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ON SOME IDEALS DEFINED BY AN ARITHMETIC SEQUENCE

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Abstract. This paper investigates properties of ideals in the affine and homogeneous projective coordinate rings of the plane, defined using arithmetic sequence

$$\{a_\ell = a + \ell d \mid \ell \geq 0\}$$

for some positive integers a and d. Specifically, we study two types of ideals: $I(a,d)$ is generated by $D(a,d)$ in $\mathbb{K}[x,y]$ and $J(a,d)$ is generated by $E(a,d)$ in $\mathbb{K}[x,y,z]$ where

$$D(a,d) = \{f_\ell = x^{a_\ell} - y^{a_{\ell+1}} \mid \ell \geq 0\}$$

and

$$E(a,d) = \{F_\ell = x^{a_\ell} z^d - y^{a_{\ell+1}} \mid \ell \geq 0\}.$$

This paper provides detailed answers to several problems, including finding finite generating sets, describing the zero locus of these ideals, and determining their Hilbert functions. Finally, the Castelnuovo-Mumford regularity and the minimal free resolution of the homogeneous coordinate ring and multi secant line are discussed.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Also let $A = \mathbb{K}[x,y]$ and $S = \mathbb{K}[x,y,z]$ be respectively the affine and the homogeneous coordinate ring of $\mathbb{A}^2$ and $\mathbb{P}^2$. The aim of this paper is to study various basic properties of some ideals of A and S defined by the arithmetic sequence

$$\{a_\ell = a + \ell d \mid \ell \geq 0\}$$

for some positive integers a and d. To be precise, let $D(a,d)$ and $E(a,d)$ be respectively the infinite subsets of A and S defined as

$$D(a,d) = \{f_\ell = x^{a_\ell} - y^{a_{\ell+1}} \mid \ell \geq 0\}$$

and

$$E(a,d) = \{F_\ell = x^{a_\ell} z^d - y^{a_{\ell+1}} \mid \ell \geq 0\}.$$

Then consider the ideal $I(a,d)$ of A generated by $D(a,d)$ and the homogeneous ideal $J(a,d)$ of S generated by $E(a,d)$. Throughout this paper, we are intended to answer for the following problems related to these ideals. In particular, we will explain how the answers to them are determined by a and d.

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(1.i) Find a finite subset of $D(a,d)$ which generates $I(a,d)$.
(1.ii) Find a minimal generating set of the radical ideal of $I(a,d)$.
(1.iii) Describe the affine algebraic set $\Gamma(a,d) := V(I(a,d)) \subset \mathbb{A}^2$.

(2.i) Find a finite subset of $E(a,d)$ which generates $J(a,d)$.
(2.ii) Find a minimal generating set of the radical ideal of $J(a,d)$.
(2.iii) Describe the projective algebraic set $\Omega(a,d) := V(J(a,d)) \subset \mathbb{P}^2$.
(2.iv) Find the Hilbert function and the minimal free resolution of the homogeneous coordinate ring of $\Omega(a,d) \subset \mathbb{P}^2$.
(2.v) Find the integers $\ell(\Omega(a,d))$ defined by $\ell(X) := \max\{\text{length}(X \cap \mathbb{P}^1) \mid \mathbb{P}^1 \subset \mathbb{P}^2\}$.

Regarding (1.i) and (2.i), Hilbert Basis Theorem says that $I(a,d)$ and $J(a,d)$ are finitely generated. Proposition 2.1 shows that

$$I(a,d) = \langle f_0 = x^a - y^{a+d}, f_1 = x^{a+d} - y^{a+2d}\rangle$$

and

$$J(a,d) = \langle F_0 = x^a z^d - y^{a+d}, F_1 = x^{a+d} z^d - y^{a+2d}\rangle.$$

Our answer to (1.ii) is closely related to (1.iii) and (2.ii). Regarding (1.ii), let

$$\Lambda(a,d) := \Gamma(a,d) - \{(0,0)\} \subset \mathbb{A}^2.$$

Theorem 3.1 says that $\Lambda(a,d)$ is a finite abelian group isomorphic to $\mathbb{Z}_{d^2/g} \times \mathbb{Z}_g$ where g is the greatest common divisor of a and d. In particular, $\Gamma(a,d)$ consists of (d^2+1) distinct points. From now on, we write $\Gamma(a,d)$ as

$$\Gamma(a,d) = \{P_0, P_1, \ldots, P_{d^2}\}$$

where $P_0 = (0,0)$. Then it turns out in Theorem 4.1 that $I(a,d)$ has the local property

$$\dim_{\mathbb{K}} \ (A/I(a,d))_{P_i} = \begin{cases} a^2+2ad & \text{for } i=0, \text{ and} \\ 1 & \text{for } i \neq 0. \end{cases} \tag{1}$$

This is proved by applying Bezout's theorem to the homogeneous ideal $J(a,d) = \langle F_0, F_1\rangle$ of S. Indeed,

$$\Omega(a,d) = \Gamma(a,d) \cup \{P_\infty := [1:0:0]\}$$

and hence $\Omega(a,d)$ has exactly one more point than $\Gamma(a,d)$. Also the intersection multiplicities

$$I(P_0, F_0 \cap F_1) \quad \text{and} \quad I(P_\infty, F_0 \cap F_1)$$

of F_0 and F_1 at P_0 and P_∞ are respectively equal to a^2+2ad and $(a+d)d$. Bezout's theorem says that

$$(a+d)(a+2d) = \sum_{i=0}^{d^2} \dim_{\mathbb{K}} \ (A/I(a,d))_{P_i} + I(P_\infty, F_0 \cap F_1)$$

and hence

$$\sum_{i=1}^{d^2} \dim_{\mathbb{K}} \ (A/I(a,d))_{P_i} = d^2,$$

which proves (2) in Section 4.

Regarding (1.*ii*) and (2.*ii*), let $a = dn + r$ where $n \geq 0$ and $1 \leq r \leq d$. Theorem 5.1 says that

$$I(\Gamma(a,d)) = \langle x^d - y^d, (x^{d-r}y^r - 1)x, (x^{d-r}y^r - 1)y \rangle$$

and

$$J(\Omega(a,d)) = \langle (x^d - y^d)y, (x^d - y^d)z, (x^{d-r}y^r - z^d)x, (x^{d-r}y^r - z^d)y \rangle.$$

Finally, we study problems (2.*iv*) and (2.*v*) in Section 6. In particular, Theorem 6.2 shows that the homogeneous ideal of $\Omega(a,d)$ is minimally generated by four forms of degree $d+1$, the Castelnuovo-Mumford regularity of $\Omega(a,d)$ is equal to $2d-1$ and $\ell(\Omega(a,d))$ is equal to $d+1$ for $d \geq 2$. When $d = 1$, the number of generators is reduced to three and calculated separately.

2. Minimal generators of $I(a,d)$ and $J(a,d)$

In this section, we will find minimal generators of the ideals $I(a,d)$ and $J(a,d)$. To this aim, we begin with proving the following fact which deals with a more general situation and tells us a lot about the generating structure of the ideals we want to study.

PROPOSITION 2.1. *For* $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_m \in \mathbb{K}[x_1, x_2, \cdots, x_n]$ *and* $\ell \geq 0$, *let*

$$W_\ell := \alpha_1^\ell \beta_1 + \cdots + \alpha_m^\ell \beta_m.$$

Then the ideal $\langle \{W_\ell \mid \ell \geq 0\} \rangle$ *of* $\mathbb{K}[x_1, x_2, \cdots, x_n]$ *is generated by* $W_0, W_1, \cdots, W_{m-1}$.

Proof. We will show that the ideal $I := \langle W_0, W_1, \cdots, W_{m-1} \rangle$ contains W_ℓ for all $\ell \geq m$. To this aim, we use the polynomial

$$G(x_1, \cdots, x_n, y) := (y - \alpha_1)(y - \alpha_2) \cdots (y - \alpha_m) \in \mathbb{K}[x_1, x_2, \cdots, x_n, y].$$

Write $G = y^m + G_1 y^{m-1} + \cdots + G_{m-1} y + G_m$ where $G_1, \cdots, G_m \in \mathbb{K}[x_1, x_2, \cdots, x_n]$.

We use induction on $\ell \geq m$. For $\ell = m$, note that

$$G(x_1, \cdots, x_n, \alpha_i) = \alpha_i^m + G_1 \alpha_i^{m-1} + \cdots + G_{m-1}\alpha_i + G_m = 0 \quad \text{for all } 1 \leq i \leq m$$

and hence

$$\sum_{i=1}^m G(x_1, \cdots, x_n, \alpha_i)\beta_i = W_m + G_1 W_{m-1} + G_2 W_{m-2} + \cdots + G_{m-1} W_1 + G_m W_0 = 0.$$

This shows that

$$W_m = -(G_1 W_{m-1} + G_2 W_{m-2} + \cdots + G_{m-1} W_1 + G_m W_0) \in \langle W_0, W_1, \cdots, W_{m-1} \rangle.$$

Now, suppose that $\ell > m$. By induction hypothesis, we may assume that

$$W_m, \ldots, W_{\ell-1} \in \langle W_0, W_1, \cdots, W_{m-1} \rangle.$$

Then

$$\sum_{i=1}^m G(x_1, \cdots, x_n, \alpha_i)\alpha_i^{\ell-m}\beta_i = W_\ell + G_1 W_{\ell-1} + G_2 W_{\ell-2} + \cdots + G_{m-1} W_{\ell-m+1} + G_m W_{\ell-m} = 0$$

and hence

$$W_\ell = -(G_1 W_{\ell-1} + G_2 W_{\ell-2} + \cdots + G_{m-1} W_{\ell-m+1} + G_m W_{\ell-m})$$

is contained in the ideal $\langle W_0, W_1, \cdots, W_{m-1}\rangle$. This completes the proof that I contains W_ℓ for all $\ell \geq m$ and hence I is equal to $\langle W_0, W_1, \cdots, W_{m-1}\rangle$. □

COROLLARY 2.2. *Let a and d be two positive integers and let $a_\ell = a+\ell d$ for $\ell \geq 0$. Then*

(a) *The ideal $I(a,d) = \langle\{f_\ell := x^{a_\ell} - y^{a_{\ell+1}} \mid \ell \geq 0\}\rangle$ of A is generated by $\{f_0, f_1\}$.*

(b) *The ideal $I_1(a,d) = \langle\{g_\ell := x^{a_\ell} + y^{a_{\ell+1}} \mid \ell \geq 0\}\rangle$ of A is generated by $\{g_0, g_1\}$.*

(c) *The ideal $I_2(a,d) = \langle\{h_\ell := x^{a_\ell} + (-1)^{\ell+1} y^{a_{\ell+1}} \mid \ell \geq 0\}\rangle$ of A is generated by $\{h_0, h_1\}$.*

(d) *The ideal $I_3(a,d) = \langle\{p_\ell := x^{a_\ell} + (-1)^{\ell} y^{a_{\ell+1}} \mid \ell \geq 0\}\rangle$ of A is generated by $\{p_0, p_1\}$.*

(e) *The ideal $J(a,d) = \langle\{F_\ell := x^{a_\ell} z^d - y^{a_{\ell+1}} | \ell \geq 0\}\rangle$ of S is generated by $\{F_0, F_1\}$.*

Proof. (a) $\sim$ (d) For each $\ell \geq 0$, write f_ℓ as

$$f_\ell = (x^d)^\ell x^a + (y^d)^\ell (-y^{a+d}).$$

Then Proposition 2.1 shows that $I(a,d)$ is generated by $\{f_0, f_1\}$. Similarly, it can be shown by Proposition 2.1 that the ideals $I_1(a,d)$, $I_2(a,d)$ and $I_3(a,d)$ are generated by $\{g_0, g_1\}$, $\{h_0, h_1\}$ and $\{p_0, p_1\}$, respectively.
(e) For each $\ell \geq 0$, write F_ℓ as

$$F_\ell = (x^d)^\ell x^a z^d + (y^d)^\ell (-y^{a+d}).$$

Then Proposition 2.1 shows that $J(a,d)$ is generated by $\{F_0, F_1\}$. □

REMARK 2.3. Let $I(a,d)^*$ be the homogenization of $I(a,d)$. Thus it contains $J(a,d)$. Also it contains

$$y^d f_0 + f_1 = x^a y^d - x^{a+d}.$$

On the other hand, $J(a,d) = \langle F_0, F_1\rangle$ by Corollary 2.2-(e) and hence any degree $(a+d)$ form in $J(a,d)$ is a constant multiple of F_0. Therefore $J(a,d)$ is a proper subset of $I(a,d)^*$.

3. The common zero set of $I(a,d)$ in $\mathbb{A}^2$

The aim of this section is to describe the common zero set of the ideal $I(a,d)$ of A in $\mathbb{A}^2$. Since $I(a,d)$ is generated by f_0 and f_1, we will describe the affine algebraic set

$$\Gamma(a,d) := V(f_0, f_1) \subset \mathbb{A}^2.$$

To this aim, we begin with defining some notations.

- $a = gs$ and $d = gt$ where $g := gcd(a,d)$
- ζ : a primitive $\frac{d^2}{g}$th root of unity
- $\Lambda(a,d) := \Gamma(a,d) - \{(0,0)\}$

THEOREM 3.1. *Let $\Gamma(a,d) \subset \mathbb{A}^2$ be as above. Then $\Gamma(a,d)$ has (d^2+1) points. Also*

$$\Lambda(a,d) = \{(\zeta^{s\ell+t\ell+tm}, \zeta^{s\ell+tm}) \mid \ell, m \in \mathbb{Z}\}$$

and hence it is an abelian group isomorphic to $\mathbb{Z}_{\frac{d^2}{g}} \times \mathbb{Z}_g$.

Proof. First we show that $\Lambda(a,d)$ is equal to $\Phi := \{(\zeta^{s\ell+t\ell+tm}, \zeta^{s\ell+tm}) \mid \ell, m \in \mathbb{Z}\}$. It is an elementary task to check that

$$f_0(\zeta^{s\ell+t\ell+tm}, \zeta^{s\ell+tm}) = f_1(\zeta^{s\ell+t\ell+tm}, \zeta^{s\ell+tm}) = 0$$

for all $\ell, m \in \mathbb{Z}$. This means that $\Phi \subseteq \Lambda(a,d)$. On the other hand, if $(\alpha,\beta) \in \Lambda(a,d)$, then

$$\alpha^a = \beta^{a+d} \quad \text{and} \quad \alpha^{a+d} = \beta^{a+2d}.$$

Thus we get

$$\alpha^a = \beta^{a+d} \quad \text{and} \quad \alpha^d = \beta^d.$$

From $\alpha^d = \beta^d$, we have

$$\frac{\alpha}{\beta} = (\zeta^{\frac{d}{g}})^{\ell} \quad \text{for some } \ell \in \mathbb{Z}.$$

Then the equality $\alpha^a = \beta^{a+d}$ means that

$$\beta^d = (\frac{\alpha}{\beta})^a = (\zeta^{\frac{d}{g}})^{a\ell} = \zeta^{\frac{ad\ell}{g}} = \zeta^{ds\ell}$$

and hence

$$\frac{\beta}{\zeta^{s\ell}} = (\zeta^{\frac{d}{g}})^m \quad \text{for some } m \in \mathbb{Z}.$$

This shows that

$$\beta = \zeta^{s\ell+tm} \quad \text{and hence} \quad \alpha = \beta \times \zeta^{d\ell/g} = \zeta^{s\ell+t\ell+tm}$$

as desired.

It is shown that $\Lambda(a,d)$ is a subset of $\mathfrak{G} \times \mathfrak{G}$ where $\mathfrak{G}$ is the cyclic group $\langle \zeta \rangle$. Now, it is easy to check that $\Lambda(a,d)$ is a subgroup of $\mathfrak{G} \times \mathfrak{G}$. Next, we consider the projection map

$$\varphi : \Lambda(a,d) \to \mathfrak{G}$$

from $\Lambda(a,d)$ to the second factor of $\mathfrak{G} \times \mathfrak{G}$. Note that φ is surjective since $\gcd(s,t) = 1$. Let (α,β) be an element in the kernel of φ. Thus $\beta = 1$ and hence $\alpha^a = \alpha^d = 1$. Thus we get $\alpha^g = 1$. This means that α is contained in the cyclic group $\langle \zeta^{\frac{d^2}{g^2}} \rangle$. Then the kernel of φ is contained in the group $K := \langle \zeta^{\frac{d^2}{g^2}} \rangle \times \{1\}$. On the other hand, every element in K satisfies the two equations f_0 and f_1. In consequence, the kernel of φ is exactly equal to K. This shows that

$$|\Lambda(a,d)| = |K| \times |\mathfrak{G}| = d^2.$$

Finally, choose two integers ℓ and m such that $s\ell + tm = 1$. Then $(\zeta^{1+t\ell}, \zeta)$ is an element of Λ which generates a subgroup H of order $\frac{d^2}{g}$. Also $H \cap K = \{(1,1)\}$. Therefore $\Lambda \cong H \times K$, which completes the proof. □

Corollary 3.2. *For each* $i = 1,2,3$, *let*

$$\Lambda_i(a,d) = V(I_i(a,d)) - \{(0,0)\}.$$

Then $\Lambda(a,d)$ *is a subgroup of* $\Lambda(2a,2d)$ *with* $[\Lambda(2a,2d) : \Lambda(a,d)] = 4$, *and* $\Lambda_i(a,d)$ *for* $i = 1,2,3$ *are exactly the other three left cosets of* $\Lambda(a,d)$ *in* $\Lambda(2a,2d)$.

Proof. Let us denote $\Lambda(a,d)$ by $\Lambda_0(a,d)$. Using Corollary 2.2, one can observe that $\Lambda_i(a,d) \cap \Lambda_j(a,d) = \emptyset$ for any $0 \le i < j \le 3$. Also, Corollary 2.2 shows that $I(2a, 2d)$ is generated by the two polynomials

$$x^{2a} - y^{2a+2d} = f_0 \times g_0 \quad \text{and} \quad x^{2a+2d} - y^{2a+4d} = f_1 \times g_1.$$

In consequence, $\Lambda(2a, 2d)$ is the disjoint union of $\Lambda_i(a,d)$'s for $0 \le i \le 3$. Also Theorem 3.1 shows that $\Lambda(2a, 2d)$ is a group isomorphic to $\mathbb{Z}_{\frac{2d^2}{g}} \times \mathbb{Z}_{2g}$. Hence $\Lambda(a,d)$ is a subgroup of $\Lambda(2a, 2d)$ such that $[\Lambda(2a, 2d) : \Lambda(a,d)] = 4$.

It remains to prove that the four left cosets of $\Lambda(a,d)$ in $\Lambda(2a, 2d)$ are exactly $\Lambda_i(a,d)$ for $0 \le i \le 3$. To this aim, consider

$$\Phi_1 := (\omega^t, \omega^t)\Lambda(a,d),\ \Phi_2 := (\omega^{s+t}, \omega^t)\Lambda(a,d) \quad \text{and} \quad \Phi_3 := (\omega^{s+2t}, \omega^{s+t})\Lambda(a,d).$$

Then it holds that $\Phi_i \subseteq \Lambda_i(a,d)$ for each $i = 1, 2, 3$. For example,

$$\begin{aligned} \Phi_1 &= (\omega^t, \omega^t)\Lambda(a,d) \\ &= \{(\omega^{2s\ell+2tm+2t\ell+t}, \omega^{2s\ell+2tm+t}) \mid \ell, m \in \mathbb{Z}\} \\ &= \{(\omega^{2s\ell+2t\ell+t(2m+1)}, \omega^{2s\ell+t(2m+1)}) \mid \ell, m \in \mathbb{Z}\} \\ &= \{(\omega^{2s\ell+2t\ell+tm'}, \omega^{2s\ell+tm'}) \mid \ell \in \mathbb{Z}, m' \in 2\mathbb{Z}+1\} \end{aligned}$$

Then we have $\Phi_1 \subseteq \Lambda_1(a,d)$ since

$$g_0(\omega^{2s\ell+2t\ell+tm}, \omega^{2s\ell+tm}) = (\omega^{2s\ell+2t\ell+tm})^a + (\omega^{2s\ell+tm})^{a+d} = 0$$

and

$$g_1(\omega^{2s\ell+2t\ell+tm}, \omega^{2s\ell+tm}) = (\omega^{2s\ell+2t\ell+tm})^{a+d} - (\omega^{2s\ell+tm})^{a+2d} = 0$$

for all $\ell \in \mathbb{Z}$ and $m \in 2\mathbb{Z}+1$. Then Φ_i's, $0 \le i \le 3$, are pairwise disjoint since so are $\Lambda_i(a,d)$'s, $0 \le i \le 3$. This completes the proof that $\Lambda_i(a,d) = \Phi_i$ for all $0 \le i \le 3$ and hence $\Lambda_i(a,d)$'s for $0 \le i \le 3$ are exactly the four left cosets of $\Lambda(a,d)$ in $\Lambda(2a, 2d)$. □

4. Local properties of $I(a,d)$ and $J(a,d)$

In this section, we will see a few basic local properties of the ideals $I(a,d)$ of $A = \mathbb{K}[x,y]$ and $J(a,d)$ of $S = \mathbb{K}[x,y,z]$.

To state our results precisely, let $\Gamma(a,d) := V(I(a,d)) \subset \mathbb{A}^2$ and let g be the greatest common divisor of a and d. Theorem 3.1 says that

$$\Lambda(a,d) := \Gamma(a,d) - \{(0,0)\}$$

is a finite abelian group isomorphic to $\mathbb{Z}_{d^2/g} \times \mathbb{Z}_g$. In particular, $\Gamma(a,d)$ consists of (d^2+1) distinct points. So, we write $\Gamma(a,d)$ as

$$\Gamma(a,d) = \{P_0, P_1, \ldots, P_{d^2}\}$$

where $P_0 = (0,0)$.

Now, let $\Omega(a,d) := V(J(a,d)) \subset \mathbb{P}^2$ and regard $\Gamma(a,d)$ as a subset of $\mathbb{P}^2$. Then it holds that

$$\Omega(a,d) = \Gamma(a,d) \cup \{P_\infty := [1:0:0]\}$$

and hence $\Omega(a,d)$ has exactly one more point than $\Gamma(a,d)$.

For each $0 \le i \le d^2$, let $\mathcal{P}_i$ denote the maximal ideal of A correspond to point P_i. Now, let

$$I(a,d) = \mathcal{Q}_0 \cap \cdots \cap \mathcal{Q}_{d^2}$$

be the minimal primary decomposition of $I(a,d)$ where $\mathcal{Q}_i$ is the primary ideal of A such that $\sqrt{\mathcal{Q}_i} = \mathcal{P}_i$. Thus there exists an isomorphism

$$A/I(a,d) \cong \prod_{i=0}^{d^2} A/\mathcal{Q}_i.$$

Along this line, our main result in this section is the following theorem.

THEOREM 4.1. *Keep the previous notations. Then*

$$\dim_{\mathbb{K}}\ A/\mathcal{Q}_i = \begin{cases} a^2 + 2ad & \text{for } i = 0, \text{ and} \\ 1 & \text{for } 1 \le i \le d^2. \end{cases} \tag{2}$$

Therefore $\mathcal{Q}_i = \mathcal{P}_i$ *for all* $1 \le i \le d^2$ *and* $\dim_{\mathbb{K}}\ A/I(a,d) = (a+d)^2$.

To prove Theorem 4.1, we need the following lemma. Recall that

$$I(a,d) = \langle f_0 = x^a - y^{a+d}, f_1 = x^{a+d} - y^{a+2d} \rangle$$

and

$$J(a,d) = \langle F_0 = x^a z^d - y^{a+d}, F_1 = x^{a+d} z^d - y^{a+2d} \rangle.$$

Following the notation in [2, Chapter3], we will denote the intersection multiplicity of F_0 and F_1 at P_i by $I(P_i, F_0 \cap F_1)$.

LEMMA 4.2. *Keep the previous notations. Then*

$$I(P_i, F_0 \cap F_1) = \begin{cases} a^2 + 2ad & \text{for } i = 0, \\ (a+d)d & \text{for } i = \infty, \text{ and} \\ 1 & \text{for } 1 \le i \le d^2. \end{cases}$$

Proof. The dehomogenizations of F_0 and F_1 with respect to z are respectively f_0 and f_1. Thus, using the equality $x^d f_0 - f_1 = y^{a+d}(y^d - x^d)$, we have

$$\begin{aligned} I(P_0, F_0 \cap F_1) &= I(P_0, (x^d f_0 - f_1) \cap f_0) \\ &= I(P_0, y^{a+d}(y^d - x^d) \cap f_0) \\ &= (a+d) I(P_0, y \cap (x^a - y^{a+d})) + I(P_0, (y^d - x^d) \cap (x^a - y^{a+d})) \\ &= (a+d) I(P_0, y \cap x^a) + I(P_0, (y^d - x^d) \cap (x^a - y^{a+d})) \\ &= a(a+d) + I(P_0, (y^d - x^d) \cap (x^a - y^{a+d})). \end{aligned}$$

Also, $y^d - x^d$ and $x^a - y^{a+d}$ have not tangent lines in common at P_0 and hence

$$I(P_0, (y^d - x^d) \cap (x^a - y^{a+d})) = d \times a.$$

This completes the proof that $I(P_0, F_0 \cap F_1) = a^2 + 2ad$. Similarly, one can show the equality $I(P_\infty, F_0 \cap F_1) = (a+d)d$. Finally, Bézout's Theorem says that

$$\begin{aligned} (a+d)(a+2d) &= \sum_{P \in \Omega(a,d)} I(P, F_0 \cap F_1) \\ &= I(P_0, F_0 \cap F_1) + I(P_\infty, F_0 \cap F_1) + \sum_{1 \le i \le d^2} I(P_i, F_0 \cap F_1). \end{aligned}$$

Thus we have

$$\sum_{1 \le i \le d^2} I(P_i, F_0 \cap F_1) = (a+d)(a+2d) - (a^2 + 2ad) - (a+d)d = d^2.$$

Obviously, this implies that $I(P_i, F_0 \cap F_1) = 1$ for all $1 \le i \le d^2$. □

Now, we will show that $\mathcal{Q}_i = \mathcal{P}_i$ for $1 \le i \le d^2$.

Proof of Theorem 4.1. Note that

$$\dim_{\mathbb{K}} \ A/\mathcal{Q}_i = \dim_{\mathbb{K}} \ (A/I(a,d))_{\mathcal{P}_i} = I(P_i, F_0 \cap F_1).$$

Thus the equalities in (2) come immediately from Lemma 4.2. Then it follows also that

$$\dim_{\mathbb{K}} \ A/I(a,d) = \sum_{0 \le i \le d^2} A/\mathcal{Q}_i = (a+d)^2.$$

Now, we will show that $\mathcal{Q}_i = \mathcal{P}_i$ for $1 \le i \le d^2$. Indeed,

$$\dim_{\mathbb{K}} \ A/\mathcal{Q}_i = \dim_{\mathbb{K}} \ A/\mathcal{P}_i + \dim_{\mathbb{K}} \ \mathcal{P}_i/\mathcal{Q}_i$$

and so $\dim_{\mathbb{K}} \ \mathcal{P}_i/\mathcal{Q}_i = 0$. This completes the proof that $\mathcal{P}_i = \mathcal{Q}_i$ for all $1 \le i \le d^2$. □

5. The radical ideals of $\Lambda(a,d)$ and $\Omega(a,d)$

By Corollary 2.2, we have $\Gamma(a,d) = V(f_0, f_1)$ and $\Omega(a,d) = V(F_0, F_1)$. Thus it holds by Hilbert's Nullstellensatz that

$$I(\Gamma(a,d)) = \sqrt{\langle f_0, f_1 \rangle} \quad \text{and} \quad J(\Omega(a,d)) = \sqrt{\langle F_0, F_1 \rangle}$$

where for a subset $X \subset \mathbb{A}^2$ (resp. $Y \subset \mathbb{P}^2$), we denote by $I(X)$ (resp. $J(Y)$) the ideal of X in $A = \mathbb{K}[x,y]$ (resp. the homogeneous ideal of Y in $S = \mathbb{K}[x,y,z]$). The goal of this section is to find generators of the ideals related to $\Gamma(a,d)$ and $\Omega(a,d)$.

The following theorem is our main result in this section.

Theorem 5.1. *Let $a = dn + r$ where $n \ge 0$ and $1 \le r \le d$. Then*

(a) $I(\Gamma(a,d)) = \langle x^d - y^d, (x^{d-r}y^r - 1)x, (x^{d-r}y^r - 1)y \rangle$.
(b) $I(\Lambda(a,d)) = \langle x^d - y^d, x^{d-r}y^r - 1 \rangle$.
(c) $J(\Omega(a,d)) = \langle (x^d - y^d)y, (x^d - y^d)z, (x^{d-r}y^r - z^d)x, (x^{d-r}y^r - z^d)y \rangle$.
(d) $J(\Lambda(a,d)) = \langle x^d - y^d, x^{d-r}y^r - z^d \rangle$.

Proof. The proof will proceed in the order (b), (d), (c), (a).
(b) Put $M := \langle x^d - y^d, x^{d-r}y^r - 1 \rangle$. Using Theorem 3.1, we can check easily that $\Lambda(a,d)$ is contained in $V(M)$. On the other hand, $V(M)$ contains at most d^2 distinct points by Bézout's Theorem. This shows that $V(M)$ is equal to $\Lambda(a,d)$. Now, let Q_i, $1 \le i \le d^2$, be the primary component of M corresponding to the point P_i in $\Lambda(a,d)$. Then, again by Bézout's Theorem, we have

$$d^2 \le \sum_{P_i \in \Lambda(a,d)} I(P_i, (x^d - y^d) \cap (x^{d-r}y^r - 1)) \le \deg(x^d - y^d) \times \deg(x^{d-r}y^r - 1) = d^2.$$

It follows that $I(P_i, (x^d - y^d) \cap (x^{d-r}y^r - 1))$ is equal to 1 and hence Q_i is a maximal ideal for all $1 \le i \le d^2$. This completes the proof that M is a radical ideal and hence $I(\Lambda(a,d)) = M$.

(d) By (b), the homogeneous ideal $K := \langle x^d - y^d, x^{d-r}y^r - z^d\rangle$ is contained in $J(\Omega(a,d))$. Conversely, let $F \in J(\Lambda(a,d))$ be a form of degree t which is not divisible by z. By (b), we can write

$$F(x,y,1) = f(x,y)(x^d - y^d) + g(x,y)(x^{d-r}y^r - 1)$$

for some $f, g \in A$. If $f(x,y) = 0$ or $g(x,y) = 0$, then $F(x,y,z)$ is contained in K. Now, suppose that neither of $x^d - y^d$ and $x^{d-r}y^r - z^d$ divides F. Let e and s be respectively the degrees of f and g. Furthermore, we assume that s is as small as possible. Write $f = f_e + f_{<e}$ and $g = g_s + g_{<s}$. If $f_e \times (x^d - y^d) + g_s \times x^{d-r}y^r \neq 0$, then it holds that

$$F(x,y,z) = z^s f(x,y)^*(x^d - y^d) + z^e g(x,y)^*(x^{d-r}y^r - z^d)$$

and hence $F(x,y,z)$ is contained in K (cf. [3, Proposition 5, Chapter 2]). On the other hand, if

$$f_e \times (x^d - y^d) + g_s \times x^{d-r}y^r = 0$$

then $e = s$ and there exists an element $h \in A$ such that $f_s = x^{d-r}y^r h$ and $g_s = -(x^d - y^d)h$. Then

$$\begin{aligned} F(x,y,1) &= (x^{d-r}y^r h + f_{<s})(x^d - y^d) + (-(x^d - y^d)h + g_{<s})(x^{d-r}y^r - 1) \\ &= (f_{<s} + h)(x^d - y^d) + g_{<s}(x^{d-r}y^r - 1). \end{aligned}$$

Since g is chosen so that its degree s is as small as possible, it follows that $g_{<s} = 0$ and so $x^d - y^d$ divides F. This is a contradiction. Therefore, the latter case does not occur. This completes the proof.

(c) Put $T := \langle (x^d - y^d)y, (x^d - y^d)z, (x^{d-r}y^r - z^d)x, (x^{d-r}y^r - z^d)y\rangle$. Since

$$\Omega(a,d) = \Lambda(a,d) \cup \{[0:0:1]\} \cup \{[1:0:0]\},$$

we have, by (d), the following equality:

$$J(\Omega(a,d)) = \langle x^d - y^d, x^{d-r}y^r - z^d\rangle \cap \langle x, y\rangle \cap \langle y, z\rangle.$$

In particular, it holds that $T \subseteq J(\Omega(a,d))$. For the converse, we will use the equality $\langle x, y\rangle \cap \langle y, z\rangle = \langle y, xz\rangle$. Let $f \in J(\Omega(a,d))$. Then we can write

$$f = w_1(x^d - y^d) + w_2(x^{d-r}y^r - z^d) = g_1 y + g_2 xz \tag{3}$$

for some $w_1, w_2, g_1, g_2 \in S$. If we set $x = y = 0$ in (3), then we obtain $w_2(0,0,z) = 0$. Similarly, we get $w_1(x,0,0) = 0$ by sending y and z to 0 in (3). Thus

$$w_2 = h_1 x + h_2 y \quad \text{and} \quad w_1 = h_3 y + h_4 z$$

for some $h_1, h_2, h_3, h_4 \in S$. Then

$$f = (x^d - y^d)yh_3 + (x^d - y^d)zh_4 + x(x^{d-r}y^r - z^d)h_1 + y(x^{d-r}y^r - z^d)h_2$$

and so f is contained in T.

(a) Since $\Gamma(a,d) = \Omega(a,d) \cap \mathbb{A}^2$, it holds that $I(\Gamma(a,d))$ is equal to $J(\Omega(a,d))_*$, the dehomogenization of $J(\Omega(a,d))$ with respect to z. Thus we have

$$I(\Gamma(a,d)) = J(\Omega(a,d))_* = \langle x^d - y^d, (x^{d-r}y^r - 1)x, (x^{d-r}y^r - 1)y\rangle.$$

This completes the proof. □

6. Regularity, normality, multi-secant line and minimal graded free resolution of $S/J(\Omega(a,d))$

We fix a few notations, which are used in this section.

Definition and Remark 6.1. Let $\Gamma \subseteq \mathbb{P}^2$ be a finite set of d points and non-degenerate.

(a) Suppose $S(\Gamma) := S/J(\Gamma)$ has a minimal graded free resolution

$$0 \to \cdots \to F_i \to F_{i-1} \to \cdots \to F_0 \to S \to S(\Gamma) \to 0$$

with $F_i = \sum_j S(-a_{i,j})$. The *Castelnuovo-Mumford regularity* of Γ is reg(Γ):=max $\{a_{i,j} - i | i, j \geq 0\}$.

(b) We say that Γ is *k-normal* when the natural restriction map $S(\Gamma)_k \to \mathfrak{F}(\Gamma, \mathbb{K})$ is a surjective map where $\mathfrak{F}(\Gamma, \mathbb{K}) = \{f : \Gamma \to \mathbb{K} | f \text{ is a function}\}$ is the set of all $\mathbb{K}$-linear maps. We denote by $N(\Gamma)$ the smallest integer k such that Γ is k-normal.

(c) Let $h_\Gamma(t)$ be the Hilbert function of Γ in $\mathbb{P}^2$. Then the sequence $h_\Gamma(0), h_\Gamma(1), h_\Gamma(2), h_\Gamma(3), \cdots$ is monotone increasing and bounded above by d. So, Γ is k-normal if $\dim_{\mathbb{K}} S(\Gamma)_k = h_\Gamma(k) = d$.

(d) Let $m(\Gamma)$ be the minimum k such that $J(\Gamma)$ is generated by polynomials of degree k or less.

(e) Letting $\ell(\Gamma)$ denote the largest integer ℓ such that Γ admits a proper ℓ-secant line, then Γ always satisfies the following inequality, $\text{reg}(\Gamma) \geq m(\Gamma) \geq \ell(\Gamma)$.

Note that since there are two generators of $J(\Lambda(a,d))$, the finite minimal graded free resolution of $S(\Lambda(a,d)) := S/J(\Lambda(a,d))$ is well described in [2, Exercises 1C.1], so it is omitted.

Theorem 6.2. *Keep the notation in Theorem* 5.1, *let* $S(\Omega(a,d)) := S/J(\Omega(a,d))$. *Then,*

(a) *$S(\Omega(a,d))$ has a finite minimal graded free resolution as follows:*
If $d = 1$, then

(4)
$$0 \;\to\; S^2(-3) \;\overset{\varphi_2}{\to}\; S^3(-2) \;\overset{\varphi_1}{\to}\; S \;\to\; S(\Omega(a,1)) \;\to\; 0$$

with

$$\varphi_2 = \begin{pmatrix} z & y \\ -y & -y \\ -z & -x \end{pmatrix} \quad \text{and} \quad \varphi_1 = \begin{pmatrix} xy - yz & xz - yz & y^2 - yz \end{pmatrix}.$$

And if $d \geq 2$, then

(5)
$$0 \;\to\; S(-2d) \oplus S^2(-d-2) \;\overset{\varphi_2}{\to}\; S^4(-d-1) \;\overset{\varphi_1}{\to}\; S \;\to\; S(\Omega(a,d)) \;\to\; 0$$

with

$$\varphi_1 = \begin{pmatrix} (x^d - y^d)y & (x^d - y^d)z & (x^{d-r}y^r - z^d)x & (x^{d-r}y^r - z^d)y \end{pmatrix}$$

and

$$\varphi_2 = \begin{pmatrix} -x^{d-r}y^{r-1} & z & 0 \\ z^{d-1} & -y & 0 \\ x^{d-1} & 0 & -y \\ -y^{d-1} & 0 & x \end{pmatrix}.$$

(b) *The hilbert function* $h_{\Omega(a,d)}(t)$ *is given as follows:*
If $1 \leq t \leq d$,

$$h_{\Omega(a,d)}(t) = \frac{(t+1)(t+2)}{2}$$

and if $d+1 \leq t \leq 2d-2$,

$$h_{\Omega(a,d)}(t) = d^2 + 1 - \frac{(t-2d)(t-2d+3)}{2}.$$

In particular, $h_{\Omega(a,d)}(2d-2) = d^2+2$ *and hence*

$$h_{(\Omega(a,d))}(t) = d^2 + 2$$

for all $t \geq 2d-2$.

(c) *If* $d = 1$, *then Castelnuovo-Mumford regularity* $\operatorname{reg}(\Omega(a,1)) = 2$ *and if* $d \geq 2$, *then* $\operatorname{reg}(\Omega(a,d)) = 2d-1$.

(d) *If* $d = 1$, *then* $N(\Omega(a,1)) = 1, m(\Omega(a,1)) = 2$ *and* $\ell(\Omega(a,1)) = 2$. *And if* $d \geq 2$, *then* $N(\Omega(a,d)) = 2d-2, m(\Omega(a,d)) = d+1$ *and* $\ell(\Omega(a,d)) = d+1$.

Proof. (a) Case1. $d = 1$.
From Theorem 5.1, if $d = 1$, then $r = 1$. So we have

(6) $J(\Omega(a,1)) = \langle (x-y)y, (x-y)z, (y-z)x, (y-z)y \rangle = \langle (x-z)y, (x-y)z, (y-z)y \rangle.$

From (6), we can get φ_1. Now we only need to see

$$\operatorname{Ker}(\varphi_1) = \langle (z,-y,-z), (y,-y,x) \rangle.$$

Obviously, $(z,-y,-z),(y,-y,x)$ is contained in $\operatorname{Ker}(\varphi_1)$. Conversely, let $(f,g,h) \in \operatorname{Ker}(\varphi_1)$, then we get the equality

(7) $$\varphi_1(f,g,h) = f(xy-yz) + g(xz-yz) + h(y^2-yz) = 0.$$

Now, Sending y to 0 in (7), we obtain

$$g(x,0,z)xz = 0.$$

So we can write $g = \lambda y$ for some $\lambda \in S$. Substituting this to (7), we get the equation,

$$fy(x-z) + \lambda y(xz-yz) + hy(y-z) = 0$$

and hence,

(8) $$f(x-z) + \lambda(xz-yz) + h(y-z) = 0.$$

Again sending z to 0 in (8), we have

$$fx + hy = 0.$$

We will prove this in two cases.
Case 1-1. If f,h do not have z as a factor.
In this case, we can write $f = \lambda_0 y, h = -\lambda_0 x$ for some $\lambda_0 \in S$. Substituting this to (8), we have

(9) $$\lambda_0 y(x-z) + \lambda(xz-yz) - \lambda_0 x(y-z) = 0.$$

Again sending x to 0 in (9), we have $-\lambda_0 yz - \lambda yz = 0$, and hence, $\lambda = -\lambda_0$. Finally we can write

$$(f, g, h) = (\lambda_0 y, -\lambda_0 y, -\lambda_0 x) = \lambda_0(y, -y, -x).$$

Case 1-2. If f, h has z as a factor.
In this case, we can write $f = \lambda_0 z, h = \lambda_1 z$ for some $\lambda_0, \lambda_1 \in S$. Substituting this to (8), we have $\lambda_0 z(x - z) + \lambda z(x - y) + \lambda_1 z(y - z) = 0$, and hence

$$\lambda_0(x - z) + \lambda(x - y) + \lambda_1(y - z) = 0. \tag{10}$$

Sending x to 0 in (10), we get

$$-\lambda_0 z - \lambda y + \lambda_1(y - z) = 0. \tag{11}$$

If we send y to 0 in (11), we get $-\lambda_0 z - \lambda_1 z = 0$, and hence $\lambda_1 = -\lambda_0$. On the other hand, if we send z to 0 in (11), we get $-\lambda y + \lambda_1 y = 0$, and hence $\lambda = \lambda_1$. Finally we can write

$$(f, g, h) = (\lambda_0 z, -\lambda_0 y, -\lambda_0 z) = \lambda_0(z, -y, -z).$$

This concludes the proof of the first case.

Case2. $d \geq 2$.
φ_1 can be obtained from Theorem 5.1. So, we need to show that

$$\mathrm{Ker}(\varphi_1) = \langle(-x^{d-r}y^{r-1}, z^{d-1}, x^{d-1}, -y^{d-1}), (z, -y, 0, 0), (0, 0, -y, x)\rangle.$$

Obviously, $(-x^{d-r}y^{r-1}, z^{d-1}, x^{d-1}, -y^{d-1}), (z, -y, 0, 0), (0, 0, -y, x)$ is contained in $\mathrm{Ker}(\varphi_1)$.
Conversely, let $(f, g, h, w) \in \mathrm{Ker}(\varphi_1)$.
First, we will show that if $f = 0$, then $g = 0$ and $(f, g, h, w) = -\lambda(0, 0, -y, x)$ for some $-\lambda \in S$.
If $f = 0$, then we have a equation,

$$\varphi_1(f, g, h, w) = g(x^d - y^d)z + h(x^{d-r}y^r - z^d)x + w(x^{d-r}y^r - z^d)y = 0. \tag{12}$$

Sending y to 0 in (12), we get

$$g(x, 0, z)x^d z - h(x, 0, z)z^d x = 0.$$

We will prove this in two cases.
Case2-1-1. If g, h do not have y as a factor.
In this case, we can write $g = \lambda_0 z^{d-1}, h = \lambda_0 x^{d-1}$ for some $\lambda_0 \in S$.
Substituting this to (12), we have

$$\lambda_0(x^d - y^d)z^d + \lambda_0(x^{d-r}y^r - z^d)x^d + w(x^{d-r}y^r - z^d)y = 0. \tag{13}$$

Again sending z to 0 in (13), we have $\lambda_0 x^{2d-r}y^r + wx^{d-r}y^{r+1} = 0$, and hence, $x^{d-r}y^r(\lambda_0 x^d + wy) = 0$. This implies that $\lambda_0 x^d = -wy$, and therefore, λ_0 has y as a factor, and g also has y as a factor. Hence, this is a contradiction.
Case2-1-2. If g, h has y as a factor.
In this case, we can write $g = \lambda_0 y, h = \lambda_1 y$ for some $\lambda_0, \lambda_1 \in S$. Substituting this to (12), we have $\lambda_0 y(x^d - y^d)z + \lambda_1 y(x^{d-r}y^r - z^d)x + wy(x^{d-r}y^r - z^d) = 0$, and hence

$$\lambda_0(x^d - y^d)z + \lambda_1(x^{d-r}y^r - z^d)x + w(x^{d-r}y^r - z^d) = 0. \tag{14}$$

Again sending z to 0 in (14), then we have $\lambda_1 x^{d-r+1}y^r + wx^{d-r}y^r = 0$, so we can write $w = -\lambda_1 x$. Substituting this to (12), finally we get

$$\lambda_0 y(x^d - y^d)z = 0.$$

This implies that $\lambda_0 = 0$, and therefore, $g = 0$. So we can write

$$(f, g, h, w) = (0, 0, \lambda_1 y, -\lambda_1 x) = -\lambda_1(0, 0, -y, x).$$

In the case where $g = 0$, by a similar method, we can show that $f = 0$ and $(f, g, h, w) = \lambda(0, 0, -y, x)$ for some $\lambda \in S$.
Symmetrically, we will show that if $h = 0$ or $w = 0$, we get $(z, -y, 0, 0)$. If $w = 0$, then we have an equation,

$$\varphi_1(f, g, h, w) = f(x^d - y^d)y + g(x^d - y^d)z + h(x^{d-r}y^r - z^d)x = 0 \tag{15}$$

and by sending x to 0 in (15), we get

$$f(0, y, z)y^{d+1} + g(0, y, z)y^d z = 0.$$

So, we can write $f = \lambda z, g = -\lambda y$ for some $\lambda \in S$. Note that We can also write this in the case where f, g have x as factors, and the following assertion holds. Substituting f, g to (15), we get

$$h(x^{d-r}y^r - z) = 0.$$

This implies $h = 0$. So finally we can write,

$$(f, g, h, w) = (\lambda z, -\lambda y, 0, 0) = \lambda(z, -y, 0, 0).$$

In the case where $h = 0$, by a similar method, we can show that $w = 0$ and $(f, g, h, w) = \lambda(z, -y, 0, 0)$ for some $\lambda \in S$.

Now, let's assume that f, g, h, w are not all 0. Then we get the equality

$$\varphi_1(f, g, h, w) = f(x^d - y^d)y + g(x^d - y^d)z + h(x^{d-r}y^r - z^d)x + w(x^{d-r}y^r - z^d)y = 0. \tag{16}$$

We may assume that g, h do not have y factors, since if $g = g_1 y + g_2, h = h_1 y + h_2$ with g_2, h_2 have no y factors. Substituting this to (16), we get the equation,

$$(f + g_1 z)(x^d - y^d)y + g_2(x^d - y^d)z + h_2(x^{d-r}y^r - z^d)x + (w + h_1 x)(x^{d-r}y^r - z^d)y = 0.$$

Observe that g_2, h_2 have a no y factors, so we may assume that g, h have no y factors.
Now, Sending y to 0 in (16), we obtain

$$g(x, 0, z)x^d z - h(x, 0, z)z^d x = 0.$$

Since g, h don't have y factors, we can write

$$g = \lambda z^{d-1}, h = \lambda x^{d-1}$$

for some $\lambda \in S$. Substituting this to (16) and sending x to 0 in (16), we obtain

$$-f(0, y, z)y^{d+1} - \lambda z^d y^d - w(0, y, z)z^d y = 0. \tag{17}$$

We may assume that f has x factors, because if f has no x factors then

$$-fy^{d+1} = \lambda z^d y^d + wz^d y = z^d(\lambda y^d + wy)$$

then f has factor z^d and sending z to 0 in (16) and replace h to λx^{d-1}, then we have

$$w = -\lambda x^d y^{-1}.$$

This means that λ has y factors and so do h. This is contradiction to h has no y factors. So, we may assume that f has x factors.
We will prove this in two cases.

Case 2-2-1. w has x factors.
This case we obtain $\lambda = 0$ by (17), so we get the equation from (16),

$$f(x^d - y^d)y + w(x^{d-r}y^r - z^d)y = 0$$

and hence,

$$f = -\mu(x^{d-r}y^r - z^d), w = \mu(x^d - y^d)$$

for some $\mu \in S$. So we can express (f, g, h, w) by

$$(f, g, h, w) = -\mu y(-x^{d-r}y^{r-1}, z^{d-1}, x^{d-1}, -y^{d-1}) - \mu z^{d-1}(z, -y, 0, 0) - \mu x^{d-1}(0, 0, -y, x).$$

So we can get the result we want.
Case 2-2-2. w has no x factors.
In this case we have

$$w = -\lambda y^{d-1}$$

by (17). Replace g, h, w in (16), we can compute

$$f = -\lambda x^{d-r}y^{r-1}.$$

Hence,

$$(f, g, h, w) = \lambda(-x^{d-r}y^{r-1}, z^{d-1}, x^{d-1}, -y^{d-1}).$$

We complete the first assertion.

(b) Hilbert function of $S(\Omega(a, d))$ is calculated by applying [2, Corollary 1.2] to (5), we can get the Hilbert function $h_{\Omega(a,d)}(t)$.

(c) Castelnuovo-Mumford regularity $\operatorname{reg}(\Omega(a, 1)) = 2$ and $\operatorname{reg}(\Omega(a, d)) = 2d - 1$ for $d \geq 2$ can be obtained from (4) and (5), respectively.

(d) $N(\Omega(a, 1)) = 1$ and $N(\Omega(a, d)) = 2d - 2$ for all $d \geq 2$ can be obtained from (b) and $m(\Omega(a, d)) = d + 1$ for all $d \geq 1$ can be obtained from Theorem 5.1.
We know that $\Omega(a, d) = \Lambda(a, d) \cup \{[0, 0, 1], [1, 0, 0]\}$ and by theorem 5.1 we have

$$I(\Lambda(a, d)) = \langle x^d - y^d, x^{d-r}y^r - z^d \rangle$$

where $a = dn + r$ for $1 \leq r \leq d$.
Let $g := x^d - y^d$ and $h := x^{d-r}y^r - z^d$. It can be factorized as $g = L_1 \cdots L_d$ where L_i's are linear equations for $1 \leq r \leq d$. And since for each i, L_i and h meet at most d points, so g and h meet at most d^2 points. However, since $|\Lambda(a, d)| = d^2$, we can see that for each i, L_i and h meet at d different points. Therefore, we can get $\ell(\Lambda(a, d)) = d$. And L_i's meet at $[0, 0, 1]$ for all i, we deduce that $\ell(\Omega(a, d)) = d+1$. $\square$

Generally, it is known that $\operatorname{reg}(\Omega(a, d)) \geq m(\Omega(a, d)) \geq \ell(\Omega(a, d))$ (see e.g. inequality (1.1) in [5]), but this paper reveals the following relationship.
If $d = 1$, then $\operatorname{reg}(\Omega(a, 1)) = m(\Omega(a, 1)) = \ell(\Omega(a, 1)) = 2$. And if $d \geq 2$, then

$$\begin{array}{ccccc} \operatorname{reg}(\Omega(a,d)) & \geq & m(\Omega(a,d)) & = & \ell(\Omega(a,d)) \\ \| & & \| & & \| \\ 2d-1 & & d+1 & & d+1. \end{array}$$

REMARK 6.3. By plugging $U = S(\Lambda(a,d))$, $R = S$ in [6, Theorem 16.2], and using Theorem 6.2, we can obtain the Hilbert series of $S(\Omega(a,d))$ as follows.

$$\text{Hilb}_{S(\Omega(a,1))}(t) = \frac{2t^3 - 3t^2 + 1}{(1-t)^3}, (d = 1) \tag{18}$$

$$\text{Hilb}_{S(\Omega(a,d))}(t) = \frac{t^{2d} + 2t^{d+2} - 4t^{d+1} + 1}{(1-t)^3}, (d \geq 2) \tag{19}$$

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