Korean J. Math. **32** (2024), No. 4, pp. 791–799
https://dx.doi.org/10.11568/kjm.2024.32.4.791

# GENERALIZED $\eta$-DUALS OF BANACH SPACE VALUED DIFFERENCE SEQUENCE SPACES

Naveen Sharma and Sandeep Kumar*

Abstract. In the present paper, we get an opportunity to introduce and study the notion of generalized $\eta$-dual for Banach space valued difference sequence spaces, as a generalization of the classical $\alpha$-Köthe Toeplitz dual for scalar sequences. We obtain a set of necessary and sufficient conditions for $(A_k) \in E^\eta(X, \Delta)$, where $E \in \{\ell_\infty, c, c_0\}$. Moreover, we explore the notion of generalized $\eta$-dual for generalized difference sequence spaces $E(X, \Delta^r)$ and $E(X, \Delta_\nu)$, where $r \in \mathbb{N}$ and $\nu$ is a multiplier sequence.

## 1. Introduction and Preliminaries

Kizmaz [13], added to the field of sequence spaces a new idea of difference sequence spaces by introducing $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ (termed as difference sequence spaces) as follows

$$\ell_\infty(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) = (x_k - x_{k+1}) \in \ell_\infty\},$$

$$c(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) = (x_k - x_{k+1}) \in c\},$$

$$c_0(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) = (x_k - x_{k+1}) \in c_0\},$$

i.e.,

$$E(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in E\} \text{ for } E \in \{\ell_\infty, c, c_0\}$$

where $c_0$, $c$, $\ell_\infty$ are Banach spaces of null, convergent and bounded sequences of scalars, normed by $\|x\|_\infty = \sup_k |x_k|$ and $\omega$ is the space of scalar sequences.

It is observed that $E(\Delta)$ are Banach spaces with the norm

$$\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty \quad \text{for} \quad x = (x_k) \in E(\Delta) \quad \text{where} \quad \Delta x = (\Delta x_k) = (x_k - x_{k+1}).$$

Et and Çolak [7] generalized the above concept by introducing $E(\Delta^n)$ as follows

$$E(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in E\} \text{ for } E \in \{\ell_\infty, c, c_0\}$$

Received October 7, 2024. Revised November 21, 2024. Accepted December 5, 2024.
2010 Mathematics Subject Classification: 46A45, 40A05.
Key words and phrases: Difference sequence spaces, Köthe Toeplitz duals, Generalized Köthe Toeplitz duals.
∗ Corresponding author.

where $\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})$ for all $k \in \mathbb{N}$ and $\Delta^0 x_k = x_k$. These spaces turn out to be complete when equipped with the norm

$$\|x\|_{\Delta^n} = \sum_{i=1}^{n} |x_i| + \|\Delta^n x\|_\infty \quad \text{for} \quad x = (x_k) \in E(\Delta^n).$$

Obviously, for $n = 1$ the work of Et and Colak [7], reduces to that of Kizmaz [13]. Using a multiplier sequence, Gnanaseelan and Srivastva [9] introduced the following sequence spaces

$$\ell_\infty(\Delta_\nu) = \{x = (x_k) \in \omega : (\nu_k(x_k - x_{k+1})) \in \ell_\infty\},$$
$$c(\Delta_\nu) = \{x = (x_k) \in \omega : (\nu_k(x_k - x_{k+1})) \in c\},$$
$$c_0(\Delta_\nu) = \{x = (x_k) \in \omega : (\nu_k(x_k - x_{k+1})) \in c_0\},$$

i.e.,

$$E(\Delta_\nu) = \{x = (x_k) \in \omega : (\nu_k \Delta x_k) \in E\} \text{ for } E \in \{\ell_\infty, c, c_0\}$$

where $\nu = (\nu_k)$ is a sequence of non zero complex numbers and

(1) $$\frac{|\nu_k|}{|\nu_{k+1}|} = 1 + O\left(\frac{1}{k}\right),$$

(2) $$k^{-1}|\nu_k| \sum_{i=1}^{k} |\nu_i^{-1}| = O(1) \text{ for each k, and}$$

(3)
$(k|\nu_k^{-1}|)$ is a monotonically increasing sequence of positive numbers tending to infinity

The spaces $E(\Delta_\nu)$ for $\nu = (1, 1, 1, \ldots)$ are noting but the spaces $E(\Delta)$ of Kizmaz and have Banach space structure when equipped with norm $\|x\|_{\Delta_\nu} = |\nu_1 x_1| + \sup_k |\nu_k(x_k - x_{k+1})|$.

For more insight into difference sequence spaces and its various generalizations, one may refer to [1–4, 6, 8, 16–21].

The theory of sequence spaces is considered to be incomplete without a touch to the concept of dual spaces. Credit of introducing dual spaces goes to G. Köthe and O. Toeplitz [14]. For a real or complex sequence space E,

$$E^\alpha = \left\{ a = (a_k) \in \omega : \sum_{k=1}^{\infty} |a_k x_k| < \infty \quad \text{for each } x = (x_k) \in E \right\}$$

and

$$E^\beta = \left\{ a = (a_k) \in \omega : \sum_{k=1}^{\infty} a_k x_k < \infty \quad \text{for each } x = (x_k) \in E \right\}$$

are called $\alpha-$, $\beta$-duals spaces of $E$, respectively.
Kizmaz [13] observed that

$$[\ell_\infty(\Delta)]^\alpha = [c_0(\Delta)]^\alpha = [c(\Delta)]^\alpha = \left\{ a = (a_k) \in \omega : \sum_k k\,|a_k| < \infty \right\}.$$

Also we have in view of [7, 9, 11]

$$[\ell_\infty(\Delta^r)]^\alpha = [c(\Delta^r)]^\alpha = [c_0(\Delta^r)]^\alpha = \left\{ a = (a_k) \in \omega : \sum_k k^r |a_k| < \infty \right\}$$

and

$$[\ell_\infty(\Delta_\nu)]^\alpha = [c(\Delta_\nu)]^\alpha = [c_0(\Delta_\nu)]^\alpha = \left\{ a = (a_k) \in \omega : \sum_k k \left|\nu_k^{-1}\right| |a_k| < \infty \right\}.$$

The above introduced notion of Köthe Toeplitz duals was further generalized by Maddox [15] and Gupta et al. [10] termed as generalized Köthe Toeplitz duals (or operator duals) as follows

Consider Banach spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ with $\theta$ as zero element. By $B(X,Y)$, we notate the class of bounded linear operators from $X$ to $Y$ which turn out to be Banach space with usual operator norm and $\omega(X)$ as the space of $X$-valued (Banach space valued ) sequences. Then for any nonempty subset $E(X)$ of $\omega(X)$

$$[E(X)]^\alpha = \left\{ A = (A_k) : \sum_{k=1}^{\infty} \|A_k x_k\| < \infty \quad \text{for each } x = (x_k) \in E(X) \right\}$$

and

$$[E(X)]^\beta = \left\{ A = (A_k) : \sum_{k=1}^{\infty} A_k x_k \text{ converges in Y for each } \ x = (x_k) \in E(X) \right\}$$

are termed as generalized $\alpha$-, $\beta$-Köthe Toeplitz dual spaces of $E(X)$ respectively. Here $(A_k)$ is a sequence of linear (not necessarly bounded) operators from $X$ to $Y$. Due to the completeness of $Y$, $[E(X)]^\alpha \subset [E(X)]^\beta$.

It is to be noted that, the generalized dual spaces $[E(X)]^\alpha$ and $[E(X)]^\beta$ reduce to classical dual spaces $E^\alpha$ and $E^\beta$ for the case $X = Y = \mathbb{C}$, because in this case the operator $A_k$ may be identified with scalar $a_k$. Maddox [12] investigated generalized Köthe Toeplitz duals, for the sequence spaces $c_0(X), c(X)$ and $\ell_\infty(X)$ (the Banach spaces of null, convergent and bounded X-valued sequences respectively) normed by $\|x\|_\infty = \sup_k \|x_k\|$. It was shown that $[\ell_\infty(X)]^\alpha = [c(X)]^\alpha = [c_0(X)]^\alpha$ which is a natural generalization of the scalar case $\ell_\infty^\alpha = c^\alpha = c_0^\alpha = \ell_1$.

Bhardwaj and Gupta [5] applied the above introduced duality notion of Maddox [15] for newly introduced Banach space valued difference sequence spaces $E(X,\Delta), E(X,\Delta_\nu)$ and $E(X,\Delta^r)$ where

$$E(X,\Delta) = \{x = (x_k) \in \omega(X) : (\Delta x_k) \in E(X)\},$$
$$E(X,\Delta_\nu) = \{x = (x_k) \in \omega(X) : (\nu_k(x_k - x_{k+1}) \in E(X)\}$$

and

$$E(X,\Delta^r) = \{x = (x_k) \in \omega(X) : (\Delta^r x_k) \in E(X)\} \text{ for } E \in \{\ell_\infty,\, c,\, c_0\}$$

and computed only their generalized $\beta$-Köthe Toeplitz duals.

In this paper, we introduce and study the notion of generalized $\eta$-dual for Banach space valued sequence spaces, as a generalization of the classical $\alpha$-Köthe Toeplitz dual of scalar cases. The generalized $\eta$-duals of the $X$-valued (Banach space valued) difference sequence spaces $E(X,\Delta)$, $E(X,\Delta_\nu)$ are obtained which is a generalization of the existing results for duals of the classical difference sequence spaces $E(\Delta)$ and $E(\Delta_\nu)$ of scalars, $E \in \{\ell_\infty, c, c_0\}$. Apart from this, we compute the generalized $\eta$-duals for $E(X, \Delta^r)$, $r \geq 0$ integer and observe that the results agree with corresponding results for scalar cases.

## 2. Generalized $\eta$-dual spaces of sequence spaces $c_0(X,\Delta)$, $c(X,\Delta)$ and $\ell_\infty(X,\Delta)$

In the present section, along with introducing a notion of generalized $\eta$-dual for Banach space valued( that is $X$-valued ) sequence space $E(X) \subset \omega(X)$, we compute generalized $\eta$-duals for $c_0(X,\Delta), c(X,\Delta)$ and $\ell_\infty(X,\Delta)$.

Definition 2.1. Let $E(X) \subset \omega(X)$. Then generalized $\eta$-dual of $E(X)$ is denoted by $E(X)^\eta$ and defined as

$$E(X)^\eta = \left\{ A = (A_k) : \sum_{k=1}^{\infty} \|A_k x_k\|^p < \infty \text{ for each } x = (x_k) \in E(X) \right\}$$

where $p \geq 1$ integer. Here $(A_k)$ is a sequence of linear (not necessarily bounded) operators from $X$ to $Y$. It is worth observing that for

(i) $[F(X)] \subset [E(X)]$ we have $[E(X)]^\eta \subset [F(X)]^\eta$.

(ii) $\eta = 1$ we have $[E(X)]^\eta = [E(X)]^\alpha$.

(iii) $\eta = 1$ and $X = \mathbb{C}$ we have $[E(X)]^\eta = E^\alpha$.

Theorem 2.2. $(A_k) \in c_0^\eta(X,\Delta)$ *iff there exists positive integer $m$ such that*

*(1)* $(A_k) \in B(X,Y)$ *for all* $k \geq m$ *and*

*(2)* $\sum_{k \geq m} k^p \|A_k\|^p < \infty$.

*Proof.* Sufficiency: Let $(i)$ and $(ii)$ holds. Let $x = (x_k) \in c_0(X,\Delta)$. Then $x_k - x_{k+1} \to \theta$ in $X$ as $k \to \infty$ and so $\sup_k \|x_k - x_{k+1}\| < \infty$. Using Lemma 1 of [5], let $M = \sup_k k^{-1}\|x_k\|$, i.e., $k^{-1}\|x_k\| \leq M$ for all $k \geq 1$. Since $\sum_{k\geq m} k^p\|A_k\|^p < \infty$, so for given $\varepsilon > 0$, there exists an integer $k_1 \geq m$ such that $\sum_{k \geq k_1} k^p \|A_k\|^p < \dfrac{\varepsilon}{M^p}$. Then

$$\begin{aligned}
\sum_{k \geq k_1} \|A_k x_k\|^p &\leq \sum_{k\geq k_1} \|A_k\|^p \|x_k\|^p \\
&= \sum_{k \geq k_1} k^p \|A_k\|^p \left(k^{-p}\|x_k\|^p\right) \\
&\leq M^p \sum_{k\geq k_1} k^p \|A_k\|^p \\
&< M^p \frac{\varepsilon}{M^p} = \varepsilon
\end{aligned}$$

and so $(A_k) \in c_0^\eta(X,\Delta)$.

Conversely, suppose $(A_k) \in c_0^\eta(X,\Delta)$ but no $m$ exists for which $(A_k) \in B(X,Y)$ for all $k \geq m$. Then there exists a strictly increasing sequence $(k_i)$ of natural numbers such that $A_{k_i} \notin B(X,Y)$ for each $i \geq 1$. Thus for each $i \geq 1$, we can find $z_i \in S$ such that $\|A_{k_i} z_i\| > i$. Define

$$x_k = \begin{cases} \dfrac{z_i}{i} & \text{for } k = k_i, i \geq 1 \\ \theta & \text{otherwise} \end{cases} \qquad k \in \mathbb{N}\,.$$

Then $x = (x_k) \in c_0(X,\Delta)$ but $\|A_{k_i} x_{k_i}\| > 1$. Consequently, $\|A_{k_i} x_{k_i}\|^p > 1$ for each $i \geq 1$, contrary to the fact that $\sum_{k\geq 1} \|A_k x_k\|^p$ converges. Hence the $A_k$'s are ultimately bounded. Now suppose, if possible, $\sum_{k\geq m} k^p \|A_k\|^p = \infty$. Then there exist natural numbers $n(1) < n(2) < \ldots$ with $n(1) \geq m$ such that for each $i \geq 1$,

$$\sum_{k=1+n(i)}^{n(i+1)} k^p \|A_k\|^p > 2^{n(i+1)p}.$$

Moreover for each $k \geq m$, there exists $z_k \in S$ such that $\|A_k\| < 2\|A_k z_k\|$. Define

$$x_k = \begin{cases} \dfrac{k z_k}{2^k} & \text{for } n(i) < k \leq n(i+1), i \geq 1 \\ \theta & \text{otherwise} \end{cases} \qquad k \in \mathbb{N}\,.$$

Then $x = (x_k) \in c_0(X,\Delta)$ and

$$\begin{aligned} \sum_{k=1+n(i)}^{n(i+1)} \|A_k x_k\|^p &= \sum_{k=1+n(i)}^{n(i+1)} \frac{k^p}{2^{kp}} \|A_k z_k\|^p \\ &> \sum_{k=1+n(i)}^{n(i+1)} \frac{k^p}{2^{kp}} \frac{1}{2^p} \|A_k\|^p \\ &> \frac{1}{2^p} \sum_{k=1+n(i)}^{n(i+1)} \frac{k^p \|A_k\|^p}{2^{n(i+1)\,p}} \\ &> \frac{1}{2^p} \end{aligned}$$

for each $i \geq 1$. Consequently, $\sum_{k\geq 1} \|A_k x_k\|^p = \infty$, which is contrary to the convergence of series $\sum_{k\geq 1} \|A_k x_k\|^p$. Hence the proof. □

THEOREM 2.3. $(A_k) \in c^\eta(X,\Delta)$ *iff there exists positive integer* $m$ *such that*
*(1)* $(A_k) \in B(X,Y)$ *for all* $k \geq m$ *and*

*(2)* $\sum_{k\geq m} k^p \|A_k\|^p < \infty$.

*Proof.* Sufficiency: Let $(i)$ and $(ii)$ holds and $x = (x_k) \in c(X,\Delta)$. Then

$$\sup_k \|x_k - x_{k+1}\| < \infty.$$

Arguing in the same way, as in sufficiency part of Theorem 2.2, we get $(A_k) \in c^\eta(X,\Delta)$.

Conversely, as $c^\eta(X,\Delta) \subset c_0^\eta(X,\Delta)$, so the necessity of $(i)$ and $(ii)$ follows from the necessity part of Theorem 2.2. □

The proof of the following runs on the similar lines as of Theorem 2.2 and hence omitted.

Theorem 2.4. $(A_k) \in \ell_\infty^\eta(X, \Delta)$ *iff there exists positive integer* $m$ *such that*
*(1)* $(A_k) \in B(X, Y)$ *for all* $k \geq m$ *and*

*(2)* $\sum_{k \geq m} k^p \|A_k\|^p < \infty$.

From the Theorem 2.2, Theorem 2.3 and Theorem 2.4 we have the following

Corollary 2.5. $c_0^\eta(X, \Delta) = c^\eta(X, \Delta) = \ell_\infty^\eta(X, \Delta)$.

Since for $X = Y = \mathbb{C}$, $A_k$ may be identified with $a_k$ so we have for $p = 1$

Corollary 2.6. $c_0^\alpha(\Delta) = c^\alpha(\Delta) = \ell_\infty^\alpha(\Delta)$.

## 3. Some further generalizations

Here in this section, we proposed to compute generalized $\eta$-dual spaces of the generalized difference sequence spaces $c_0(X, \Delta^r), c(X, \Delta^r)$ and $\ell_\infty(X, \Delta^r)$ where

$$E(X, \Delta^r) = \{x = (x_k) \in \omega(X) : (\Delta^r x_k) \in E(X)\}, \ E \in \{\ell_\infty, c, c_0\}, \ r \in \mathbb{N}.$$

Proceeding on the lines similar as in Theorem 2.2, we have obtained necessary and sufficient conditions for $(A_k) \in E^\eta(X, \Delta^r)$. Before proceeding further we recall the following simple and useful lemmas of [10].

Lemma 3.1. *If* $\sup_k \|\Delta^r x_k\| < \infty$, *then* $\sup_k k^{-1} \|\Delta^{r-1} x_k\| < \infty$, $r \in \mathbb{N}$.

Lemma 3.2. *If* $\sup_k k^{-i} \|\Delta^{r-i} x_k\| < \infty$, *then* $\sup_k k^{-(i+1)} \|\Delta^{r-(i+1)} x_k\| < \infty$ *for all* $i$, $r \in \mathbb{N}$ *and* $1 \leq i < r$.

Corollary 3.3. *If* $\sup_k k^{-1} \|\Delta^{r-1} x_k\| < \infty$, *then* $\sup_k k^{-r} \|x_k\| < \infty$.

Theorem 3.4. $(A_k) \in E^\eta(X, \Delta^r)$, $E \in \{\ell_\infty, c, c_0\}$ *iff there exists positive integer* $m$ *such that*
*(1)* $(A_k) \in B(X, Y)$ *for all* $k \geq m$ *and*

*(2)* $\sum_{k \geq m} k^{rp} \|A_k\|^p < \infty$.

Corollary 3.5. $c_0^\eta(X, \Delta^r) = c^\eta(X, \Delta^r) = \ell_\infty^\eta(X, \Delta^r)$*and hence*
*(a)* $c_0^\alpha(X, \Delta^r) = c^\alpha(X, \Delta^r) = \ell_\infty^\alpha(X, \Delta^r)$

*(b)* $c_0^\alpha(\Delta) = c^\alpha(\Delta) = \ell_\infty^\alpha(\Delta)$.

In the end of this section, making use of Lemma 3.1 of [10] we compute generalized $\eta$-dual spaces of $E(X, \Delta_\nu)$ for $E \in \{\ell_\infty, c, c_0\}$ where

$$E(X, \Delta_\nu) = \{x = (x_k) \in \omega(X) : (\nu_k(x_k - x_{k+1})) \in E(X)\}.$$

However, for the sake of completeness of the paper we are proving the lemma here.

Lemma 3.6. *If* $\sup_k \|\nu_k(x_k - x_{k+1})\| < \infty$, *then* $\sup_k k^{-1} \|\nu_k x_k\| < \infty$.

*Proof.* Let $\sup_k \|v_k(x_k - x_{k+1})\| < \infty$. We get

$$\begin{aligned}\|x_1 - x_{k+1}\| &= \|\sum_{i=1}^{k} x_i - x_{i+1}\| \leq \sum_{i=1}^{k} \|\nu_i(x_i - x_{i+1})\|\,|\nu_i^{-1}| \\ &= O(1)\sum_{i=1}^{k} |\nu_i^{-1}| \\ &= O(1)(k^{-1}|\nu_k|)\sum_{i=1}^{k} |\nu_i^{-1}|\,k\,|\nu_k^{-1}| \\ &= O(k|\nu_k^{-1}|) \quad \text{(using (2))}.\end{aligned}$$

Also $\|x_k\| = \|x_k - x_{k+1} + x_{k+1} - x_1 + x_1\| \leq \|x_k - x_{k+1}\| + \|x_{k+1} - x_1\| + \|x_1\|$ for every $k$, which implies $k^{-1}\,\|\nu_k x_k\| \leq k^{-1}\,|\nu_k|\,\|x_k - x_{k+1}\| + k^{-1}\,|\nu_k|\,\|x_{k+1} - x_1\| + k^{-1}\,|\nu_k|\,\|x_1\|$ . Using (3), we get, $k^{-1}\|\nu_k x_k\| \leq k^{-1}O(1) + O(1) + k^{-1}\,|\nu_k|\,\|x_1\|$. Hence $\sup_k k^{-1}\|\nu_k x_k\| < \infty$. □

Using Lemma 3.6, it is easy to have the following

THEOREM 3.7. *$(A_k) \in E^{\eta}(X, \Delta_\nu)$, $E \in \{\ell_\infty, c, c_0\}$ iff there exists positive integer $m$ such that*

*(1) $(A_k) \in B(X, Y)$ for all $k \geq m$ and*

*(2) $\sum_{k\geq m} k^p\,|\nu_k^{-p}|\,\|A_k\|^p < \infty$.*

COROLLARY 3.8. *(a) $c_0^{\eta}(X, \Delta_\nu) = c^{\eta}(X, \Delta_\nu) = \ell_\infty^{\eta}(X, \Delta_\nu)$.*

*(b) $c_0^{\eta}(X, \Delta) = c^{\eta}(X, \Delta) = \ell_\infty^{\eta}(X, \Delta)$.*

*(c) $c_0^{\alpha}(X, \Delta) = c^{\alpha}(X, \Delta) = \ell_\infty^{\alpha}(X, \Delta)$.*

*(d) $c_0^{\alpha}(\Delta) = c^{\alpha}(\Delta) = \ell_\infty^{\alpha}(\Delta)$.*

## Acknowledgement

The authors would like to thank the referees for a careful reading and several constructive comments that have improved the presentation of the paper.

## References

[1] B. Altay and F. Başar, *The fine spectrum and the matrix domain of the difference operator $\Delta$ on the sequence space $\ell_p$*, Comm. Math. Anal., **2** (2) (2007).
https://www.researchgate.net/publication/26489984

[2] F. Başar and B. Altay, *On the space of sequences of p-bounded variation and related matrix mappings*, Ukrainian Math. J., **55** (2003), 136–147.
https://doi.org/10.1023/A:1025080820961

[3] C. A. Bektaş and R. Çolak, *On some generalized difference sequence spaces*, Thai J. Math., **1** (3) (2005), 83–98.
https://www.researchgate.net/publication/268859083

[4] C. A. Bektaş, M. Et, and R. Çolak, *Generalized difference sequence spaces and their dual spaces*, J. Math. Anal. Appl., **292** (2) (2004), 423–432.
https://doi.org/10.1016/j.jmaa.2003.12.006

[5] V. K. Bhardwaj and S. Gupta, *On $\beta$-dual of Banach space valued difference sequence spaces*, Ukrainian Math. J., **65** (8) (2013).
https://doi.org/10.1007/s11253-014-0857-3

[6] O. Duyar, *On some new vector valued sequence spaces $E(X, \lambda, p)$*, AIMS Math., **8** (6) (2023), 13306–13316.
https://doi.org/10.3934/math.2023673

[7] M. Et and R. Çolak, *On some generalized difference sequence spaces*, Soochow J. Math., **21** (4) (1995), 377–386.
https://www.researchgate.net/publication/284263561

[8] M. Et and A. Esi, *On Köthe-Toeplitz duals of generalized difference sequence spaces*, Bull. Malays. Math. Sci. Soc., **23** (1) (2000), 25–32.
https://www.researchgate.net/publication/247009485

[9] C. Gnanaseelan and P. D. Srivastva, *The $\alpha$-, $\beta$-, $\gamma$-duals of some generalized difference sequence spaces*, Indian J. Math., **38** (2) (1996), 111–120.

[10] S. Gupta, Ritu, and M. Kumar, *Generalized $\alpha$-Köthe Toeplitz duals of certain difference sequence spaces*, Korean J. Math., **32** (2) (2024), 219–228.
https://doi.org/10.11568/kjm.2024.32.2.219

[11] Haryadi, Supama, and A. Zulijanto, *Köthe-Toeplitz duals of the Cesàro sequence spaces defined on a generalized Orlicz space*, Global J. Pure Appl. Math., **14** (4) (2018), 591–601.
https://www.ripublication.com/gjpam18/gjpamv14n4_06.pdf

[12] S. A. Khan, *Cesàro difference sequence spaces and its duals*, Int. J. Math. Appl., **11** (1) (2023), 41–48.
http://ijmaa.in/index.php/ijmaa/article/view/883

[13] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull., **24** (2) (1981), 169–176.
https://doi.org/10.4153/CMB-1981-027-5

[14] P. K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Marcel Dekker, Inc., New York and Basel.
https://books.google.co.in/books?id=-HoZAQAAIAAJ

[15] I. J. Maddox, *Infinite matrices of operators*, Lecture Notes Math., Berlin, Springer, 1980.
https://doi.org/10.1007/bfb0088196

[16] E. Malkowsky, M. Mursaleen, and S. Suantai, *The dual spaces of sets of difference sequences of order $m$ and matrix transformations*, Acta Math. Sin. (Engl. Ser.), **23** (3) (2007), 521–532.
https://doi.org/10.1007/s10114-005-0719-x

[17] E. Malkowsky and S. D. Parashar, *Matrix transformations in spaces of bounded and convergent difference sequences of order $m$*, Analysis, **17** (1) (1997), 87–98.
https://doi.org/10.1524/anly.1997.17.1.87

[18] N. Rath, *Operator duals of some sequence-spaces*, Indian J. Pure Appl. Math., **20** (10) (1989), 953–963.

[19] A. Robinson, *On functional transformations and summability*, Proc. London Math. Soc., (1950).
https://doi.org/10.1112/plms/s2-52.2.132

[20] J. K. Srivastava and B. K. Srivastava, *Generalized sequence space $c_0(X, \lambda, p)$*, Indian J. Pure Appl. Math., **27** (1996), 73–84.

[21] S. Suantai and W. Sanhan, *On $\beta$-dual of vector-valued sequence spaces of Maddox*, Int. J. Math. Math. Sci., **30** (2001), 385–392.
https://doi.org/10.1155/s0161171202012772

**Naveen Sharma**
Department of Mathematics, D.A.V. College Muzaffarnagar,
Chaudhary Charan Singh University, Meerut (India)
*E-mail*: ns2000dav@gmail.com

**Sandeep Kumar**
Department of Mathematics, D.A.V. College Muzaffarnagar,
Chaudhary Charan Singh University, Meerut (India)
*E-mail*: sandeepkharb1989@gmail.com