\wp)}{6\Gamma(\ell-2)}\int_0^1(1-\bar{\Upsilon})^{\ell-3}\times(\mathfrak{S}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))-\mathfrak{S}(\bar{\Upsilon},\digamma(\bar{\Upsilon})))d\bar{\Upsilon}\\
&\quad+\frac{\wp^2}{2\Gamma(\ell-1)}\int_0^1(1-\bar{\Upsilon})^{\ell-2}\times(\mathfrak{S}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))-\mathfrak{S}(\bar{\Upsilon},\digamma(\bar{\Upsilon})))d\bar{\Upsilon}\\
&\quad+\frac{1}{\Gamma(\ell)}\int_0^{\wp}(\wp-\bar{\Upsilon})^{\ell-1}\times(\mathfrak{S}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))-\mathfrak{S}(\bar{\Upsilon},\digamma(\bar{\Upsilon})))d\bar{\Upsilon}\\
&\quad+\frac{\varrho+\varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1(\mathfrak{T}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))-\mathfrak{T}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))d\bar{\Upsilon}\\
&\quad+\frac{\zeta+\nu\wp}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1(\mathfrak{X}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))-\mathfrak{X}(\bar{\Upsilon},\digamma(\bar{\Upsilon}))d\bar{\Upsilon}\Big|\\
&\le\Big|\frac{\wp^2(3-\wp)}{6\Gamma(\ell-2)}\Big|\int_0^1(1-\bar{\Upsilon})^{\ell-3}\times\big|\mathscr{G}(\bar{\Upsilon})-\digamma(\bar{\Upsilon})\big|d\bar{\Upsilon}+\epsilon\\
&\quad+\Big|\frac{\wp^2}{2\Gamma(\ell-1)}\Big|\int_0^1(1-\bar{\Upsilon})^{\ell-2}\times\big|\mathscr{G}(\bar{\Upsilon})-\digamma(\bar{\Upsilon})\big|d\bar{\Upsilon}\\
&\quad+\Big|\frac{1}{\Gamma(\ell)}\Big|\int_0^{\wp}(\wp-\bar{\Upsilon})^{\ell-1}\times\big|\mathscr{G}(\bar{\Upsilon})-\digamma(\bar{\Upsilon})\big|d\bar{\Upsilon}\\
&\quad+\Big|\frac{\varrho+\varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\Big|\int_0^1\big|\mathscr{G}(\bar{\Upsilon}))-\digamma(\bar{\Upsilon})\big|d\bar{\Upsilon}\\
&\quad+\Big|\frac{\zeta+\nu\wp}{\nu(\varrho+\varkappa)+\zeta\varkappa}\Big|\int_0^1\big|\mathscr{G}(\bar{\Upsilon}))-\digamma(\bar{\Upsilon})\big|d\bar{\Upsilon},
\end{aligned}$$

where

$$\Big|\mathscr{G}(\wp) - \frac{\wp^2(3-\wp)}{6\Gamma(\ell-2)}\int_0^1 (1-\bar{\Upsilon})^{\ell-3}\times \mathfrak{S}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))d\bar{\Upsilon} + \frac{\wp^2}{2\Gamma(\ell-1)}\int_0^1 (1-\bar{\Upsilon})^{\ell-2}\times \mathfrak{S}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))d\bar{\Upsilon}$$

$$+\frac{1}{\Gamma(\ell)}\int_0^{\wp} (\wp-\bar{\Upsilon})^{\ell-1}\times \mathfrak{S}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))d\bar{\Upsilon} + \frac{\varrho+\varkappa(1-\wp)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1 \mathfrak{T}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))d\bar{\Upsilon}$$

$$+\frac{\zeta+\nu\wp}{\nu(\varrho+\varkappa)+\zeta\varkappa}\int_0^1 \mathfrak{X}(\bar{\Upsilon},\mathscr{G}(\bar{\Upsilon}))d\bar{\Upsilon}\Big| \le \epsilon,$$

and by the using $(A_1)-(A_3)$, we obtain

$$\big|\mathscr{G}(\wp) - F(\wp)\big| \le \epsilon + \left\{\frac{1}{2\Gamma(\ell-1)} + \frac{1}{\Gamma(\ell)} + \frac{(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa} + \frac{(\zeta+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\}\big|\mathscr{G}(\wp) - F(\wp)\big|$$

$$+\frac{1}{\Gamma(\ell)}\int_0^{\wp}(\wp-\bar{\Upsilon})^{\ell-1}\times\big|\mathscr{G}(\bar{\Upsilon}) - F(\bar{\Upsilon})\big|d\bar{\Upsilon}.$$

Let $u(\wp) = \big|\mathscr{G}(\wp) - F(\wp)\big|$, then one has

$$\big|\mathscr{G}(\wp) - F(\wp)\big| \le \epsilon + \left\{\frac{1}{2\Gamma(\ell-1)} + \frac{1}{\Gamma(\ell)} + \frac{(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa} + \frac{(\zeta+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\}\big|\mathscr{G}(\wp) - F(\wp)\big|$$

$$+\frac{1}{\Gamma(\ell)}\int_0^{\wp}(\wp-\bar{\Upsilon})^{\ell-1}F(\bar{\Upsilon})d\bar{\Upsilon}.$$

By using Lemma 2.16, it becomes

$$\big|\mathscr{G}(\wp) - F(\wp)\big| \le \epsilon\left\{\frac{1}{2\Gamma(\ell-1)} + \frac{1}{\Gamma(\ell)} + \frac{(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa} + \frac{(\zeta+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\}e^{\frac{1}{\Gamma(\ell+1)}},$$

if we put

$$C_w := \left\{\frac{1}{2\Gamma(\ell-1)} + \frac{1}{\Gamma(\ell)} + \frac{(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa} + \frac{(\zeta+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\}e^{\frac{1}{\Gamma(\ell+1)}}.$$

This implies that

$$\big|\mathscr{G}(\wp) - F(\wp)\big| \le C_w\epsilon,$$

where

$$C_w = \left\{\frac{1}{2\Gamma(\ell-1)} + \frac{1}{\Gamma(\ell)} + \frac{(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa} + \frac{(\zeta+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\}e^{\frac{1}{\Gamma(\ell+1)}} > 0,$$

then $\big|\mathscr{G}(\wp) - F(\wp)\big| \le C_w\epsilon$. Thus, the problem `Eq.` (1) is Ulam-Hyers stability. □

4. Applications

This section addresses the explanation and usefulness of the main results by presenting the three main examples.

Example 4.1. Let

$$(8)\qquad \begin{cases} {}^cD^{7/4}\, F(\wp) = \dfrac{e^{-5\wp}F(\wp)}{(39\sqrt{\pi}+e^{-5\wp})(1+z(\wp))}, \quad \wp\in[0,1],\\ F(0) - 2F'(0) = \int_0^1 \dfrac{1}{23}F(\bar{\Upsilon})d\bar{\Upsilon}, \quad 2F(1) + 2F'(1) = \int_0^1 \dfrac{1}{34}F(\bar{\Upsilon})d\bar{\Upsilon}, \end{cases}$$

where,

$$\ell=\frac{7}{4},\quad \mathfrak{S}(\wp,F)=\frac{e^{-5\wp}F(\wp)}{(39\sqrt{\pi}+e^{-5\wp})(1+F(\wp))},\ \mathfrak{T}(\wp,F)=\frac{1}{23}F(\bar{\Upsilon}),$$
$$\mathfrak{X}(\wp,F)=\frac{1}{34}F(\bar{\Upsilon}),\ \text{for } \wp\in[0,1],$$
$$\text{and}\quad \nu=1,\quad \zeta=2,\quad \varkappa=\varrho=2.$$

Now, for F_1, $F_2\in\mathbb{R}$ and $\wp\in[0,1]$,

$$\begin{aligned}|\mathfrak{S}(\wp,F_1)-\mathfrak{S}(\wp,F_2)| &= \Big|\frac{e^{-5\wp}F_1(\wp)}{(39\sqrt{\pi}+e^{-5\wp})(1+F_1(\wp))}-\frac{e^{-5\wp}F_2(\wp)}{(39\sqrt{\pi}+e^{-5\wp})(1+F_2(\wp))}\Big|\\ &\leq \frac{e^{-5\wp}}{(39\sqrt{\pi}+e^{-5\wp})}\Big|\frac{F_1-F_2}{(1+F_1)(1+F_2)}\Big|\\ &\leq \frac{1}{(39\sqrt{\pi}+1)}\big|F_1-F_2\big|\\ |\mathfrak{T}(\wp,F_1)-\mathfrak{T}(\wp,F_2)| &\leq \frac{1}{23}|F_1-F_2|,\\ |\mathfrak{X}(\wp,F_1)-\mathfrak{X}(\wp,F_2)| &\leq \frac{1}{34}|F_1-F_2|,\end{aligned}$$

therefore, (A_1), (A_2) and (A_3) satisfied with $k=\frac{1}{39\sqrt{\pi}}$, $k_1=\frac{1}{23}$, and $k_2=\frac{1}{34}$, further

$$\mu=\frac{1}{39\sqrt{\pi}}\left\{\frac{1}{2\Gamma(\ell-1)}+\frac{1}{\Gamma(\ell)}+\frac{1}{\Gamma(\ell+1)}\right\}+\frac{1}{23}\left\{\frac{(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\}+\frac{1}{34}\left\{\frac{(\zeta+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\}$$
$$\mu=\frac{1}{39\sqrt{\pi}}\left\{\frac{1}{2\Gamma\left(\frac{3}{4}\right)}+\frac{1}{\Gamma\left(\frac{7}{4}\right)}+\frac{1}{\Gamma\left(\frac{11}{4}\right)}\right\}+\frac{1}{23}\left\{\frac{(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\}+\frac{1}{34}\left\{\frac{(\zeta+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\}$$
$$\mu=0.03359+0.02173+0.01470$$
$$\mu=0.07002<1.$$

It follows that BVP (4.1) must have a unique solution, as shown by Theorem 3.2. Furthermore, we can observe that $C_w\approx 2.6481287032>0$. So by Theorem 3.5, the considered BVP (4.1) is UHS.

Example 4.2. Let

(9)
$$\begin{cases} {}^{c}D^{3/2}\,F(\wp)=\dfrac{e^{-2\wp\cos^2 F(\wp)}\wp F(\wp)}{31(1+\wp)(1+z(\wp))},\quad \wp\in[0,1],\\ F(0)-F'(0)=\int_0^1\dfrac{|F(\bar{\Upsilon})|e^{-\bar{\Upsilon}}}{13(1+|F(\bar{\Upsilon})|)}d\bar{\Upsilon},\quad F(1)+F'(1)=\int_0^1\dfrac{|F(\bar{\Upsilon})|\cos(\bar{\Upsilon})}{29}d\bar{\Upsilon},\end{cases}$$

where,

$$\ell = \frac{3}{2}, \quad \mathfrak{S}(\wp, F) = \frac{e^{-2\wp\cos^2 F(\wp)}\wp F(\wp)}{31(1+\wp)(1+F(\wp))}, \quad \mathfrak{T}(F) = \frac{|F(\bar{\Upsilon})|e^{-\bar{\Upsilon}}}{13(1+|F(\bar{\Upsilon})|)}$$

$$\mathfrak{X}(F) = \frac{|F(\bar{\Upsilon})|\cos(\bar{\Upsilon})}{29} \quad \text{for} \quad \wp \in [0,1],$$

$$\textit{and} \quad \nu = \zeta = \varkappa = \varrho = 1.$$

Now, for F_1, $F_2 \in \mathbb{R}$ and $\wp \in [0,1]$,

$$\begin{aligned}
|\mathfrak{S}(\wp, F_1) - \mathfrak{S}(\wp, F_2)| &= \Big|\frac{e^{-2\wp\cos^2 F_1(\wp)}\wp F_1(\wp)}{31(1+\wp)(1+F_1(\wp))} - \frac{e^{-2\wp\cos^2 F_2(\wp)}\wp F_2(\wp)}{31(1+\wp)(1+F_2(\wp))}\Big| \\
&\leq \frac{e^{-2\wp}\wp}{31(1+\wp)}\Big|\frac{F_1(\wp)}{1+F_1(\wp)} - \frac{F_2(\wp)}{1+F_2(\wp)}\Big| \\
&\leq \frac{1}{31}\Big|\frac{F_1(\wp)}{1+F_1(\wp)} - \frac{F_2(\wp)}{1+F_2(\wp)}\Big| \\
&\leq \frac{1}{31}\big|F_1(\wp) - F_2(\wp)\big|. \\
|\mathfrak{T}(\wp, F_1) - \mathfrak{T}(\wp, F_2)| &= \frac{e^{-\wp}}{13}\Big|\frac{|F_1(\wp)|}{(1+|F_1(\wp)|)} - \frac{|F_2(\wp)|}{(1+|F_2(\wp)|)}\Big| \\
&\leq \frac{e^{-\wp}}{13}\Big|\frac{F_1(\wp)}{1+F_1(\wp)} - \frac{F_2(\wp)}{1+F_2(\wp)}\Big| \\
&\leq \frac{e^{-\wp}}{13}\Big|\frac{F_1(\wp) - F_2(\wp)}{(1+F_1(\wp))(1+F_2(\wp))}\Big| \\
&\leq \frac{1}{13}\big|F_1(\wp) - F_2(\wp)\big|,
\end{aligned}$$

and

$$\begin{aligned}
|\mathfrak{X}(\wp, F_1) - \mathfrak{X}(\wp, F_2)| &= \frac{\cos\wp}{29}\big||F_1(\wp)| - |F_2(\wp)|\big| \\
&\leq \frac{1}{29}|F_1 - F_2|.
\end{aligned}$$

Therefore, (A_1), (A_2) and (A_3) satisfied with $k = \frac{1}{31}$, $k_1 = \frac{1}{13}$, and $k_2 = \frac{1}{29}$, further,

$$\mu = \frac{1}{31}\left\{\frac{1}{2\Gamma(\ell-1)} + \frac{1}{\Gamma(\ell)} + \frac{1}{\Gamma(\ell+1)}\right\} + \frac{1}{13}\left\{\frac{(\varrho+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\} + \frac{1}{29}\left\{\frac{(\zeta+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\},$$

$$\mu = \frac{1}{31}\left\{\frac{1}{2\Gamma\left(\frac{1}{2}\right)} + \frac{1}{\Gamma\left(\frac{3}{2}\right)} + \frac{1}{\Gamma\left(\frac{5}{2}\right)}\right\} + \frac{1}{13}\left\{\frac{(\varrho+\varkappa)}{\alpha(\varrho+\varkappa)+\zeta\varkappa}\right\} + \frac{1}{29}\left\{\frac{(\zeta+\varkappa)}{\nu(\varrho+\varkappa)+\zeta\varkappa}\right\},$$

$\mu < 1$.

It follows that BVP (4.2) must have a unique solution, as shown by Theorem 3.2. Furthermore, we can observe that $C_w \approx 2.64275456 > 0$. So by Theorem 3.5, the considered BVP (4.2) is UHS.

EXAMPLE 4.3. Let

$$(10)\qquad \begin{cases} {}^{c}D^{7/4}\, F(\wp) = \dfrac{e^{-9\wp}}{37+\cos F(\wp)}, \quad \wp\in[0,1], \\ \int_0^1 \dfrac{F(\bar{\Upsilon})}{31(1+F(\bar{\Upsilon}))}d\bar{\Upsilon}, \quad F(1)+F'(1) = \int_0^1 \dfrac{\sin\bar{\Upsilon}}{39}d\bar{\Upsilon}, \end{cases}$$

where

$$\ell = \frac{7}{4}, \quad \mathfrak{S}(\wp, F) = \frac{e^{-9\wp}}{37+\cos F(\wp)}, \quad \mathfrak{T}(F) = \frac{F}{31(1+F)},$$
$$\mathfrak{X}(F) = \frac{\sin F}{39}, \quad \text{for} \quad \wp \in[0,1],$$
$$\text{and} \quad \nu = \zeta = \varkappa = \varrho = 1,$$

clearly

$$\begin{aligned} \big|\mathfrak{S}(\wp, F)\big| &= \Big|\frac{e^{-9\wp}}{37+\cos F(\wp)}\Big| \\ &\leq \frac{1}{36}, \\ \big|\mathfrak{T}(F)\big| &= \frac{F}{31(1+F)} \\ &\leq \frac{1}{31}, \\ \text{and} \quad \big|\mathfrak{X}(F)\big| &= \Big|\frac{\sin F}{39}\Big| \\ &\leq \frac{1}{39}, \end{aligned}$$

i.e., (A_1), (A_2) and (A_3) satisfied with $\kappa = \frac{1}{36}$, $\Lambda = \frac{1}{31}$ and $\eta = \frac{1}{39}$.

It follows that BVP (4.3) must have a unique solution, as shown by Theorem 3.2. Furthermore, we can observe that $C_w \approx 5.2688479473 > 0$. So by Theorem 3.5, the considered BVP (4.3) is UHS.

5. Conclusions and future work

This research provides a comprehensive examination of the EU of solutions for Caputo FDEs subject to IBCs. Through applying fundamental fixed point theorems, namely Schaefer's and BCP, we have demonstrated significant results, these findings not only guarantee the existence of solutions but also ensure their uniqueness. By precisely identifying the conditions in Schaefer's fixed point theorems applied, the existence of solutions that conform to the integral boundary conditions might be demonstrated by us. Moreover, we established UHS for fractional-order BVP using Grönwall inequality and verified the uniqueness of these solutions using the BCP. Lastly, three examples have been given to illustrate the key results.

In the future, employing a contractive iteration approach is expected to enhance the rate of convergence of successive approximations in numerical computations, providing better accuracy with fewer iterations. This type of problem may also be investigated numerically in future work, and in this sense, the present study offers several promising directions for further research.

References

[1] S. Abbas, M. Benchohra, N. Hamidi, and J. Henderson, *Caputo–Hadamard fractional differential equations in Banach spaces*, Fract. Calc. Appl. Anal. **21** (4) (2018), 1027–1045.
https://doi.org/10.1515/fca-2018-0056

[2] M. I. Abbas, *Existence and uniqueness of solution for a boundary value problem of fractional order involving two Caputo's fractional derivatives*, Adv. Differ. Equ. **2015** (1) (2015), 1–19.

[3] S. Agarwal and L. N. Mishra, *Attributes of residual neural networks for modeling fractional differential equations*, Heliyon **10** (19) (2024), e38332.
https://doi.org/10.1016/j.heliyon.2024.e38332

[4] A. Alsaedi, M. Alghanmi, B. Ahmad, and B. Alharbi, *Uniqueness of solutions for a ψ-Hilfer fractional integral boundary value problem with the p-Laplacian operator*, Demonstr. Math. **56** (1) (2023), 20220195.
https://doi.org/10.1515/dema-2022-0195

[5] M. Alam, A. Khan, and M. Asif, *Analysis of implicit system of fractional order via generalized boundary conditions*, Math. Methods Appl. Sci. **46** (9) (2023), 10554–10571.
https://doi.org/10.1002/mma.9139

[6] M. Alam, A. Zada, and T. Abdeljawad, *Stability analysis of an implicit fractional integro-differential equation via integral boundary conditions*, Alex. Eng. J. **87** (2024), 501–514.
https://doi.org/10.1016/j.aej.2023.12.055

[7] M. Ansari and L. N. Mishra, *Common solution to a coupled system of fractional differential equations and nonlinear integral equations via weakly altering distance functions and w-distance*, Adv. Stud.: Euro-Tbil. Math. J. **18** (2) (2025), 149–174.
https://doi.org/10.32513/asetmj/193220082518210

[8] A. Antony and L. N. Mishra, *The role of residual neural networks for advancing fractional differential equations*, Bol. Soc. Paran. Mat. **43** (2025), 1–24.

[9] T. M. Atanackovic, S. Pilipovic, B. Stankovic, and D. Zorica, *Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes*, John Wiley & Sons, Inc., London (2014).

[10] D. Baleanu, J. Alzabut, J. Jonnalagadda, Y. Adjabi, and M. Matar, *A coupled system of generalized Sturm–Liouville problems and Langevin fractional differential equations in the framework of nonlocal and nonsingular derivatives*, Adv. Differ. Equ. **2020** (1) (2020), 1–30.
https://doi.org/10.1186/s13662-020-02690-1

[11] H. Bazgir and B. Ghazanfari, *Existence of solutions for fractional integro-differential equations with non-local boundary conditions*, Math. Comput. Appl. **23** (3) (2018), 36.
https://doi.org/10.3390/mca23030036

[12] M. Benchohra and J. E. Lazreg, *Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative*, Stud. Univ. Babeş-Bolyai Math. **62** (1) (2017), 27–38.

[13] I. A. Bhat, L. N. Mishra, and V. N. Mishra, *Comparative analysis of nonlinear Urysohn functional integral equations via Nyström method*, Appl. Math. Comput. **494** (2025), 129287.
https://doi.org/10.1016/j.amc.2025.129287

[14] I. A. Bhat, L. N. Mishra, V. N. Mishra, M. Abdel-Aty, and M. Qasymeh, *A comprehensive analysis for weakly singular nonlinear functional Volterra integral equations using discretization techniques*, Alex. Eng. J. **104** (2024), 564–575.
https://doi.org/10.1016/j.aej.2024.08.017

[15] C. Cesarano, R. Garra, and F. Maltese, *Exact solutions for nonlinear PDEs via Hermite polynomials*, Int. J. Appl. Math. **37** (4) (2024), 421–426.
https://doi.org/10.12732/ijam.v37i4.2

[16] C. R. Chen, M. Bohner, and B. G. Jia, *Ulam–Hyers stability of Caputo fractional difference equations*, Math. Methods Appl. Sci. **42** (18) (2019), 7461–7470. https://doi.org/10.1002/mma.5869

[17] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin–Heidelberg (1985).

[18] T. H. Gronwall, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. Math. **20** (1919), 292–296.

[19] H. Khan, W. Chen, and H. Sun, *Analysis of positive solution and Hyers–Ulam stability for a class of singular fractional differential equations with p-Laplacian in Banach space*, Math. Methods Appl. Sci. **41** (9) (2018), 3430–3440. https://doi.org/10.1002/mma.4835

[20] T. Kherraz, M. Benbachir, M. Lakrib, M. E. Samei, M. K. Kaabar, and S. A. Bhanotar, *Existence and uniqueness results for fractional boundary value problems with multiple orders of fractional derivatives and integrals*, Chaos Solit. Fractals **166** (2023), 113007. https://doi.org/10.1016/j.chaos.2022.113007

[21] F. Mainardi, *Fractional diffusive waves in viscoelastic solids*, Nonlinear Waves in Solids **137** (1995), 93–97.

[22] K. Mathiyalagan and G. Sangeetha, *Second-order sliding mode control for nonlinear fractional-order systems*, Appl. Math. Comput. **383** (2020), 125264. https://doi.org/10.1016/j.amc.2020.125264

[23] F. Mirzaee and S. Alipour, *Cubic B-spline approximation for linear stochastic integro-differential equation of fractional order*, J. Comput. Appl. Math. **366** (2020), 112440. https://doi.org/10.1016/j.cam.2019.112440

[24] F. Mirzaee, E. Solhi, and S. Naserifar, *Approximate solution of stochastic Volterra integro-differential equations by using moving least squares scheme and spectral collocation method*, Appl. Math. Comput. **410** (2021), 126447. https://doi.org/10.1016/j.amc.2021.126447

[25] A. K. Nain, R. K. Vats, and S. K. Verma, *Existence and uniqueness results for positive solutions of Hadamard type fractional BVP*, J. Interdiscip. Math. **22** (5) (2019), 697–710. https://doi.org/10.1080/09720502.2019.1661603

[26] S. K. Paul and L. N. Mishra, *Asymptotic stability and approximate solutions to quadratic functional integral equations containing ψ-Riemann–Liouville fractional integral operator*, Comput. Math. Math. Phys. **65** (2) (2025), 320–338. https://doi.org/10.1134/S096554252470204X

[27] S. K. Paul and L. N. Mishra, *Approximation of solutions through the Fibonacci wavelets and measure of noncompactness to nonlinear Volterra–Fredholm fractional integral equations*, Korean J. Math. **32** (1) (2024), 137–162. https://doi.org/10.11568/kjm.2024.32.1.137

[28] S. K. Paul, L. N. Mishra, and V. N. Mishra, *Results on integral inequalities for a generalized fractional integral operator unifying two existing fractional integral operators*, Nonlinear Anal.: Model. Control **29** (6) (2024), 1080–1105. https://doi.org/10.15388/namc.2024.29.37848

[29] M. Rabbani, A. Deep, and Deepmala, *On some generalized non-linear functional integral equations of two variables via measures of noncompactness and numerical method to solve it*, Math. Sci. (2021), 1–8. https://doi.org/10.1007/s40096-020-00367-0

[30] S. Rezapour, S. Kumar, M. Q. Iqbal, A. Hussain, and S. Etemad, *On two abstract Caputo multi-term sequential fractional boundary value problems under the integral conditions*, Math. Comput. Simul. **194** (2022), 365–382. https://doi.org/10.1016/j.matcom.2021.11.018

[31] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, London (1974).

[32] X. Su, *Boundary value problem for a coupled system of nonlinear fractional differential equations*, Appl. Math. Lett. **22** (1) (2009), 64–69. https://doi.org/10.1016/j.aml.2008.03.001

[33] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer Science Business Media (2011).
http://link.springer.com/book/10.1007/978-3-642-14003-7
[34] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York (1964).
[35] J. Wang, K. Shah, and A. Ali, *Existence and Hyers–Ulam stability of fractional nonlinear impulsive switched coupled evolution equations*, Math. Methods Appl. Sci. **41** (6) (2018), 2392–2402.
https://doi.org/10.1002/mma.4748
[36] F. Wang, I. Ahmad, H. Ahmad, M. D. Alsulami, K. S. Alimgeer, C. Cesarano, and T. A. Nofal, *Meshless method based on RBFs for solving three-dimensional multi-term time fractional PDEs arising in engineering phenomenons*, J. King Saud Univ. Sci. **33** (8) (2021), 101604.
https://doi.org/10.1016/j.jksus.2021.101604

Muntazeer Ansari
Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India
E-mail: muntazeeransari1995@gmail.com

Lakshmi Narayan Mishra
Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India
E-mail: lakshminarayanmishra04@gmail.com, lakshminarayan.mishra@vit.ac.in