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DECIMAL EXPANSION OF THE SQUARE ROOT OF A NONNEGATIVE INTEGER

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Abstract. For positive integers n and k, with $k \leq 2n$, let

$$\sqrt{n^2+k} = n_t \cdots n_1.a_1a_2a_3\cdots$$

be the decimal expansion of $\sqrt{n^2+k}$. In this paper, we introduce a systematic method of how to calculate the value of a_i for all $i = 1, 2, \ldots$.

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $[\![a, b]\!] = \{m \in \mathbb{N}_0 \mid a \leq m \leq b\}$ for any $a, b \in \mathbb{N}_0$, with $a < b$, and $A_n = \{n^2, n^2+1, \ldots, n^2+2n\}$ for all $n \in \mathbb{N}_0$. Then $A_n = [\![n^2, n^2+2n]\!]$, $|A_n| = 2n+1$ and $\{A_n \mid n \in \mathbb{N}_0\}$ is a partition of $\mathbb{N}_0$, so $a \in \mathbb{N}_0$ if and only if $a \in A_n$ for some unique $n \in \mathbb{N}_0$. Now let

$$\sqrt{n^2+k} = n_t \ldots n_1.a_1a_2\cdots$$

be the decimal expansion of $\sqrt{n^2+k}$ for integers $n, k \in \mathbb{N}_0$, with $k \leq 2n$. In this paper, we introduce a systematic method of how to calculate the value of a_i for all $i \in \mathbb{N}_0$. We first prove a theorem by which we can systematically classify the value of a_1 by dividing n into five cases, i.e., $n \equiv i \pmod 5$ for $i \in [\![0, 4]\!]$. We then give a simple corollary of the theorem which can be used to obtain the values of $a_2, a_3, \ldots$ in order.

Throughout this note we use the following notations.

Notation. For a nonnegative integer n, let

(a) $\varphi_n : A_n \to [\![0, 9]\!]$ be a function defined by $\varphi_n(x) =$ the number at the first decimal place of $\sqrt{x}$ and
(b) $\underline{n}(y) = |\varphi_n^{-1}(\{y\})|$ for each $y \in [\![0, 9]\!]$.

In this note we must keep it in mind that if $a \in A_n$ is such that

$$\sqrt{a} = m = (m-1).999\cdots$$

for some integer m, then $\varphi_n(a) = 0$ but not 9, i.e., $\varphi_n(a) \neq 9$. For example, $\varphi_n(n^2) = 0$ but $\varphi_n(n^2) \neq 9$ for all $n \in \mathbb{N}_0$.

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Now, let $\mathbb{R}$ be the set of real numbers and $[x]$ be the greatest integer less than or equal to a real number $x \in \mathbb{R}$, so if $\mathbb{Z}$ is the set of integers, then $[\ \ ] : \mathbb{R} \to \mathbb{Z}$, called the greatest integer function, is a function. It is easy to see that $[n + x] = n + [x]$ for any $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. See [1, Section 6.3] for some basic properties of the greatest integer function. We first give a simple lemma which plays a key role in the proof of the results in this paper.

Lemma 1. *Let $y \in [\![0, 9]\!]$, $n \in \mathbb{N}_0$, and $k \in [\![0, 2n]\!]$. Then $\varphi_n(n^2 + k) = y$ if and only if $0.2yn + 0.01y^2 \le k < 0.2(y + 1)n + 0.01(y + 1)^2$.*

Proof. ($\Rightarrow$) It is clear that $\sqrt{n^2 + k} = n + \theta$ for a suitable choice of a real number θ, with $0 \le \theta < 1$. So if $\varphi_n(n^2 + k) = y$, then $0.1y \le \theta < 0.1(y + 1)$, and hence $0.01y^2 \le \theta^2 < 0.01(y + 1)^2$ and $0.2yn \le 2n\theta < 0.2(y + 1)n$. Moreover, $k = 2n\theta + \theta^2$ by the equality of $\sqrt{n^2 + k} = n + \theta$, so we have

$$0.2yn + 0.01y^2 \le k < 0.2(y + 1)n + 0.01(y + 1)^2.$$

($\Leftarrow$) Let $[a, b) = \{x \in \mathbb{R} \mid a \le x < b\}$ be as usual for $a, b \in \mathbb{R}$, with $a < b$. Then the result can be proved by noting that

$$\{[0.2yn + 0.01y^2, 0.2(y + 1)n + 0.01(y + 1)^2) \mid y \in [\![0, 9]\!]\}$$

is a partition of $[0, 2n + 1)$. □

We are now ready to give the main result of this paper.

Theorem 2. *For a nonnegative integer n, the following statements hold.*

(1) *φ_n is increasing.*

(2) *φ_n is surjective if and only if $n \ge 5$.*

(3) *$(\underline{n + 5})(y) = \underline{n}(y) + 1$ for each $y \in [\![0, 9]\!]$.*

Proof. (1) Let $a, b \in A_n$, with $a < b$. Then $n^2 \le a < b < (n + 1)^2$, and hence $n \le \sqrt{a} < \sqrt{b} < n + 1$. Thus, $\varphi_n(a) \le \varphi_n(b)$.

(2) If $n \le 4$, then $|A_n| = 2n + 1 \le 9$, and hence $\varphi_n(A_n) \subsetneq [\![0, 9]\!]$. Thus, if φ_n is surjective, then $n \ge 5$. Conversely, assume that $n \ge 5$. Then we have to consider the three cases of when $y = 0, y \in [\![1, 8]\!]$ and $y = 9$ by Lemma 1 and the properties of the greatest integer function $[\ \ ]$.

Case 1. $y = 0$. Then, by Lemma 1, $\underline{n}(y) = [0.2n + 0.01] + 1 \ge [1.01] + 1 = 2$, where the first inequality follows because $n \ge 5$.

Case 2. $y \in [\![1, 8]\!]$. Then none of $0.2yn + 0.01y^2$ and $0.2(y + 1)n + 0.01(y + 1)^2$ is an integer, and hence, by Lemma 1,

$$\begin{aligned} \underline{n}(y) &= [0.2(y + 1)n + 0.01(y + 1)^2] - [0.2yn + 0.01y^2] \\ &\ge [0.2(y + 1)n + 0.01(y + 1)^2 - 0.2yn - 0.01y^2] \\ &= [0.2n + 0.02y + 0.01] \\ &\ge [0.2n] \ge 1, \end{aligned}$$

where the last inequality follows because $n \ge 5$.

Case 3. $y = 9$. Then, by Lemma 1, $\underline{n}(y) = (2n + 1) - ([1.8n + 0.81] + 1) = 2n - [1.8n + 0.81] \ge 2n - [2n - 1 + 0.81] = 2n - (2n - 1 + [0.81]) = 1$, where the third inequality follows from that $n \ge 5$ implies $1.8n \le 2n - 1$.

Therefore, by Case 1, 2 and 3, φ_n is surjective.

(3) Let $y \in [\![0,9]\!]$. By a simple calculation, $(\underline{n+5})(y) = \underline{n}(y) + 1$ for all $n \in [\![0,4]\!]$ and $y \in [\![0,9]\!]$, so we assume that $n \geq 5$. Then, as in the case of the proof of (2) above, we have three cases to prove.

Case 1. $y = 0$. Then $(\underline{n+5})(y) = [0.2(n+5)+0.01]+1 = [0.2n+1+0.01]+1 = ([0.2n+0.01]+1)+1 = \underline{n}(y)+1$ by Lemma 1.

Case 2. $y \in [\![1,8]\!]$. Then, by Lemma 1,

$$\begin{aligned}(\underline{n+5})(y) &= [0.2(y+1)(n+5)+0.01(y+1)^2] - [0.2y(n+5)+0.01y^2] \\ &= [0.2(y+1)n+0.01(y+1)^2+y+1] - [0.2yn+0.01y^2+y] \\ &= ([0.2(y+1)n+0.01(y+1)^2] - [0.2yn+0.01y^2]) + 1 \\ &= \underline{n}(y)+1.\end{aligned}$$

Case 3. $y = 9$. Then $(\underline{n+5})(y) = 2(n+5) - [1.8(n+5)+0.81] = (2n+10) - [1.8n+0.81+9] = (2n - [1.8n+0.81]) + 1 = \underline{n}(y)+1$ by Lemma 1. □

The following corollary is an application of Theorem 2. We can use this result to classify the number at the first decimal place of $\sqrt{a}$ for all $a \in \mathbb{N}_0$.

Corollary 3. *Let n be a nonnegative integer. Then the following statements hold.*

(1) $(\underline{5n})(l) = \begin{cases} n+1, & l = 0 \\ n, & l \in [\![1,9]\!]. \end{cases}$

(2) $(\underline{5n+1})(l) = \begin{cases} n+1, & l = 0,4,7 \\ n, & l = 1,2,3,5,6,8,9. \end{cases}$

(3) $(\underline{5n+2})(l) = \begin{cases} n+1, & l = 0,2,4,6,8 \\ n, & l = 1,3,5,7,9. \end{cases}$

(4) $(\underline{5n+3})(l) = \begin{cases} n+1, & l = 0,1,3,4,6,7,8 \\ n, & l = 2,5,9. \end{cases}$

(5) $(\underline{5n+4})(l) = \begin{cases} n+1, & l \in [\![0,8]\!] \\ n, & l = 9. \end{cases}$

Proof. This can be proved by a simple calculation and Theorem 2(3). □

We next give a very useful method by which, together with Corollary 3, we can calculate the numbers at all of the decimal places of $\sqrt{a}$ for each $a \in A_n$.

Corollary 4. *Let n and k be positive integers, with $k \in [\![1,2n]\!]$, and*

$$\sqrt{n^2+k} = n_t \cdots n_1.a_1 a_2 \cdots$$

be the decimal expansion of $\sqrt{n^2+k}$, so $n_i, a_j \in [\![0,9]\!]$ and $n = n_t \cdots n_1 = n_t \times 10^{t-1} + \cdots + n_2 \times 10 + n_1$. For an integer $r \in \mathbb{N}$, with $r \geq 2$, let

- $a = n \times 10^{r-1}$,
- $b = a_1 \times 10^{r-2} + \cdots + a_{r-2} \times 10 + a_{r-1}$,
- $N = a + b$,
- $K = k \times (10^{r-1})^2 - 2ab - b^2$,
- $a_{r-1} = 5\delta + i$ *for* $i \in [\![0,4]\!]$ *and* $\delta \in \{0,1\}$, *and*

$\cdot\ m = \begin{cases} 2n+\delta, & r=2 \\ 2(n\times 10^{r-2}+a_1\times 10^{r-3}+\cdots+a_{r-2})+\delta, & r\geq 3. \end{cases}$

Then the following statements hold.

(1) $N = 5m+i$,

(2) $0\leq K\leq 2N$,

(3) $\sqrt{N^2+K} = n_t\cdots n_1a_1\cdots a_{r-1}.a_ra_{r+1}\cdots$, *which is the decimal expansion of* $\sqrt{N^2+K}$, *and*

(4) $a_r = \varphi_N(N^2+K)$.

Proof. (1) and (2) are clear.

(3) Note that, by a simple calculation,

$$\begin{aligned}\sqrt{N^2+K} &= \sqrt{(10^{r-1}n)^2+(10^{r-1})^2k} \\ &= 10^{r-1}\sqrt{n^2+k} \\ &= n_t\cdots n_1a_1\cdots a_{r-1}.a_ra_{r+1}\cdots.\end{aligned}$$

Thus, $n_t\cdots n_1a_1\cdots a_{r-1}.a_ra_{r+1}\cdots$ is the decimal expansion of $\sqrt{N^2+K}$.

(4) This follows directly from (2) and (3) above. □

There are too many cases we have to consider in order to classify the value of a_r in Corollary 4 as in Corollary 3. However, if r is sufficiently large, there is almost a 100% chance that the value of a_r will become $[\frac{5(K-1)}{N}]$ by Corollary 3.

The following corollary is a special case of Corollary 4 in which the value of a_r can be easily calculated.

Corollary 5. *Let the notation be as in Corollary 4, and assume that* $10^{r-1}\leq[\frac{2n}{k}]$. *Then the following statements are satisfied.*

(1) $N = n\times 10^{r-1}$,

(2) $K = k\times(10^{r-1})^2$,

(3) $a_1=\cdots=a_{r-1}=0$, *and*

(4) $a_r = q$ *if and only if* $q(2\cdot 10^{r-2}n)+1\leq(10^{r-1})^2k\leq(q+1)(2\cdot 10^{r-2}n)$.

Proof. (1), (2), and (3) If $10^{r-1}\leq[\frac{2n}{k}]$, then $k\times 10^{r-1}\leq 2n$. Hence, $N = n\times 10^{r-1}$ and $K = k\times(10^{r-1})^2$, which implies that $a_1\times 10^{r-2}+\cdots+a_{r-2}\times 10+a_{r-1}=0$. Thus, $a_1=\cdots=a_{r-1}=0$.

(4) Since $r\geq 2$, $2\cdot 10^{r-2}n$ is a positive integer and $10^{r-1}n = 5(2\cdot 10^{r-2}n)$. Moreover, $K = (10^{r-1})^2k\leq 2(10^{r-1}n) = 2N$ by assumption. Thus, the result follows directly from (1) above, Corollary 3(1) and Theorem 2(1). □

Next, we give a concrete example of how to use the result of this paper to calculate the decimal expansion of $\sqrt{n}$ for $n\in\mathbb{N}_0$.

Example 6. Let

$$\sqrt{26} = 5.a_1a_2a_3\cdots$$

be the decimal expansion of $\sqrt{26}$; in this case, $n=5$ and $k=1$ in Corollary 4. We now use the results of this note to calculate the values of a_1, a_2, a_3 and a_4.

◇ $a_1 = 0$ by Corollary 5(3) (note that $10^{2-1}\leq[\frac{2\cdot 5}{1}]$).

- ⋄ $a_2 = 9$ by the inequalities of $9 \cdot (2 \cdot 10^{r-2}n) + 1 \leq (10^{r-1})^2 k \leq (9+1)(2 \cdot 10^{r-2}n)$ in Corollary 5(4) (note that $10^{2-1} \leq [\frac{2 \cdot 5}{1}]$).
- ⋄ In Corollary 4, if $r = 3, n = 5, k = 1$, then $N = 5 \times 101 + 4$, $K = 919$, $m = 101$, and $919 = 9m + 10$. Hence, by Corollary 3(5), $a_3 = 9$.
- ⋄ In Corollary 4, if $r = 4, n = 5, k = 1$, then $N = 5 \times 1019 + 4$, $K = 199$, $m = 1019$, and $199 < 1019$. Hence, by Corollary 3(5), $a_4 = 0$.

In fact, $\sqrt{26} = 5.0990195 \cdots$.

Let the notation be as in Corollary 4. Then the results of this paper say that if the values of $a_1, \ldots, a_r$ are obtained, then we can use these values to calculate the value of a_{r+1} for all $r \in \mathbb{N}$. Even though we don't know how practical this method is for calculating the decimal expansion of $\sqrt{m}$ for an integer $m \in \mathbb{N}_0$, it is an interesting result of finding that there is some regularity in that expansion.

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