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NOVEL INSIGHTS ON λ-STATISTICALLY CONVERGENT SEQUENCES IN NEUTROSOPHIC n-NORMED LINEAR SPACES

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Abstract. In this research paper, we introduce a thorough exploration of λ-statistical convergence and develop some of its interesting properties in neutrosophic n-normed linear spaces. Also, we define the notion of λ-statistical Cauchy sequence and prove that every neutrosophic n-normed linear space is λ-statistically complete. Furthermore, we study the concept of $[V, \lambda]$-summability associated with λ-statistical convergence with regard to neutrosophic n-norm. And we obtain the relation between statistical convergence and λ-statistical convergence within this specific framework.

1. Introduction

As a momentous generalization of ordinary convergence structure of sequences of real numbers, the concept of statistical convergence was first explored independently by Fast [9], Steinhaus [43] and Schoenberg [39]. After that, this significant concept is being brought up in different directions by various researchers like [4, 8, 14–16, 18–22, 27–32, 34, 35, 37, 38, 41, 44, 46, 47] and one may have many more references therein.

Zadeh [48] is the first prominent pioneering of the introduction of fuzzy set theory as an extension of classical set theory. Since then, it is being developed and applied in various branches of engineering and science, namely [3, 10, 12, 26]. Later on, this concept has been intriguingly generalized into new notions as its extension which may be referred as [1, 2, 7, 45]. As a comprehensive generalization of these concepts, Smarandache [42] defined a new idea named as neutrosophic set. Later on, Bera and Mahapatra explored the notion of neutrosophic soft linear space [5] and neutrosophic soft normed linear space [6]. In recent times, Kirişci and Şimşek [17] defined neutrosophic normed space and in this significant framework, different types of summability method were studied. In 2023, Murtaza et al. [36] defined the excellent idea named as neutrosophic 2-normed linear space which is a momentous generalization of neutrosophic normed space and studied statistical convergence and statistical completeness. In very recent time, Kumar et al. [24] introduced the concept of neutrosophic n-normed linear space and studied convergence structure and defined Cauchy sequence within this space.

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In 2000, Mursaleen [33] first introduced the concept of λ-statistical convergence of real sequences developed it alongside $[V, \lambda]$-summability and $[C, 1]$-summability method. After that, this significant idea is being carried out by various researchers in different directions. In this research article, we define and study the notion of λ-statistical convergence and λ-statistical Cauchy sequence with respect to neutrosophic n-norm and develop some of its interesting properties.

2. Preliminaries

In this section, we provide an overview of basic definitions and terminology which will be useful to describe our main results. Throughout the study $\mathbb{N}$ stands for the set of all natural numbers.

Definition 2.1. Let $\mathcal{K} \subset \mathbb{N}$. Then the natural density of $\mathcal{K}$, denoted by $\delta(\mathcal{K})$, is defined as

$$\delta(\mathcal{K}) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in \mathcal{K}\}|,$$

provided the limit exists, where the vertical bars denote the cardinality of the enclosed set.

A real sequence $\{w_k\}$ is named to be statistically convergent to $\xi \in \mathbb{R}$ (set of all reals) if for each $\varepsilon > 0$,

$$\delta\{k \in \mathbb{N} : |w_k - \xi| \geq \varepsilon\} = 0.$$

In this case, we write $\mathcal{S} - \lim w_k = \xi$ or $w_k \xrightarrow{\mathcal{S}} \xi$ and ξ is called $\mathcal{S}$-limit of $\{w_k\}$. Or, it can be restated as

$$\lim_n \frac{1}{n} |\{k \leq n : |w_k - \xi| \geq \varepsilon\}| = 0.$$

Let $\lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean of a sequence $w = \{w_k\}$ is defined by $\sigma_n(w) := \frac{1}{\lambda_n} \sum_{k\in\mathcal{I}_n} w_k$ where $\mathcal{I}_n = [n - \lambda_n + 1, n]$. A sequence $w = \{w_k\}$ is said to be (V, λ)-summable [25] to a number ξ if $\sigma_n(w) \to \xi$ as $n \to \infty$.

If $\lambda_n = n$, then (V, λ)-summability reduces to $(C, 1)$-summability. We denote

$$[V, \lambda] = \left\{ w = \{w_k\} : \exists\ \xi \in \mathbb{R},\ \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in\mathcal{I}_n} |w_k - \xi| = 0 \right\}$$

and

$$[C, 1] = \left\{ w = \{w_k\} : \exists\ \xi \in \mathbb{R},\ \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |w_k - \xi| = 0 \right\}$$

for the sets of sequences $w = \{w_k\}$ which are strongly (V, λ)-summable and strongly $(C, 1)$-summable to ξ, i.e. $w_k \to \xi[V, \lambda]$ and $w_k \to \xi[C, 1]$.

Let $\mathcal{K} \subset \mathbb{N}$. Then the λ-density of $\mathcal{K}$, denoted by $\delta_\lambda(\mathcal{K})$, is defined as

$$\delta_\lambda(\mathcal{K}) = \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in \mathcal{I}_n : k \in \mathcal{K}\}|,$$

provided the limit exists, where the vertical bars denote the cardinality of the enclosed set. It is worth being noted that λ-density of a finite set is always zero. Also, $\mathcal{A} \subseteq \mathcal{B} \implies \delta_\lambda(\mathcal{A}) \leq \delta_\lambda(\mathcal{B})$.

A sequence $\{w_k\} \in \mathbb{R}$ is named to be λ-statistically convergent [33] or $\mathcal{S}_\lambda$-convergent to $\xi \in \mathbb{R}$ if for $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in \mathcal{I}_n : |w_k - \xi| \geq \varepsilon\}| = 0,$$

i.e., $\delta_\lambda(\mathcal{K}_\varepsilon) = 0$ where $\mathcal{K}_\varepsilon = \{k \in \mathcal{I}_n : |w_k - \xi| \geq \varepsilon\}$.

Definition 2.2. [40] A binary operation $\boxdot : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$, where $\mathcal{J} = [0,1]$ is named to be a continuous t-norm if for each $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathcal{J}$, the below conditions hold:

1. $\boxdot$ is associative and commutative;
2. $\boxdot$ is continuous;
3. $\nu_1 \boxdot 1 = \nu_1$ for all $\nu_1 \in \mathcal{J}$;
4. $\nu_1 \boxdot \nu_2 \leq \nu_3 \boxdot \nu_4$ whenever $\nu_1 \leq \nu_3$ and $\nu_2 \leq \nu_4$.

Definition 2.3. [40] A binary operation $\oplus : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$, where $\mathcal{J} = [0,1]$ is named to be a continuous t-conorm if for each $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathcal{J}$, the below conditions hold:

1. $\oplus$ is associative and commutative;
2. $\oplus$ is continuous;
3. $\nu_1 \oplus 0 = \nu_1$ for all $\nu_1 \in \mathcal{J}$;
4. $\nu_1 \oplus \nu_2 \leq \nu_3 \oplus \nu_4$ whenever $\nu_1 \leq \nu_3$ and $\nu_2 \leq \nu_4$.

Example 2.4. [23] The continuous t-norms are $\nu_1 \boxdot \nu_2 = \min\{\nu_1, \nu_2\}$ and $\nu_1 \boxdot \nu_2 = \nu_1.\nu_2$. On the other hand, continuous t-conorms are $\nu_1 \oplus \nu_2 = \max\{\nu_1, \nu_2\}$ and $\nu_1 \oplus \nu_2 = \nu_1 + \nu_2 - \nu_1.\nu_2$.

Definition 2.5. [11] Let $n \in \mathbb{N}$ and $\mathcal{W}$ be a real vector space having dimension $d \geq n$ (d is finite or infinite). A real valued function $\|\cdot, \ldots, \cdot\|$ on $\underbrace{\mathcal{W} \times \mathcal{W} \times \ldots \times \mathcal{W}}_{n\ times} = \mathcal{W}^n$, gratifying the below four axioms:

1. $\|w_1, w_2, \ldots, w_n\| = 0$ if and only if $w_1, w_2, \ldots, w_n$ are linearly dependent;
2. $\|w_1, w_2, \ldots, w_n\|$ remains invariant under any permutation of $w_1, w_2, \ldots, w_n$;
3. $\|w_1, w_2, \ldots, w_{n-1}, \kappa w_n\| = |\kappa|\, \|w_1, w_2, \ldots, w_{n-1}, w_n\|$ for $\kappa \in \mathbb{R}$ (set of real numbers);
4. $\|w_1, w_2, \ldots, w_{n-1}, \tau + \omega\| \leq \|w_1, w_2, \ldots, w_{n-1}, \tau\| + \|w_1, w_2, \ldots, w_{n-1}, \omega\|$.

is called an n-norm on $\mathcal{W}$ and the pair $(\mathcal{W}, \|\cdot, \ldots, \cdot\|)$ is named to be an n-normed linear space.

As an illustration of n-normed linear space we take $\mathcal{W} = \mathbb{R}^n$ equipped with the Euclidean norm

$$\|w_1, w_2, \ldots, w_n\| = abs \left(\begin{vmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{vmatrix} \right)$$

where $w_i = (w_{i1}, w_{i2}, \ldots, w_{in}) \in \mathbb{R}^n$. For instance, we get $\|w_1, w_2, \ldots, w_n\| \geq 0$ in an n-normed linear space.

Definition 2.6. [24] Let $\mathcal{W}$ be a vector space over $\mathcal{F}$ and $\boxdot$ and $\oplus$ be continuous t-norm and t-conorm respectively. Let $\Im, \Re, \wp$ be the functions from $\mathcal{W}^n \times (0, \infty)$ to $[0,1]$. Then a six tuple $(\mathcal{W}, \Im, \Re, \wp, \boxdot, \oplus)$ is named to be a neutrosophic n-normed linear space (in short Nn-NLS), $(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) \in \mathcal{W}^n \times (0, \infty) \to [0,1]$, if the below conditions hold:

1. $\Im(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) + \Re(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) + \wp(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) \leq 3$;
2. $\Im(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) > 0$;
3. $\Im(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) = 1$ if and only if w_j are linearly dependent, $1 \leq j \leq n$;
4. $\Im(w_1, w_2, \dots, w_{n-1}, w_n; \zeta)$ is invariant under any permutation of $w_1, w_2, \dots, w_n$;
5. $\Im(w_1, w_2, \dots, w_{n-1}, \kappa w_n; \zeta) = \Im(w_1, w_2, \dots, w_{n-1}, w_n; \frac{\zeta}{|\kappa|})$, $\kappa \neq 0$ and $\kappa \in \mathcal{F}$;
6. $$\begin{aligned}\Im(w_1, w_2, \dots, w_{n-1}, w_n + w_n'; \zeta + \tau) &\geq \Im(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) \\ &\boxdot \Im(w_1, w_2, \dots, w_{n-1}, w_n'; \tau);\end{aligned}$$
7. $\Im(w_1, w_2, \dots, w_{n-1}, w_n; \zeta)$ is non-decreasing continuous in ζ;
8. $\lim_{\zeta \to \infty} \Im(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) = 1$ and $\lim_{\zeta \to 0} \Im(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) = 0$;
9. $\Re(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) > 0$;
10. $\Re(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) = 0$ if and only if w_j are linearly dependent, $1 \leq j \leq n$;
11. $\Re(w_1, w_2, \dots, w_{n-1}, w_n; \zeta)$ is invariant under any permutation of $w_1, w_2, \dots, w_n$;
12. $\Re(w_1, w_2, \dots, w_{n-1}, \kappa w_n; \zeta) = \Re(w_1, w_2, \dots, w_{n-1}, w_n; \frac{\zeta}{|\kappa|})$, $\kappa \neq 0$ and $\kappa \in \mathcal{F}$;
13. $$\begin{aligned}\Re(w_1, w_2, \dots, w_{n-1}, w_n + w_n'; \zeta + \tau) &\leq \Re(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) \\ &\oplus \Re(w_1, w_2, \dots, w_{n-1}, w_n'; \tau);\end{aligned}$$
14. $\Re(w_1, w_2, \dots, w_{n-1}, w_n; \zeta)$ is non-increasing continuous in ζ;
15. $\lim_{\zeta \to \infty} \Re(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) = 0$ and $\lim_{\zeta \to 0} \Re(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) = 1$;
16. $\wp(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) > 0$;
17. $\wp(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) = 0$ if and only if w_j are linearly dependent, $1 \leq j \leq n$;
18. $\wp(w_1, w_2, \dots, w_{n-1}, w_n; \zeta)$ is invariant under any permutation of $w_1, w_2, \dots, w_n$;
19. $\wp(w_1, w_2, \dots, w_{n-1}, \kappa w_n; \zeta) = \wp(w_1, w_2, \dots, w_{n-1}, w_n; \frac{\zeta}{|\kappa|})$, $\kappa \neq 0$ and $\kappa \in \mathcal{F}$;
20. $$\begin{aligned}\wp(w_1, w_2, \dots, w_{n-1}, w_n + w_n'; \zeta + \tau) &\leq \wp(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) \\ &\oplus \wp(w_1, w_2, \dots, w_{n-1}, w_n'; \tau);\end{aligned}$$
21. $\wp(w_1, w_2, \dots, w_{n-1}, w_n; \zeta)$ is non-increasing continuous in ζ;
22. $\lim_{\zeta \to \infty} \wp(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) = 0$ and $\lim_{\zeta \to 0} \wp(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) = 1$;

In the sequel, we shall use the notation $\mathcal{X}$ for neutrosophic n-normed linear space instead of $(\mathcal{W}, \Im, \Re, \wp, \boxdot, \oplus)$ and we denote $\mathcal{N}_n$ to mean neutrosophic n-norm on $\mathcal{X}$.

Example 2.7. [24] Let $(\mathcal{W}, \|w_1, w_2, \dots, w_n\|)$ be an n-normed linear space. Also, let $\nu_1 \boxdot \nu_2 = \min(\nu_1, \nu_2)$ and $\nu_1 \oplus \nu_2 = \max(\nu_1, \nu_2)$ for every $\nu_1, \nu_2 \in [0,1]$. If we define $\Im, \Re$ and $\wp$ as

$$\Im(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) = \frac{\zeta}{\zeta + \|w_1, w_2, \dots, w_{n-1}, w_n\|}$$

$$\Re(w_1, w_2, \dots, w_{n-1}, w_n; \zeta) = \frac{\|w_1, w_2, \dots, w_{n-1}, w_n\|}{\zeta + \|w_1, w_2, \dots, w_{n-1}, w_n\|}$$

$$\text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = \frac{\|w_1, w_2, \ldots, w_{n-1}, w_n\|}{\zeta}.$$

Then $(\mathcal{W}, \Im, \Re, \wp, \boxdot, \oplus)$ is a neutrosophic n-normed linear space.

DEFINITION 2.8. [24] Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then $\{w_k\}$ is named to be convergent to $\upsilon \in \mathcal{W}$ with respect to $\mathcal{N}_n$ if for every $\sigma > 0, \zeta > 0$ and $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$ there exists $k_0 \in \mathbb{N}$ such that $\Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) > 1 - \sigma$ and $\Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma$, $\wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma$ for all $k \geq k_0$. In this scenario, it is denoted as $\mathcal{N}_n - \lim w_k = \upsilon$ or $w_k \xrightarrow{\mathcal{N}_n} \upsilon$.

DEFINITION 2.9. [24] Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then $\{w_k\}$ is named to be Cauchy sequence with respect to $\mathcal{N}_n$ (in short $\mathcal{N}_n$-Cauchy) if for every $\sigma > 0, \zeta > 0$ there exists $k_0 \in \mathbb{N}$ such that $\Im(w_1, w_2, \ldots, w_{n-1}, w_k - w_m; \zeta) > 1 - \sigma$ and $\Re(w_1, w_2, \ldots, w_{n-1}, w_k - w_m; \zeta) < \sigma$, $\wp(w_1, w_2, \ldots, w_{n-1}, w_k - w_m; \zeta) < \sigma$ for all $k, m \geq k_0$.

THEOREM 2.10. [24] Let $\{w_k\}$ be a sequence in $\mathcal{W}$. Then $\{w_k\}$ is convergent in $(\mathcal{W}, \|\cdot, \ldots, \cdot\|)$ iff $\{w_k\}$ is convergent in $\mathcal{X}$ with respect to neutrosophic n-norm as defined in Example 2.7.

DEFINITION 2.11. [13] Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then $\{w_k\}$ is named to be statistically convergent to $\upsilon \in \mathcal{W}$ with respect to $\mathcal{N}_n$ (in short $\mathcal{S}(\mathcal{N}_n)$-convergence) if for every $\sigma \in (0, 1), \zeta > 0$ and $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$ such that

$$\begin{gathered}\delta(\{k \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma \\ \text{or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \\ \text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\}) = 0.\end{gathered}$$

or, it can be restated as

$$\begin{gathered}\lim_{s \to \infty} \tfrac{1}{s} |\{k \leq s : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma \\ \text{or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \\ \text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\}| = 0.\end{gathered}$$

In this scenario, it is denoted as $\mathcal{S}(\mathcal{N}_n) - \lim w_k = \upsilon$ or $w_k \xrightarrow{\mathcal{S}(\mathcal{N}_n)} \upsilon$. And, υ is called $\mathcal{S}(\mathcal{N}_n)$-limit of $\{w_k\}$.

3. λ-statistical convergence in Nn-NLS

In this section, we define the concept of λ-statistical convergence with respect to $\mathcal{N}_n$ and explored some interesting results related to this notion.

DEFINITION 3.1. Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then $\{w_k\}$ is named to be λ-statistically convergent to $\upsilon \in \mathcal{W}$ with respect to $\mathcal{N}_n$ (in short $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergent) if for every $\sigma \in (0, 1)$,
$\zeta > 0$ and nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$ such that

$$\begin{gathered}\lim_{n \to \infty} \tfrac{1}{\lambda_n} |\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma \\ \text{or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \\ \text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\}| = 0,\end{gathered}$$

i.e., $\delta_\lambda(\mathcal{A}(\sigma,\zeta)) = 0$ where

$$\begin{gathered}\mathcal{A}(\sigma,\zeta) = \{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \le 1-\sigma \\ \text{or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma \\ \text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\}.\end{gathered}$$

In this scenario, it is denoted as $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$ or $w_k \xrightarrow{\mathcal{S}_\lambda(\mathcal{N}_n)} \upsilon$. And, υ is called $\mathcal{S}_\lambda(\mathcal{N}_n)$-limit of $\{w_k\}$.

REMARK 3.2. From Definition 3.1, we have if $\lambda_n = n$, then the notion of $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergence coincides with the notion of $\mathcal{S}(\mathcal{N}_n)$-convergence.

Based on the Definition 3.1 and using the concept of λ-density, we obtain the following lemma.

LEMMA 3.3. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, for every $\sigma \in (0,1), \zeta > 0$ and nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$, the below properties are gratified:*

1. $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$;

2.
$$\begin{aligned}&\delta_\lambda(\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \le 1-\sigma\}) \\ &= \delta_\lambda(\{k \in \mathcal{I}_n : \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\}) \\ &= \delta_\lambda(\{k \in \mathcal{I}_n : \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\}) = 0;\end{aligned}$$

3.
$$\begin{gathered}\delta_\lambda\left(\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) > 1-\sigma \right.\\ \textit{and } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma, \\ \left.\wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma\}\right) = 1;\end{gathered}$$

4.
$$\begin{aligned}&\delta_\lambda(\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) > 1-\sigma\}) \\ &= \delta_\lambda(\{k \in \mathcal{I}_n : \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma\}) \\ &= \delta_\lambda(\{k \in \mathcal{I}_n : \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma\}) = 1;\end{aligned}$$

5.
$$\begin{gathered}\mathcal{S}_\lambda(\mathcal{N}_n) - \lim_k \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) = 1 \textit{ and} \\ \mathcal{S}_\lambda(\mathcal{N}_n) - \lim_k \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) = 0, \\ \mathcal{S}_\lambda(\mathcal{N}_n) - \lim_k \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) = 0.\end{gathered}$$

THEOREM 3.4. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. If $\mathcal{S}_\lambda(\mathcal{N}_n)$-limit of $\{w_k\}$ exists, then it is uniquely determined.*

Proof. If possible, let $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon_1$ and $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon_2$ where $\upsilon_1 \ne \upsilon_2$. Let $\sigma \in (0,1)$ be given. Choose $\varpi \in (0,1)$ such that $(1-\varpi) \boxdot (1-\varpi) > 1-\sigma$ and $\varpi \oplus \varpi < \sigma$. For any $\zeta > 0$ and nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$, we define the following sets:

$$\mathcal{A}_{\Im,1}(\varpi,\zeta) = \left\{k \in \mathcal{I}_n : \Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}\right) \le 1-\varpi\right\};$$

$$\mathcal{A}_{\Im,2}(\varpi,\zeta) = \left\{k \in \mathcal{I}_n : \Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_2; \frac{\zeta}{2}\right) \le 1-\varpi\right\};$$

$$\mathcal{A}_{\Re,1}(\varpi,\zeta) = \left\{k \in \mathcal{I}_n : \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}\right) \ge \varpi\right\};$$

$$\mathcal{A}_{\Re,2}(\varpi,\zeta) = \left\{k \in \mathcal{I}_n : \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_2; \frac{\zeta}{2}\right) \ge \varpi\right\};$$

$$\mathcal{A}_{\wp,1}(\varpi,\zeta)=\left\{k\in\mathcal{I}_n:\wp\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon_1;\frac{\zeta}{2}\right)\geq\varpi\right\};$$
$$\mathcal{A}_{\wp,2}(\varpi,\zeta)=\left\{k\in\mathcal{I}_n:\wp\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon_2;\frac{\zeta}{2}\right)\geq\varpi\right\}.$$

Since $\mathcal{S}_\lambda(\mathcal{N}_n)-\lim w_k=\upsilon_1$, by Lemma 3.3, we have

$$\delta_\lambda(\mathcal{A}_{\Im,1}(\varpi,\zeta))=\delta_\lambda(\mathcal{A}_{\Re,1}(\varpi,\zeta))=\delta_\lambda(\mathcal{A}_{\wp,1}(\varpi,\zeta))=0. \tag{1}$$

Again, since $\mathcal{S}_\lambda(\mathcal{N}_n)-\lim w_k=\upsilon_2$, so

$$\delta_\lambda(\mathcal{A}_{\Im,2}(\varpi,\zeta))=\delta_\lambda(\mathcal{A}_{\Re,2}(\varpi,\zeta))=\delta_\lambda(\mathcal{A}_{\wp,2}(\varpi,\zeta))=0. \tag{2}$$

Let

$$\begin{aligned}\mathcal{A}_{(\Im,\Re,\wp)}(\sigma,\zeta)&=[\mathcal{A}_{\Im,1}(\varpi,\zeta)\cup\mathcal{A}_{\Im,2}(\varpi,\zeta)]\cap[\mathcal{A}_{\Re,1}(\varpi,\zeta)\cup\mathcal{A}_{\Re,2}(\varpi,\zeta)]\\&\cap[\mathcal{A}_{\wp,1}(\varpi,\zeta)\cup\mathcal{A}_{\wp,2}(\varpi,\zeta)].\end{aligned}$$

Then $\delta_\lambda\left(\mathcal{A}_{(\Im,\Re,\wp)}(\sigma,\zeta)\right)=0$ i.e., $\delta_\lambda\left(\mathbb{N}\setminus\mathcal{A}_{(\Im,\Re,\wp)}(\sigma,\zeta)\right)=1$. So, there exists an element $k\in\mathbb{N}\setminus\mathcal{A}_{(\Im,\Re,\wp)}(\sigma,\zeta)$. Hence, there are three cases to consider:

Case-1 : If $k\in\mathbb{N}\setminus(\mathcal{A}_{\Im,1}(\varpi,\zeta)\cup\mathcal{A}_{\Im,2}(\varpi,\zeta))$, then

$$\begin{aligned}&\Im(w_1,w_2,\dots,w_{n-1},\upsilon_1-\upsilon_2;\zeta)\\&\geq\Im\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon_1;\frac{\zeta}{2}\right)\boxdot\Im\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon_2;\frac{\zeta}{2}\right)\\&>(1-\varpi)\boxdot(1-\varpi)>1-\sigma.\end{aligned}$$

For arbitrariness of $\sigma\in(0,1)$, $\Im(w_1,w_2,\dots,w_{n-1},\upsilon_1-\upsilon_2;\zeta)=1$ which yields $\upsilon_1=\upsilon_2$.

Case-2 : If $k\in\mathbb{N}\setminus(\mathcal{A}_{\Re,1}(\varpi,\zeta)\cup\mathcal{A}_{\Re,2}(\varpi,\zeta))$, then

$$\begin{aligned}&\Re(w_1,w_2,\dots,w_{n-1},\upsilon_1-\upsilon_2;\zeta)\\&\leq\Re\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon_1;\frac{\zeta}{2}\right)\oplus\Re\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon_2;\frac{\zeta}{2}\right)\\&<\varpi\oplus\varpi<\sigma.\end{aligned}$$

For arbitrariness of $\sigma\in(0,1)$, $\Re(w_1,w_2,\dots,w_{n-1},\upsilon_1-\upsilon_2;\zeta)=0$ for every $\zeta>0$, which yields $\upsilon_1=\upsilon_2$.

Case-3 : If $k\in\mathbb{N}\setminus(\mathcal{A}_{\wp,1}(\varpi,\zeta)\cup\mathcal{A}_{\wp,2}(\varpi,\zeta))$, then by similar technique, we can have

$$\wp(w_1,w_2,\dots,w_{n-1},\upsilon_1-\upsilon_2;\zeta)=0\text{ i.e., }\upsilon_1=\upsilon_2.$$

Therefore, in the above three cases, we get $\upsilon_1=\upsilon_2$. Hence $\mathcal{S}_\lambda(\mathcal{N}_n)$-limit of $\{w_k\}$ is unique. □

Now, we find out the relation between $\mathcal{N}_n$-convergence and $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergence.

THEOREM 3.5. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. If $\mathcal{N}_n-\lim w_k=\upsilon$ then $\mathcal{S}_\lambda(\mathcal{N}_n)-\lim w_k=\upsilon$.*

Proof. Let $\mathcal{N}_n-\lim w_k=\upsilon$. Then, for every $\sigma>0,\zeta>0$ and nonzero $w_1,w_2,\dots,w_{n-1}\in\mathcal{W}$ there exists $k_0\in\mathbb{N}$ such that

$$\Im(w_1,w_2,\dots,w_{n-1},w_k-\upsilon;\zeta)>1-\sigma\quad\text{and}\quad\Re(w_1,w_2,\dots,w_{n-1},w_k-\upsilon;\zeta)<\sigma,$$
$$\wp(w_1,w_2,\dots,w_{n-1},w_k-\upsilon;\zeta)<\sigma$$

for all $k \geq k_0$. Therefore, the set

$$\{k \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma \text{ or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\}$$

contains finite number of terms. Since λ-density of a finite set is zero,

$$\delta_\lambda(\{k \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma \text{ or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\}) = 0.$$

So, $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$. □

But, in general, the converse of the Theorem 3.5 is not true which can be illustrated as given below.

EXAMPLE 3.6. Let $\mathcal{W} = \mathbb{R}^n$ with

$$\|w_1, w_2, \ldots, w_n\| = abs\left(\begin{vmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{vmatrix}\right)$$

where $w_i = (w_{i1}, w_{i2}, \ldots, w_{in}) \in \mathbb{R}^n$. We take continuous t-norm and t-conorm as $\nu_1 \boxdot \nu_2 = \min\{\nu_1, \nu_2\}$ and $\nu_1 \oplus \nu_2 = \max\{\nu_1, \nu_2\}$ for every $\nu_1, \nu_2 \in [0, 1]$. We take Nn-NLS as defined in Example 2.7. Now, we define a sequence $\{w_k\} \in \mathcal{W}$ by

$$w_k = \begin{cases} (k, \underbrace{0, \ldots, 0}_{n-1 \text{ times}}), & \text{if } n - \left[\sqrt{\lambda_n}\right] + 1 \leq k \leq n, \\ (\underbrace{0, 0, \ldots, 0}_{n \text{ times}}) = \mathbf{0}, & \text{otherwise} \end{cases}.$$

Then for any $\sigma \in (0, 1)$, $\zeta > 0$ and nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$, we have

$$\begin{aligned} \mathcal{A}(\sigma, \zeta) =& \{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \mathbf{0}; \zeta) \leq 1 - \sigma \text{ or} \\ & \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \mathbf{0}; \zeta) \geq \sigma \text{ and} \\ & \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \mathbf{0}; \zeta) \geq \sigma\} \\ =& \Big\{k \in \mathcal{I}_n : \|w_1, w_2, \ldots, w_{n-1}, w_k - \mathbf{0}\| \geq \frac{\zeta\sigma}{1 - \sigma} > 0 \text{ or} \\ & \|w_1, w_2, \ldots, w_{n-1}, w_k - \mathbf{0}\| \geq \zeta\sigma > 0\} \\ =& \left\{k \in \mathcal{I}_n : w_k = (k, \underbrace{0, \ldots, 0}_{n-1 \text{ times}})\right\} \\ =& \left\{k \in \mathcal{I}_n : n - \left[\sqrt{\lambda_n}\right] + 1 \leq k \leq n\right\}. \end{aligned}$$

So,

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\mathcal{A}(\sigma, \zeta)| \leq \lim_{n \to \infty} \frac{\left[\sqrt{\lambda_n}\right]}{\lambda_n} = 0.$$

Therefore, $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \mathbf{0}$. However, the sequence $\{w_k\}$ is not ordinary convergent with respect to n-norm $\|\cdot, \ldots, \cdot\|$ and hence, using Theorem 2.10, the sequence $\{w_k\}$ is not ordinary convergent with respect to $\mathcal{N}_n$.

Now, We discuss on the linearity of the following:

$$\mathcal{ST}_\lambda(\mathcal{N}_n) = \{\{w_k\} \in \mathcal{W} : \exists\ \upsilon \in \mathcal{W},\ \mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon\}.$$

THEOREM 3.7. *$\mathcal{ST}_\lambda(\mathcal{N}_n)$ is a linear space.*

Proof. First we suppose that $\{w_k\}$ and $\{l_k\}$ belong to $\mathcal{ST}_\lambda(\mathcal{N}_n)$. To prove the Theorem, we have to show that $\{w_k\} + \{l_k\} = \{w_k + l_k\} \in \mathcal{ST}_\lambda(\mathcal{N}_n)$ and for any $\kappa \in \mathbb{R}$, $\kappa\{w_k\} = \{\kappa w_k\}$ $(\forall k \in \mathbb{N})$ $\in \mathcal{ST}_\lambda(\mathcal{N}_n)$.

Let $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon_1$ and $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim l_k = \upsilon_2$. Let $\sigma \in (0,1)$ be given. Choose $\varpi \in (0,1)$ such that $(1-\varpi) \boxdot (1-\varpi) > 1-\sigma$ and $\varpi \oplus \varpi < \sigma$. For any $\zeta > 0$ and nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$, we define the following sets:

$$\begin{aligned}
\mathcal{A}(\sigma,\zeta) = \{k \in \mathcal{I}_n : \ &\Im(w_1, w_2, \ldots, w_{n-1}, w_k + l_k - (\upsilon_1+\upsilon_2); \zeta) > 1-\sigma \\
&\text{and } \Re(w_1, w_2, \ldots, w_{n-1}, w_k + l_k - (\upsilon_1+\upsilon_2); \zeta) < \sigma, \\
&\wp(w_1, w_2, \ldots, w_{n-1}, w_k + l_k - (\upsilon_1+\upsilon_2); \zeta) < \sigma\}; \\
\mathcal{A}_1(\varpi,\zeta) = \{k \in \mathcal{I}_n : \ &\Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_1; \tfrac{\zeta}{2}) > 1-\varpi \\
&\text{and } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_1; \tfrac{\zeta}{2}) < \varpi, \\
&\wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_1; \tfrac{\zeta}{2}) < \varpi\}, \\
\mathcal{A}_2(\varpi,\zeta) = \{k \in \mathcal{I}_n : \ &\Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_2; \tfrac{\zeta}{2}) > 1-\varpi \\
&\text{and } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_2; \tfrac{\zeta}{2}) < \varpi, \\
&\wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_2; \tfrac{\zeta}{2}) < \varpi\}.
\end{aligned}$$

By the hypothesis, $\mathcal{A}_1(\varpi,\zeta) \cap \mathcal{A}_2(\varpi,\zeta) \neq \emptyset$. So, let $k \in \mathcal{A}_1(\varpi,\zeta) \cap \mathcal{A}_2(\varpi,\zeta)$. Then, we can have

$$\begin{aligned}
&\Im(w_1, w_2, \ldots, w_{n-1}, w_k + l_k - (\upsilon_1+\upsilon_2); \zeta) \\
&\geq \Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}\right) \boxdot \Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_2; \frac{\zeta}{2}\right) \\
&> (1-\varpi) \boxdot (1-\varpi) > 1-\sigma
\end{aligned}$$

and

$$\begin{aligned}
&\Re(w_1, w_2, \ldots, w_{n-1}, w_k + l_k - (\upsilon_1+\upsilon_2); \zeta) \\
&\leq \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}\right) \oplus \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_2; \frac{\zeta}{2}\right) \\
&< \varpi \oplus \varpi < \sigma; \\
&\wp(w_1, w_2, \ldots, w_{n-1}, w_k + l_k - (\upsilon_1+\upsilon_2); \zeta) \\
&\leq \wp\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}\right) \oplus \wp\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_2; \frac{\zeta}{2}\right) \\
&< \varpi \oplus \varpi < \sigma.
\end{aligned}$$

Therefore, we see $\mathcal{A}(\sigma,\zeta) \subseteq \mathcal{A}_1(\varpi,\zeta) \cap \mathcal{A}_2(\varpi,\zeta)$, i.e., $\mathcal{A}_1^c(\varpi,\zeta) \cup \mathcal{A}_2^c(\varpi,\zeta) \subseteq \mathcal{A}^c(\sigma,\zeta)$. So,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, (w_k + l_k) - (\upsilon_1+\upsilon_2); \zeta) \leq 1-\sigma \\
&\quad \text{or } \Re(w_1, w_2, \ldots, w_{n-1}, (w_k + l_k) - (\upsilon_1+\upsilon_2); \zeta) \geq \sigma \\
&\quad \text{and } \wp(w_1, w_2, \ldots, w_{n-1}, (w_k + l_k) - (\upsilon_1+\upsilon_2); \zeta) \geq \sigma\}| \\
&\leq \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}) \leq 1-\varpi \\
&\quad \text{or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}) \geq \varpi
\end{aligned}$$

$$\text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}) \geq \varpi\}|$$
$$+ \lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon_2; \frac{\zeta}{2}) \leq 1 - \varpi$$
$$\text{or } \Re(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon_2; \frac{\zeta}{2}) \geq \varpi$$
$$\text{and } \wp(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon_2; \frac{\zeta}{2}) \geq \varpi\}| = 0.$$

Since sequence of positive numbers never converges to a negative number, $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k + l_k = \upsilon_1 + \upsilon_2$. Therefore $\{w_k\} + \{l_k\} \in \mathcal{ST}_\lambda(\mathcal{N}_n)$.
Suppose that $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$. If $\kappa = 0$, then it is obvious $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim \kappa w_k = \kappa\upsilon$. So, let $\kappa \neq 0$. We take the following sets:

$$\begin{gathered}\mathcal{A}(\sigma,\zeta) = \{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, \kappa w_k - \kappa\upsilon; \zeta) > 1 - \sigma \\ \text{and } \Re(w_1, w_2, \ldots, w_{n-1}, \kappa w_k - \kappa\upsilon; \zeta) < \sigma, \\ \wp(w_1, w_2, \ldots, w_{n-1}, \kappa w_k - \kappa\upsilon; \zeta) < \sigma\},\end{gathered}$$

$$\begin{gathered}\mathcal{B}(\sigma,\zeta) = \{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) > 1 - \sigma \\ \text{and } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma, \\ \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma\}.\end{gathered}$$

By the hypothesis, $\delta_\lambda(\mathcal{B}(\sigma,\zeta)) = 1$. So, there is an element $k \in \mathcal{B}(\sigma,\zeta)$. Then, we have

$$\begin{aligned}\Im(w_1, w_2, \ldots, w_{n-1}, \kappa w_k - \kappa\upsilon; \zeta) &= \Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{|\kappa|}\right) \\ &\geq \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \boxdot \Im\left(\underbrace{0, \cdots, 0}_{n\ times}; \frac{\zeta}{|\kappa|} - \zeta\right) \\ &= \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \boxdot 1 \\ &= \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \\ &> 1 - \sigma \\ &\text{and} \\ \Re(w_1, w_2, \ldots, w_{n-1}, \kappa w_k - \kappa\upsilon; \zeta) &= \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{|\kappa|}\right) \\ &\leq \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \oplus \Re\left(\underbrace{0, \cdots, 0}_{n\ times}; \frac{\zeta}{|\kappa|} - \zeta\right) \\ &= \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \oplus 0 \\ &= \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \\ &< \sigma; \\ \wp(w_1, w_2, \ldots, w_{n-1}, \kappa w_k - \kappa\upsilon; \zeta) &= \wp\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{|\kappa|}\right) \\ &\leq \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \oplus \wp\left(\underbrace{0, \cdots, 0}_{n\ times}; \frac{\zeta}{|\kappa|} - \zeta\right)\end{aligned}$$

$$= \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \oplus 0$$
$$= \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta)$$
$$< \sigma.$$

This shows that $k \in \mathcal{B}(\sigma, \zeta) \subseteq \mathcal{A}(\sigma, \zeta) \implies \mathcal{A}^c(\sigma, \zeta) \subseteq \mathcal{B}^c(\sigma, \zeta)$. This gives

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, \kappa w_k - \kappa\upsilon; \zeta) \leq 1 - \sigma$$
$$\text{or } \Re(w_1, w_2, \ldots, w_{n-1}, \kappa w_k - \kappa\upsilon; \zeta) \geq \sigma, \text{ and } \wp(w_1, w_2, \ldots, w_{n-1}, \kappa w_k - \kappa\upsilon; \zeta) \geq \sigma\}|$$
$$\leq \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma$$
$$\text{or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\}| = 0.$$

Therefore, $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim \kappa w_k = \kappa\upsilon$, i.e., $\kappa\{w_k\} \in \mathcal{ST}_\lambda(\mathcal{N}_n)$. This completes the proof. □

Theorem 3.8. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$ if and only if there exists a subset $\mathcal{M} = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subseteq \mathbb{N}$ such that $\delta_\lambda(\mathcal{M}) = 1$ and $\mathcal{N}_n - \lim_{p\to\infty} w_{k_p} = \upsilon$.*

Proof. First, suppose that $\mathcal{S}_{\mathcal{N}_n} - \lim w_k = \upsilon$. Then, for any $\zeta > 0, j \in \mathbb{N}$ and for all nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$,

$$\delta_\lambda(\mathcal{Y}_{\mathcal{N}_n}(j, \zeta)) = 1 \text{ and } \delta_\lambda(\mathcal{T}_{\mathcal{N}_n}(j, \zeta)) = 0 \tag{3}$$

where

$$\mathcal{Y}_{\mathcal{N}_n}(j, \zeta) = \{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) > 1 - \tfrac{1}{j}$$
$$\text{and } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \tfrac{1}{j},\ \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \tfrac{1}{j}\}$$

and

$$\mathcal{T}_{\mathcal{N}_n}(j, \zeta) = \{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \tfrac{1}{j}$$
$$\text{or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \tfrac{1}{j},\ \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \tfrac{1}{j}\}.$$

It follows that $\mathcal{Y}_{\mathcal{N}_n}(j+1, \zeta) \subseteq \mathcal{Y}_{\mathcal{N}_n}(j, \zeta)$. It is enough to prove that

$$\mathcal{N}_n - \lim_{k\to\infty} w_k = \upsilon, \text{ for those } k \in \mathcal{Y}_{\mathcal{N}_n}(j, \zeta).$$

If possible, let $\{w_k\}_{k\in\mathcal{Y}_{\mathcal{N}_n}(j,\zeta)}$ is not convergent with respect to $\mathcal{N}_n$. Then for some $\sigma \in (0, 1)$,

$$\Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma,$$
$$\Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \text{ and}$$
$$\wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma$$

holds good except for at most a finite number of terms $k \in \mathcal{Y}_{\mathcal{N}_n}(j, \zeta)$. Let

$$\mathcal{D}_{\mathcal{N}_n}(\sigma, \zeta) = \{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) > 1 - \sigma$$
$$\text{and } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma,\ \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma\}$$

where $\sigma > \frac{1}{j}$. Then $\delta_\lambda(\mathcal{D}_{\mathcal{N}_n}(\sigma, \zeta)) = 0$ because λ-density of a finite set is zero. As, $\sigma > \frac{1}{j}$, $\mathcal{Y}_{\mathcal{N}_n}(j, \zeta) \subseteq \mathcal{D}_{\mathcal{N}_n}(\sigma, \zeta)$ and hence $\delta_\lambda(\mathcal{Y}_{\mathcal{N}_n}(j, \zeta)) = 0$ which contradicts (3). Therefore, for $k \in \mathcal{Y}_{\mathcal{N}_n}(j, \zeta)$, $\mathcal{N}_n - \lim_{k\to\infty} w_k = \upsilon$.

Conversely suppose that there exists a subset $\mathcal{M} = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subseteq \mathbb{N}$ such that $\delta_\lambda(\mathcal{M}) = 1$ and $\mathcal{N}_n - \lim_{p\to\infty} w_{k_p} = \upsilon$. Then for every $\sigma \in (0,1), \zeta > 0$ and nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$ there exists $p_0 \in \mathbb{N}$ such that

$$\Im(w_1, w_2, \ldots, w_{n-1}, w_{k_p} - \upsilon; \zeta) > 1 - \sigma \text{ and } \Re(w_1, w_2, \ldots, w_{n-1}, w_{k_p} - \upsilon; \zeta) < \sigma,$$
$$\wp(w_1, w_2, \ldots, w_{n-1}, w_{k_p} - \upsilon; \zeta) < \sigma \text{ for all } p \geq p_0.$$

Thus, if we let

$$\mathcal{A}(\sigma,\zeta) = \{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma$$
$$\text{or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\},$$

then it is easy to show that $\mathcal{A}(\sigma,\zeta) \subseteq \mathbb{N}\backslash\{k_{p_0+1}, k_{p_0+2}, \ldots\}$. So, $\delta_\lambda(\mathcal{A}(\sigma,\zeta)) = 1-1 = 0$ i.e., $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$. This ends the proof. □

Theorem 3.9. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$ if and only if there exists a sequence $\{l_k\} \in \mathcal{W}$ such that $\mathcal{N}_n - \lim l_k = \upsilon$ and $\delta_\lambda(\{k \in \mathbb{N} : w_k = l_k\}) = 1$.*

Proof. First suppose that $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$. Then, by the fact of Theorem 3.8, there exists a subset $\mathcal{M} = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subseteq \mathbb{N}$ such that $\delta_\lambda(\mathcal{M}) = 1$ and $\mathcal{N}_n - \lim_{p\to\infty} w_{k_p} = \upsilon$. Now, we define a sequence $\{l_k\} \in \mathcal{W}$ as follows:

$$l_k = \begin{cases} w_k, & \text{if } k \in \mathcal{M} \\ \upsilon, & \text{otherwise} \end{cases},$$

which gives our desired result.

Conversely, suppose that there exists a sequence $\{l_k\} \in \mathcal{W}$ such that $\mathcal{N}_n - \lim l_k = \upsilon$ and $\delta_\lambda(\{k \in \mathbb{N} : w_k = l_k\}) = 1$. Then, for every $\sigma > 0, \zeta > 0$ and nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$ there exists $k_0 \in \mathbb{N}$ such that

$$\Im(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon; \zeta) > 1 - \sigma \text{ and } \Re(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon; \zeta) < \sigma,$$
$$\wp(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon; \zeta) < \sigma$$

for all $k \geq k_0$, i.e., the set

$$\mathcal{A}(\sigma,\zeta) = \{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon; \zeta) \leq 1 - \sigma$$
$$\text{or } \Re(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon; \zeta) \geq \sigma\}$$

is finite. Hence $\delta_\lambda(\mathcal{A}(\sigma,\zeta)) = 0$. Also, by the hypothesis, $\delta_\lambda(\{k \in \mathbb{N} : w_k \neq l_k\}) = 0$. Now, we can have

$$\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma$$
$$\text{or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\}$$
$$\subseteq \mathcal{A}(\sigma,\zeta) \cup \{k \in \mathbb{N} : w_k \neq l_k\}.$$

Since λ-density of a set can never be negative,

$$\delta_\lambda\left(\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma\right.$$
$$\left.\text{or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\}\right) = 0.$$

Consequently $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$. This completes the proof. □

4. Relation between $\mathcal{S}(\mathcal{N}_n)$-convergence and $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergence

First we define a notation as follows:

$$\mathcal{ST}(\mathcal{N}_n) = \{\{w_k\} \in \mathcal{W} : \exists\, \upsilon \in \mathcal{W},\ \mathcal{S}(\mathcal{N}_n) - \lim w_k = \upsilon\}.$$

In this section, we study the inclusion relationship between the sets $\mathcal{ST}(\mathcal{N}_n)$ and $\mathcal{ST}_\lambda(\mathcal{N}_n)$. From the nature of the sequence $\{\lambda_n\}$, it is clear that $\left\{\frac{\lambda_n}{n}\right\}$ is bounded. We can easily prove $\mathcal{ST}_\lambda(\mathcal{N}_n) \subseteq \mathcal{ST}(\mathcal{N}_n)$.

THEOREM 4.1. *$\mathcal{ST}(\mathcal{N}_n) \subseteq \mathcal{ST}_\lambda(\mathcal{N}_n)$ if $\liminf_n \frac{\lambda_n}{n} > 0$.*

Proof. Let $\{w_k\} \in \mathcal{ST}(\mathcal{N}_n)$. Then, there exists $\upsilon \in \mathcal{W}$ such that $\mathcal{S}(\mathcal{N}_n) - \lim w_k = \upsilon$. Now, it is obvious that for any $\sigma \in (0,1)$, $\zeta > 0$ and nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$,

$$\begin{aligned}
&\{k \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \le 1 - \sigma \text{ or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\\
&\text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\}\\
&\supseteq \{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \le 1 - \sigma \text{ or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\\
&\text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\}.
\end{aligned}$$

So,

$$\begin{aligned}
&\frac{1}{n}|\{k \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \le 1 - \sigma \text{ or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\\
&\text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\}|\\
&\ge \frac{1}{n}|\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \le 1 - \sigma \text{ or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\\
&\text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\}|\\
&= \left(\frac{\lambda_n}{n}\right) . \frac{1}{\lambda_n}|\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \le 1 - \sigma \text{ or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta)\\
&\ge \sigma \text{ and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\}|.
\end{aligned}$$

Letting $n \to \infty$ and by the condition, we can arrive at $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$, i.e., $\{w_k\} \in \mathcal{ST}_\lambda(\mathcal{N}_n)$. This completes the proof. □

5. $[V, \lambda]$-summability in Nn-NLS

In this section, we define $[V, \lambda]$-summability with respect to $\mathcal{N}_n$ and prove some of its interesting properties associated with $\mathcal{ST}_\lambda(\mathcal{N}_n)$. First we recall some definitions from [13].

DEFINITION 5.1. [13] For $\sigma \in (0,1), \zeta > 0, \alpha \in \mathcal{W}$ and every $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$, the open ball (also named as $\mathcal{N}_n$-open ball) centered at α and of radius σ with respect to ζ, denoted by $\mathcal{B}(\alpha, \sigma; \zeta)$, is defined by

$$\begin{aligned}
\mathcal{B}(\alpha, \sigma; \zeta) = \{w \in \mathcal{W} : \Im(w_1, w_2, \ldots, w_{n-1}, \alpha - w; \zeta) > 1 - \sigma\\
\text{and } \Re(w_1, w_2, \ldots, w_{n-1}, \alpha - w; \zeta) < \sigma,\ \wp(w_1, w_2, \ldots, w_{n-1}, \alpha - w; \zeta) < \sigma\}.
\end{aligned}$$

DEFINITION 5.2. [13] A subset $\mathscr{O}$ of $\mathcal{W}$ is called open with respect to $\mathcal{N}_n$ (named as $\mathcal{N}_n$-open set) if for each $\alpha \in \mathscr{O}$ there exists an $\mathcal{N}_n$-open ball of some radius which is contained in $\mathscr{O}$.

Let $\mathcal{X}$ be an Nn-NLS. If we take a collection $\mathcal{T}_{\mathcal{N}_n}$ as $\mathcal{T}_{\mathcal{N}_n} = \{\mathscr{O} \subset \mathcal{W} : \mathscr{O}$ is an $\mathcal{N}_n-$ open set$\}$. Then $\mathcal{T}_{\mathcal{N}_n}$ becomes a topology on $\mathcal{X}$. A subset $\mathcal{U}$ of $\mathcal{W}$ is named to be bounded with respect to $\mathcal{N}_n$ (denoted as $\mathcal{N}_n$-bounded) if there exist $\zeta > 0$ and $\sigma \in (0,1)$ such that for each $\alpha \in \mathcal{U}$, $\Im(w_1, w_2, \ldots, w_{n-1}, \alpha; \zeta) > 1-\sigma$ and $\Re(w_1, w_2, \ldots, w_{n-1}, \alpha; \zeta) < \sigma$, $\wp(w_1, w_2, \ldots, w_{n-1}, \alpha; \zeta) < \sigma$ holds for every $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$ [13].

Here, we define some topological notions with regard to $\mathcal{N}_n$ which will be needed to develop our results.

Definition 5.3. Let $\mathcal{Y} \subseteq \mathcal{W}$ and $\upsilon \in \mathcal{W}$. Then, υ is named to be a limit point ($\mathcal{N}_n$-limit) of $\mathcal{Y}$ with respect to $\mathcal{N}_n$ ($\mathcal{N}_n$-limit point) if for every $\mathcal{N}_n$-open ball centered at υ contains at least one point of $\mathcal{Y}$ different from υ.

$\mathcal{Y}$ is named to be closed in $\mathcal{W}$ with regard to $\mathcal{N}_n$ ($\mathcal{N}_n$-closed set) if it contains all of its $\mathcal{N}_n$-limit point. Throughout our discussion $\overline{\mathcal{A}}$ denotes closure of $\mathcal{A}$ with respect to $\mathcal{N}_n$.

Now, we define strong $[V, \lambda]$-summable of a sequence with regard to $\mathcal{N}_n$.

Definition 5.4. Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then $\{w_k\}$ is named to be strongly (V, λ)-summable to $\upsilon \in \mathcal{W}$ with respect to $\mathcal{N}_n$ if for every $\sigma > 0, \zeta > 0$ and nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$ there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n} \sum_{k \in \mathcal{I}_n} \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) > 1 - \sigma \text{ and}$$
$$\frac{1}{\lambda_n} \sum_{k \in \mathcal{I}_n} \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma,$$
$$\frac{1}{\lambda_n} \sum_{k \in \mathcal{I}_n} \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma$$

for all $n \geq n_0$. In this scenario, it is denoted as $(V, \lambda)_{\mathcal{N}_n} - \lim w_k = \upsilon$ or $w_k \xrightarrow{(V,\lambda)_{\mathcal{N}_n}} \upsilon$.

We denote

$$[V, \lambda](\mathcal{N}_n) = \{\{w_k\} \in \mathcal{W} : \exists\ \upsilon \in \mathcal{W},\ (V, \lambda)_{\mathcal{N}_n} - \lim w_k = \upsilon\}.$$

Now, we define the concept of boundedness of a sequence with regard to $\mathcal{N}_n$.

Definition 5.5. A sequence $\{w_k\}$ in a Nn-NLS $\mathcal{X}$ is said to be bounded with respect to $\mathcal{N}_n$ (in short $\mathcal{N}_n$-bounded) if there exist $\zeta_0 > 0$ and $\sigma \in (0,1)$ such that for all positive integers k,

$$\Im(w_1, w_2, \ldots, w_{n-1}, w_k; \zeta_0) > 1 - \sigma \text{ and } \Re(w_1, w_2, \ldots, w_{n-1}, w_k; \zeta_0) < \sigma,$$
$$\wp(w_1, w_2, \ldots, w_{n-1}, w_k; \zeta_0) < \sigma$$

holds for every nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$.

We denote $\ell_\infty(\mathcal{N}_n)$ to mean the set of all $\mathcal{N}_n$-bounded sequence.

Theorem 5.6. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. $(V, \lambda)_{\mathcal{N}_n} - \lim w_k = \upsilon \implies \mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$ and $[V, \lambda](\mathcal{N}_n) \subsetneqq \mathcal{ST}_\lambda(\mathcal{N}_n)$.*

Proof. Let $\zeta > 0$, $\sigma \in (0,1)$ and $(V,\lambda)_{\mathcal{N}_n} - \lim w_k = \upsilon$. Then for every nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$, we can have

$$\begin{aligned}
&\sum_{k\in\mathcal{I}_n} \Big[\Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta),\ \text{or}\\
&\qquad \Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta),\\
&\qquad \wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\Big]\\
&= \sum_{\substack{k\in\mathcal{I}_n\\ \Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\le 1-\sigma\\ \text{or } \Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\ge\sigma,\\ \wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\ge\sigma}} \Big[\Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta),\ \text{or}\\
&\qquad \Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta),\\
&\qquad \wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\Big]\\
&+ \sum_{\substack{k\in\mathcal{I}_n\\ \Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)> 1-\sigma\\ \text{and } \Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)<\sigma,\\ \wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)<\sigma}} \Big[\Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta),\ \text{or}\\
&\qquad \Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta),\\
&\qquad \wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\Big]\\
&\ge \sum_{\substack{k\in\mathcal{I}_n\\ \Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\le 1-\sigma\\ \text{or } \Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\ge\sigma,\\ \wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\ge\sigma}} \Big[\Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta),\ \text{or}\\
&\qquad \Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta),\\
&\qquad \wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\Big]\\
&\ge \sigma\cdot\big|\{k\in\mathcal{I}_n : \Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\le 1-\sigma\\
&\qquad \text{or } \Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\ge\sigma\\
&\qquad \text{and } \wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\ge\sigma\}\big|.
\end{aligned}$$

Hence $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$. Now, we show $[V,\lambda](\mathcal{N}_n) \subsetneqq \mathcal{ST}_\lambda(\mathcal{N}_n)$. For this, we take N$n$-NLS, the same sequence as defined in Example 3.6 and following the same procedure we can arrive at $(V,\lambda)_{\mathcal{N}_n} - \lim w_k \neq 0$. □

Theorem 5.7. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon \implies (V,\lambda)_{\mathcal{N}_n} - \lim w_k = \upsilon$ if $\{w_k\} \in \ell_\infty(\mathcal{N}_n)$.*

Proof. Let $\{w_k\} \in \ell_\infty(\mathcal{N}_n)$ such that $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$. We shall show that $(V,\lambda)_{\mathcal{N}_n} - \lim w_k = \upsilon$. Since $\{w_k\} \in \ell_\infty(\mathcal{N}_n)$, there exists $\mathcal{Q} > 0$ such that

$$\begin{aligned}
&\Im(w_1,w_2,\ldots,w_{n-1},w_k;\zeta) > 1-\mathcal{Q} \text{ and } \Re(w_1,w_2,\ldots,w_{n-1},w_k;\zeta) < \mathcal{Q},\\
&\wp(w_1,w_2,\ldots,w_{n-1},w_k;\zeta) < \mathcal{Q}
\end{aligned}$$

holds for every nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$. Let $\sigma \in (0,1)$ be arbitrarily selected. Then, we have

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in\mathcal{I}_n} \Big[\Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta),\ \text{or}\\
&\qquad \Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta),\\
&\qquad \wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\Big]
\end{aligned}$$

$$= \frac{1}{\lambda_n} \sum_{\substack{k\in\mathcal{I}_n \\ \Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\leq 1-\sigma \\ \text{or } \Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\geq\sigma, \\ \wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)\geq\sigma}} \begin{bmatrix} \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta), \text{ or} \\ \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta), \\ \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \end{bmatrix}$$

$$+ \frac{1}{\lambda_n} \sum_{\substack{k\in\mathcal{I}_n \\ \Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)>1-\sigma \\ \text{and } \Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)<\sigma, \\ \wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta)<\sigma}} \begin{bmatrix} \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta), \text{ or} \\ \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta), \\ \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \end{bmatrix}$$

$$\begin{aligned} \leq \frac{\mathcal{Q}}{\lambda_n} |\{k \in \mathcal{I}_n : &\Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma \\ &\text{or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \\ &\text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\}| + \sigma. \end{aligned}$$

Since $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$ and $\sigma \in (0, 1)$ be arbitrary, we get $(V, \lambda)_{\mathcal{N}_n} - \lim w_k = \upsilon$. Hence proved. $\square$

Corollary 5.8. *From Theorem 5.6 and 5.7, we can have* $\mathcal{ST}_\lambda(\mathcal{N}_n) \cap \ell_\infty(\mathcal{N}_n) = [V, \lambda](\mathcal{N}_n) \cap \ell_\infty(\mathcal{N}_n)$.

Theorem 5.9. *Let* $\{w_k\}$ *be a sequence in a Nn-NLS* $\mathcal{X}$. *Then* $m(\mathcal{ST}_\lambda(\mathcal{N}_n))$, *the space of all bounded* λ*-statistically convergent sequences, is a closed linear subspace of* $\ell_\infty(\mathcal{N}_n)$.

Proof. Let $l = \{l_k\} \in \overline{m(\mathcal{ST}_\lambda(\mathcal{N}_n))}$. Then there exists an $\mathcal{N}_n$-open ball centered at l and of radius $\sigma \in (0, 1)$ with respect to $\zeta > 0$ such that $\mathcal{B}(l, \sigma; \zeta) \cap m(\mathcal{ST}_\lambda(\mathcal{N}_n)) \neq \emptyset$. Choose $\varpi \in (0, 1)$ such that $(1 - \varpi) \boxdot (1 - \varpi) > 1 - \sigma$ and $\varpi \oplus \varpi < \sigma$. So, there is an element $w = \{w_k\} \in \mathcal{B}(l, \sigma; \zeta) \cap m(\mathcal{ST}_\lambda(\mathcal{N}_n))$. Let $\mathcal{S}_\lambda(\mathcal{N}_n) - \lim w_k = \upsilon$. Then, there exists a $\mathcal{M} \subseteq \mathbb{N}$ such that $\delta_\lambda(\mathcal{M}) = 1$ and for all $k \in \mathcal{M}$

$$\Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}) > 1 - \varpi \text{ and } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}) < \varpi,$$
$$\wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}) < \varpi$$

holds good for every nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$. Again, since $w \in \mathcal{B}(l, \sigma; \zeta)$,

$$\Im(w_1, w_2, \ldots, w_{n-1}, w_k - l_k; \frac{\zeta}{2}) > 1 - \varpi \text{ and } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - l_k; \frac{\zeta}{2}) < \varpi,$$
$$\wp(w_1, w_2, \ldots, w_{n-1}, w_k - l_k; \frac{\zeta}{2}) < \varpi$$

holds good for all positive integers k. Now, we have

$$\begin{aligned} &\Im(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon; \zeta) \\ &\geq \Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) \boxdot \Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - l_k; \frac{\zeta}{2}\right) \\ &> (1 - \varpi) \boxdot (1 - \varpi) > 1 - \sigma \end{aligned}$$

and

$$\begin{aligned}&\Re(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon; \zeta)\\ &\leq \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) \oplus \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - l_k; \frac{\zeta}{2}\right)\\ &< \varpi \oplus \varpi < \sigma,\end{aligned}$$

$$\begin{aligned}&\wp(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon; \zeta)\\ &\leq \wp\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) \oplus \wp\left(w_1, w_2, \ldots, w_{n-1}, w_k - l_k; \frac{\zeta}{2}\right)\\ &< \varpi \oplus \varpi < \sigma.\end{aligned}$$

Since $\delta_\lambda(\mathcal{M}) = 1$,

$$\delta_\lambda(\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon; \zeta) > 1 - \sigma \\ \text{and } \Re(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon; \zeta) < \sigma,\ \wp(w_1, w_2, \ldots, w_{n-1}, l_k - \upsilon; \zeta) < \sigma\}) = 1.$$

Hence $l \in m(\mathcal{ST}_\lambda(\mathcal{N}_n))$. This completes the proof. □

6. λ-statistically completeness in Nn-NLS

In this section, we proceed with definig λ-statistically Cauchy sequence and discussing λ-statistically completeness with respect to $\mathcal{N}_n$.

Definition 6.1. Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then $\{w_k\}$ is named to be λ-statistically Cauchy sequence with respect to $\mathcal{N}_n$ (in short $\mathcal{S}_\lambda(\mathcal{N}_n)$-Cauchy sequence) if for every $\sigma \in (0, 1), \zeta > 0$ and nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$ there exists $k_0 = k_0(\sigma) \in \mathbb{N}$ such that

$$\delta_\lambda(\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \leq 1 - \sigma \text{ or} \\ \Re(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq \sigma\}) = 0.$$

Or, it can be restated as

$$\lim_{n\to\infty} \tfrac{1}{\lambda_n}|\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \leq 1 - \sigma \text{ or} \\ \Re(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq \sigma\}| = 0.$$

Theorem 6.2. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. If $\{w_k\}$ is $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergent, it is $\mathcal{S}_\lambda(\mathcal{N}_n)$-Cauchy sequence.*

Proof. Let $\{w_k\}$ is $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergent to υ. Let $\sigma \in (0, 1)$ be given. Choose $\varpi \in (0, 1)$ such that $(1 - \varpi) \boxdot (1 - \varpi) > 1 - \sigma$ and $\varpi \oplus \varpi < \sigma$. Then for any $\zeta > 0$ and nonzero $w_1, w_2, \ldots, w_{n-1} \in \mathcal{W}$, λ-density of each of

$$\begin{aligned}\mathcal{A}_1 &= \left\{k \in \mathcal{I}_n : \Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) > 1 - \varpi\right\},\\ \mathcal{A}_2 &= \left\{k \in \mathcal{I}_n : \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \varpi\right\}\\ \text{and } \mathcal{A}_3 &= \left\{k \in \mathcal{I}_n : \wp\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \varpi\right\}\end{aligned}$$

is 1. Let $\mathcal{B} = \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$. Then $\delta_\lambda(\mathcal{B}) = 1$. So, let $k \in \mathcal{B}$ and thus, choose $k_0 \in \mathcal{B}$. Then

$$\begin{aligned}
&\Im(w_1, w_2, \dots, w_{n-1}, w_k - w_{k_0}; \zeta)\\
&\geq \Im\left(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) \boxdot \Im\left(w_1, w_2, \dots, w_{n-1}, w_{k_0} - \upsilon; \frac{\zeta}{2}\right)\\
&> (1-\varpi) \boxdot (1-\varpi) > (1-\sigma)
\end{aligned}$$

and

$$\begin{aligned}
&\Re(w_1, w_2, \dots, w_{n-1}, w_k - w_{k_0}; \zeta)\\
&\leq \Re\left(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) \oplus \Re\left(w_1, w_2, \dots, w_{n-1}, w_{k_0} - \upsilon; \frac{\zeta}{2}\right)\\
&< \varpi \oplus \varpi < \sigma;
\end{aligned}$$

$$\begin{aligned}
&\wp(w_1, w_2, \dots, w_{n-1}, w_k - w_{k_0}; \zeta)\\
&\leq \wp\left(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) \oplus \wp\left(w_1, w_2, \dots, w_{n-1}, w_{k_0} - \upsilon; \frac{\zeta}{2}\right)\\
&< \varpi \oplus \varpi < \sigma.
\end{aligned}$$

Therefore,

$$\begin{gathered}
\delta_\lambda(\{k \in \mathcal{I}_n : \Im(w_1, w_2, \dots, w_{n-1}, w_k - w_{k_0}; \zeta) \leq 1-\sigma \text{ or}\\
\Re(w_1, w_2, \dots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \dots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq \sigma\}) = 0,
\end{gathered}$$

i.e., $\{w_k\}$ is $\mathcal{S}_\lambda(\mathcal{N}_n)$-Cauchy sequence. □

THEOREM 6.3. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. If $\{w_k\}$ is $\mathcal{S}_\lambda(\mathcal{N}_n)$-Cauchy sequence, it is $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergent.*

Proof. Let $\{w_k\}$ be $\mathcal{S}_\lambda(\mathcal{N}_n)$-Cauchy sequence but not $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergent. Then for $\sigma \in (0,1), \zeta > 0$ and nonzero $w_1, w_2, \dots, w_{n-1} \in \mathcal{W}$ there exists $k_0 = k_0(\sigma) \in \mathbb{N}$ such that $\delta_\lambda(\mathcal{K}) = 0$ where

$$\begin{gathered}
\mathcal{K} = \{k \in \mathcal{I}_n : \Im(w_1, w_2, \dots, w_{n-1}, w_k - w_{k_0}; \zeta) \leq 1-\sigma \text{ or}\\
\Re(w_1, w_2, \dots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \dots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq \sigma\}) = 0.
\end{gathered}$$

And, $\delta_\lambda(\mathcal{M}) = 0$ where

$$\begin{gathered}
\mathcal{M} = \{k \in \mathcal{I}_n : \Im(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \tfrac{\zeta}{2}) > 1-\sigma \text{ and}\\
\Re(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \tfrac{\zeta}{2}) < \sigma, \text{ and } \wp(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \tfrac{\zeta}{2}) < \sigma\}.
\end{gathered}$$

Since

$$\Im(w_1, w_2, \dots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq 2\Im\left(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) > 1-\sigma$$

and

$$\Re(w_1, w_2, \dots, w_{n-1}, w_k - w_{k_0}; \zeta) \leq 2\Re\left(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \sigma$$

$$\wp(w_1, w_2, \dots, w_{n-1}, w_k - w_{k_0}; \zeta) \leq 2\wp\left(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \sigma,$$

if

$$\Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \tfrac{\zeta}{2}\right) > \tfrac{1-\sigma}{2}$$
$$\text{and } \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \tfrac{\zeta}{2}\right) < \tfrac{\sigma}{2}, \wp\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \tfrac{\zeta}{2}\right) < \tfrac{\sigma}{2}.$$

It follows that

$$\delta_\lambda(\{k \in \mathcal{I}_n : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) > 1 - \sigma$$
$$\text{and } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) < \sigma,\ \wp(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) < \sigma\}) = 0$$

which means $\delta_\lambda(\mathcal{K}^c) = 0$ that implies $\delta(\mathcal{K}) = 1$ by which we arrive at a contradiction. Hence $\{w_k\}$ is $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergent to $\upsilon \in \mathcal{W}$. It ends the proof. □

Definition 6.4. A Nn-NLS is named to be λ-statistically complete (in short $\mathcal{S}_\lambda(\mathcal{N}_n)$-complete) if every $\mathcal{S}_\lambda(\mathcal{N}_n)$-Cauchy sequence is $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergent.

Remark 6.5. In the light of Theorem 6.2 and 6.3, we conclude every Nn-NLS is $\mathcal{S}_\lambda(\mathcal{N}_n)$-complete.

Now, on the basis of Theorems 3.8, 6.2 and 6.3 we state an equivalent result.

Theorem 6.6. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then the below properties are gratified:*

1. *$\{w_k\}$ is $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergent;*
2. *$\{w_k\}$ is $\mathcal{S}_\lambda(\mathcal{N}_n)$-Cauchy;*
3. *There exists a subset $\mathcal{C} = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subset \mathbb{N}$ such that $\delta_\lambda(\mathcal{C}) = 1$ and the subsequence $\{w_{k_p}\}$ is an ordinary Cauchy sequence with regard to $\mathcal{N}_n$.*

Conclusion and future developments

In this research paper, we have introduced $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergence and $\mathcal{S}_\lambda(\mathcal{N}_n)$-Cauchy sequence and obtained that every Nn-NLS is $\mathcal{S}_\lambda(\mathcal{N}_n)$-complete. Also, we have defined $[V, \lambda]$-summability with respect to $\mathcal{N}_n$ and proved that $(V, \lambda)(\mathcal{N}_n)$-convergence implies $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergence. And, the converse is true whenever the sequence belongs to $\ell_\infty(\mathcal{N}_n)$. Also, the relation between $\mathcal{S}(\mathcal{N}_n)$ and $\mathcal{S}_\lambda(\mathcal{N}_n)$-convergence has been obtained. In future, based on this research work, one can generalize this notion in the context of ideal and nurture it related to multiple sequences. Also, this idea can be used in the field of convergence related problems in many branches of science and engineering.

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