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A CHARACTERIZATION OF S_1-PROJECTIVE MODULES

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Abstract. Recently, Zhao, Pu, Chen, and Xiao introduced and investigated novel concepts regarding S-torsion exact sequences, S-torsion commutative diagrams, and S_i-projective modules (for $i = 1, 2$) in the context of a commutative ring R and a multiplicative subset S of R. Their research included various results, such as proving that an R-module is S_1-projective if it is S-torsion isomorphic to a projective module. In this paper, we further examine properties of S-torsion exact sequences and S-torsion commutative diagrams, and we establish the equivalence between an R-module being S_1-projective and its S-torsion isomorphism to a projective module.

1. Introduction

In this paper, we assume that all rings are commutative with a non-zero identity and that all modules are unitary. For a ring R, we denote by $\mathrm{Nil}(R)$ the ideal of all nilpotent elements of R, and by $Z(R)$ the set of all zero-divisors of R. A ring R is called a PN-ring if $\mathrm{Nil}(R)$ is a prime ideal of R, and a ZN-ring if $Z(R) = \mathrm{Nil}(R)$. An ideal I of R is said to be *nonnil* if $I \nsubseteq \mathrm{Nil}(R)$. A nonempty subset S of R is said to be a *multiplicative subset* if $1 \in S$, $0 \notin S$, and for each $a, b \in S$, we have $ab \in S$.

Recall from [2] that a prime ideal P of R is said to be *divided* if it is comparable to any ideal of R. Set

$$\mathcal{H} := \{R \mid R \text{ is a commutative ring, and } \mathrm{Nil}(R) \text{ is a divided prime ideal of } R\}.$$

If $R \in \mathcal{H}$, then R is called a ϕ-ring. A ϕ-ring is called a *strongly* ϕ-ring if it is also a ZN-ring. Recall from [1] that for a ϕ-ring R with total ring of quotients $T(R)$, the map $\phi : T(R) \to R_{\mathrm{Nil}(R)}$ defined by $\phi\left(\frac{b}{a}\right) = \frac{b}{a}$ is a ring homomorphism, and the image of R, denoted by $\phi(R)$, is a strongly ϕ-ring. The classes of ϕ-rings and strongly ϕ-rings provide useful extensions of integral domains to commutative rings with zero divisors. In 2002, Badawi [3] generalized the notion of Noetherian rings to that of *nonnil-Noetherian rings*, in which all nonnil ideals are finitely generated. He showed that a ϕ-ring R is nonnil-Noetherian if and only if $\phi(R)$ is nonnil-Noetherian, if and only if $R/\mathrm{Nil}(R)$ is a Noetherian domain. Many well-known notions of integral domains

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have the corresponding analogues in the class of ϕ-rings, such as valuation domains, Dedekind domains, Prüfer domains, Noetherian domains, coherent domains, Bézout domains and Krull domains. For more on ϕ-rings, see Badawi's survey article [4]. For further study of ring-theoretic characterizations of ϕ-rings. To study module-theoretic characterizations over ϕ-rings, the authors of [6, 9, 12, 15, 16, 18, 20] introduced nonnil-injective modules, ϕ-projective modules, and ϕ-flat modules, and characterized nonnil-Noetherian rings, ϕ-von Neumann regular rings, nonnil-coherent rings, ϕ-coherent rings, ϕ-Dedekind rings, and ϕ-Prüfer rings. Additional information about ϕ-rings from a module-theoretic point of view can be found in the interesting survey article [7].

Let M be an R-module and S a multiplicative subset of R. Define

$$\mathrm{tor}_S(M) := \{x \in M \mid sx = 0 \text{ for some } s \in S\}.$$

If $\mathrm{tor}_S(M) = M$, then M is called an *S-torsion module*; if $\mathrm{tor}_S(M) = 0$, then M is called an *S-torsion-free module*. Denote by $\mathcal{T}$ (resp., $\mathcal{F}$) the class of all S-torsion modules (resp., S-torsion-free modules). Then $(\mathcal{T}, \mathcal{F})$ forms a hereditary torsion theory.

Recall that the authors of [11, 17] defined a ring homomorphism $\phi : R \to R_S$ by $\phi(r) = \frac{r}{1}$ for every $r \in R$ and a module homomorphism $\psi : M \to M_S$ by $\psi(x) = \frac{x}{1}$ for every $x \in M$. Then $\psi(M)$ is a $\phi(R)$-module. If $f : M \to N$ is a homomorphism of R-modules, then f naturally induces a $\phi(R)$-homomorphism $\widetilde{f} : \psi(M) \to \psi(N)$ such that $\widetilde{f}\left(\frac{x}{1}\right) = \frac{f(x)}{1}$ for $x \in M$. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of R-modules and homomorphisms is called *S-torsion exact* if the $\phi(R)$-sequence $\psi(A) \xrightarrow{\widetilde{f}} \psi(B) \xrightarrow{\widetilde{g}} \psi(C)$ is exact. An R-module P is said to be *S_2-projective* (resp., *S_2-free*) if $\psi(P)$ is projective (resp., free) as a $\phi(R)$-module.

Note that if R is a PN-ring, then the notion of ϕ-projective modules introduced in [17, 19] coincides with that of S_2-projective modules when $S := R \setminus \mathrm{Nil}(R)$.

Let R be a ring, S a multiplicative subset of R, and $f : A \to B$ a homomorphism of R-modules. Define

$$\mathrm{Ker}_S(f) := \{a \in A \mid sf(a) = 0 \text{ for some } s \in S\} \quad \text{and}$$

$$\mathrm{Im}_S(f) := \{b \in B \mid sb = sf(a) \text{ for some } a \in A \text{ and } s \in S\}.$$

Note that $\mathrm{Ker}_S(f)$ is a submodule of A, called the *S-kernel* of f, and $\mathrm{Im}_S(f)$ is a submodule of B, called the *S-image* of f. We set $\mathrm{Coker}_S(f) := B/\mathrm{Im}_S(f)$. It is easy to verify that $\mathrm{Ker}(f) + \mathrm{tor}_S(A) \subseteq \mathrm{Ker}_S(f)$ and $\mathrm{Im}(f) + \mathrm{tor}_S(B) = \mathrm{Im}_S(f)$.

Let A, B, C, and D be R-modules, and let $f : A \to B$, $g : B \to D$, $h : A \to C$, and $k : C \to D$ be homomorphisms of R-modules. Then the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ C & \xrightarrow{k} & D \end{array}$$

is said to be *S-torsion commutative* if $\mathrm{Im}_S(gf - kh) = \mathrm{tor}_S(D)$, equivalently, $A = \mathrm{Ker}_S(gf - kh)$. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of R-modules and homomorphisms is called an *S-torsion complex* (resp., an *S-torsion exact sequence*) if and only if $\mathrm{Im}_S(f) \subseteq \mathrm{Ker}_S(g)$ (resp., $\mathrm{Im}_S(f) = \mathrm{Ker}_S(g)$) according to [11, Theorem 2.4]. A homomorphism $f : A \to B$ of R-modules is called an *S-torsion monomorphism* if $\mathrm{Ker}_S(f) = \mathrm{tor}_S(A)$, equivalently, $0 \to A \xrightarrow{f} B$ is an S-torsion exact sequence. The homomorphism f is called an *S-torsion epimorphism* if $\operatorname{Im}_S(f) = B$ (i.e., $\operatorname{Coker}_S(f) = 0$), equivalently, $A \xrightarrow{f} B \to 0$ is an S-torsion exact sequence. Moreover, f is called an *S-torsion isomorphism* if there exists a homomorphism $g : B \to A$ such that $\operatorname{Im}_S(\mathbf{1}_A - gf) = \operatorname{tor}_S(A)$ and $\operatorname{Im}_S(\mathbf{1}_B - fg) = \operatorname{tor}_S(B)$. If there exists an S-torsion isomorphism $f : A \to B$, we say that A and B are *S-torsion isomorphic*, denoted by $A \overset{S}{\simeq} B$.

Note that if $f : A \to B$ is an S-torsion isomorphism, then f is both an S-torsion monomorphism and an S-torsion epimorphism. It is interesting to note that although a homomorphism f of R-modules is both an S-torsion monomorphism and an S-torsion epimorphism, f is not necessarily an S-torsion isomorphism (see [11]).

According to [11], an R-module P is said to be *S_1-projective* if for any diagram of module homomorphisms

$$\begin{array}{ccccc} & & P & & \\ & \overset{h}{\swarrow} & \downarrow f & & \\ B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

with the bottom row S-torsion exact, there exists a homomorphism $h : P \to B$ that makes this diagram S-torsion commutative. Also, an R-module F_0 is said to be S_1-*free* if it is S-torsion isomorphic to a free module.

Note that if R is a PN-ring, then the notion of nonnil-projective modules introduced in [10] and that of S_1-projective modules are the same where $S = R \setminus Nil(R)$. Let $S \subseteq T$ be two multiplicative subsets of R. Then every S_1-projective module is a T_1-projective module, and we have equivalence in the case where S and T have the same saturation. In particular, if S^* is the saturation of S in R, then an R-module P is S_1-projective if and only if it is S_1^*-projective.

Note that if there exists $s \in S \cap \operatorname{Nil}(R)$, then there exists a positive integer n such that $0 = s^n \in S$, which is a contradiction. Hence, we always have $S \cap \operatorname{Nil}(R) = \emptyset$. Therefore, if R is a PN-ring, then every S_1-projective module is nonnil-projective.

According to [11, Theorem 3.7], an R-module is S_1-projective if and only if it is a direct summand of an S_1-free module. If an R-module P is S-torsion isomorphic to a projective module, then P is S_1-projective (cf. [10, Corollary 3.9]). However, they did not show that an S_1-projective module and a module that is S-torsion isomorphic to a projective module are necessarily the same, nor did they provide examples to distinguish them. One of the main goals of this paper is to address this question.

In Section 2, we study some new properties of S-torsion commutative diagrams and S-torsion exact sequences. In the final section, we prove that an R-module is S_1-projective if and only if it is S-torsion isomorphic to a projective module.

2. On S-exacte sequences

Let R be a ring and S a multiplicative subset of R. Define

$$\mathcal{I}_S(R) := \{I \mid I \text{ is an ideal of } R \text{ such that } I \cap S \neq \emptyset\}$$

and

$$\mathcal{I}_S^f(R) := \{I \mid I \text{ is a finitely generated ideal of } R \text{ such that } I \cap S \neq \emptyset\}.$$

We begin this article with the following theorem, which characterizes when an R-module is S-torsion-free.

THEOREM 2.1. *Let R be a commutative ring, S a multiplicative subset of R, and M an R-module. Then the following statements are equivalent.*

1. *M is S-torsion-free.*
2. $\mathrm{Hom}_R(R/J, M) = 0$ *for any* $J \in \mathcal{I}_S(R)$.
3. $\mathrm{Hom}_R(R/J, M) = 0$ *for any* $J \in \mathcal{I}_S^f(R)$.
4. *The natural homomorphism:*
$$\lambda : M \to \mathrm{Hom}_R(J, M) \text{ defined by } \lambda(x)(r) = rx,$$
for $x \in M$ and $r \in J$, is a monomorphism for any $J \in \mathcal{I}_S(R)$ (or $J \in \mathcal{I}_S^f(R)$).
5. $\mathrm{Hom}_R(B, M) = 0$ *for any R/J-module B, where $J \in \mathcal{I}_S(R)$ (or $J \in \mathcal{I}_S^f(R)$).*

Proof. (1) $\Rightarrow$ (2) Assume that M is S-torsion-free. If $f \in \mathrm{Hom}_R(R/J, M)$, then set $x := f(\overline{1})$. Thus $Jx = 0$, and so $x = 0$. Therefore, $f = 0$ and consequently $\mathrm{Hom}_R(R/J, M) = 0$.

(2) $\Rightarrow$ (3) This is straightforward.

(3) $\Rightarrow$ (1) Let $x \in M$ and $I \in \mathcal{I}_S(R)$ such that $Ix = 0$. Then there exists a $J \in \mathcal{I}_S^f(R)$ such that $J \subseteq I$ and $Jx = 0$. The map $f : R/J \to M$ defined by $f(\overline{r}) = rx$ for $r \in R$, is well-defined. If $\mathrm{Hom}_R(R/J, M) = 0$ for any $J \in \mathcal{I}_S^f(R)$, then $x = f(\overline{1}) = 0$.

(2) $\Leftrightarrow$ (4) Consider the exact sequence of R-modules:
$$0 \to \mathrm{Hom}_R(R/J, M) \to \mathrm{Hom}_R(R, M) = M \to \mathrm{Hom}_R(J, M).$$
Then λ is a monomorphism if and only if $\mathrm{Hom}_R(R/J, M) = 0$.

(4) $\Rightarrow$ (5) Let F be a free R/J-module such that $\delta : F \to B$ is an epimorphism. Then there is an exact sequence $0 \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(F, M)$. Since $\mathrm{Hom}_R(F, M) \cong \prod \mathrm{Hom}_R(R/J, M) = 0$, it follows that $\mathrm{Hom}_R(B, M) = 0$.

(5) $\Rightarrow$ (2) This follows by setting $B := R/J$. □

Let N be an R-module. Then for any family $\{M_i\}$ of R-modules, we have the following natural homomorphisms from [13]:
$$\theta_1 : \prod_{i\in\Gamma} \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R\left(N, \prod_{i\in\Gamma} M_i\right),$$
$$\theta_1([f_i])(x) = [f_i(x)] \text{ for } x \in N \text{ and } f_i \in \mathrm{Hom}_R(N, M_i)$$
and
$$\theta_2 : \bigoplus_{i\in\Gamma} \mathrm{Hom}_R(N, M_i) \cong \mathrm{Hom}_R\left(N, \bigoplus_{i\in\Gamma} M_i\right),$$
$$\theta_2([f_i])(x) = [f_i(x)] \text{ for } x \in N \text{ and finite non-zero } f_i \in \mathrm{Hom}_R(N, M_i).$$

1. If N is finitely generated, then θ_1 is an isomorphism.
2. If N is finitely presented, then θ_2 is an isomorphism.

Considering $N := R/J$ for $J \in \mathcal{I}_S^f(R)$ in the above homomorphisms, we obtain the following result.

COROLLARY 2.2. *Let R be a commutative ring, S a multiplicative subset of R, and $\{M_i \mid i \in \Gamma\}$ a family of R-modules. Then $\prod_{i\in\Gamma} M_i$ is S-torsion-free if and only if all M_i are S-torsion-free, if and only if $\bigoplus_{i\in\Gamma} M_i$ is S-torsion-free.*

PROPOSITION 2.3. *An R-module M is S-torsion if and only if $\mathrm{Ann}_R(x) \in \mathcal{I}_S(R)$ for all $x \in M$.*

Proof. Since M is S-torsion if and only if for any $x \in M$ there exists $s \in S$ such that $sx = 0$, we can conclude the result immediately. □

The following result follows directly from the fact that $(\mathcal{T}, \mathcal{F})$ is a (hereditary) torsion theory. For completeness, however, we provide its proof.

THEOREM 2.4. *1. A module M is S-torsion if and only if $\mathrm{Hom}_R(M, N) = 0$ for any S-torsion-free module N.*
2. A module N is S-torsion-free if and only if $\mathrm{Hom}_R(M, N) = 0$ for any S-torsion module M.

Proof. (1) Assume that M is S-torsion and let $f \in \mathrm{Hom}_R(M, N)$. Then $\mathrm{Im}(f)$ is an S-torsion submodule of N. Since N is S-torsion-free, we have $f(M) = 0$, and hence $f = 0$. Conversely, set $T := \mathrm{tor}_S(M)$ and $N := M/T$. Then N is S-torsion-free. Thus the natural homomorphism $\pi : M \to N$ is the zero homomorphism since $\mathrm{Hom}_R(M, N) = 0$. Therefore, $N = 0$, that is, $M = \mathrm{tor}_S(M)$, and so M is S-torsion.

(2) Assume that N is S-torsion-free. By (1), we have $\mathrm{Hom}_R(M, N) = 0$ for any S-torsion module M. Conversely, set $M := \mathrm{tor}_S(N)$. Then $\mathrm{Hom}_R(M, N) = 0$. Thus the inclusion homomorphism $M \to N$ is the zero homomorphism. Therefore, $M = 0$, and so N is S-torsion-free. □

COROLLARY 2.5. *Let R be a commutative ring, S a multiplicative subset of R, and $\{M_i \mid i \in \Gamma\}$ a family of S-torsion modules. Then $\bigoplus_{i\in\Gamma} M_i$ is S-torsion.*

Proof. This follows immediately by Theorem 2.4 using the following isomorphism

$$\mathrm{Hom}_R\left(\bigoplus_{i\in\Gamma} M_i, N\right) \cong \prod_{i\in\Gamma} \mathrm{Hom}_R\left(M_i, N\right)$$

for any R-module N. □

PROPOSITION 2.6. *Let R and T be rings, $f : R \to T$ a monomorphism of rings, and S a multiplicative subset of R. If M is an S-torsion R-module, then $M \otimes_R T$ is an $f(S)$-torsion T-module.*

Proof. Note that if $I \in \mathcal{I}_S(R)$, then $f(I) \in \mathcal{I}_{f(S)}(T)$. So we can easily deduce the result using Proposition 2.3. □

COROLLARY 2.7. *If M is an S-torsion R-module, then $M[x] = M \otimes_R R[x]$ as an $R[x]$-module is also an S-torsion module.*

We now give an analog of the Five Lemma in S-torsion theory.

THEOREM 2.8. *Consider the following S-torsion commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
D & \xrightarrow{h} & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{k} & E \\
\delta\downarrow & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \mu\downarrow \\
D' & \xrightarrow{h'} & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{k'} & E'
\end{array}$$

1. *If α and γ are S-torsion monomorphisms and δ is an S-torsion epimorphism, then β is an S-torsion monomorphism.*
2. *If α and γ are S-torsion epimorphisms and μ is an S-torsion monomorphism, then β is an S-torsion epimorphism.*

Proof. (1) Let $b \in \mathrm{Ker}_S(\beta)$. Then there exists $t_1 \in S$ such that $t_1\beta(b) = 0$. On the other hand, there exists $t_2 \in S$ such that $t_2\gamma \circ g(b) = t_2g' \circ \beta(b)$. Hence $t_1t_2\gamma \circ g(b) = t_2g'(t_1\beta(b)) = 0$. Therefore, $g(b) \in \mathrm{Ker}_S(\gamma)$. Since γ is an S-torsion monomorphism, there exists $t_3 \in S$ such that $t_3g(b) = 0$, and so $b \in \mathrm{Ker}_S(g) = \mathrm{Im}_S(f)$. Then $t_4b = t_4f(a)$ for some $a \in A$ and $t_4 \in S$. Hence

$$t_4(\beta \circ f(a) - f' \circ \alpha(a)) = t_4(\beta(b) - f'(\alpha(a))).$$

Since $a \in A$, it follows that $t_5(f' \circ \alpha(a) - \beta \circ f(a)) = 0$ for some $t_5 \in S$. Therefore,

$$\begin{aligned} 0 &= t_1t_4t_5(f' \circ \alpha(a) - \beta \circ f(a)) \\ &= -t_1t_5t_4(\beta(b) + f' \circ \alpha(a)) \\ &= -t_5t_4\beta(t_1b) + t_1t_4t_5f' \circ \alpha(a) \\ &= t_1t_4t_5f' \circ \alpha(a). \end{aligned}$$

Hence $\alpha(a) \in \mathrm{Ker}_S(f') = \mathrm{Im}_S(h)$, and so $t_6\alpha(a) = t_6h'(x')$ for some $t_6 \in S$ and $x' \in D'$. Since δ is an S-torsion epimorphism, there exist $x \in D$ and $t_7 \in S$ such that $t_7\delta(x) = t_7x'$. Hence

$$\begin{aligned} t_6t_7\alpha(a) &= t_7t_6h'(x') \\ &= t_6h'(t_7x') \\ &= t_6h'(t_7\delta(x)) \\ &= t_6t_7h' \circ \delta(x). \end{aligned}$$

On the other hand, since $x \in D$, it follows that $t_8h'\delta(x) = t_8\alpha h(x)$ for some $t_8 \in S$. So $t_6t_7t_8\alpha(a) = t_6t_7t_8h' \circ \delta(x) = t_6t_7t_8\alpha \circ h(x)$, and hence $t_6t_7t_8\alpha(a - h(x)) = 0$. Therefore, $a - h(x) \in \mathrm{Ker}_S(\alpha) = \mathrm{tor}_S(A)$, and hence there exists $t_9 \in S$ such that $t_9a = t_9h(x)$. Since $h(x) \in \mathrm{Im}\,(h) \subseteq \mathrm{Im}_S(h) = \mathrm{Ker}_S(f)$, we get $t_{10}f \circ h(x) = 0$ for some $t_{10} \in S$. Then

$$\begin{aligned} t_4t_9t_{10}b &= t_9t_{10}t_4f(a) \\ &= t_{10}t_4f(t_9h(a)) \\ &= t_4t_9t_{10}f \circ h(a) = 0. \end{aligned}$$

Therefore, $tb = 0$ with $t := t_4t_9t_{10} \in S$, and so $b \in \mathrm{tor}_S(B)$. Thus β is an S-torsion monomorphism.

(2) Let $b' \in B'$. Since γ is an S-torsion epimorphism, there exist $c \in C$ and $t_1 \in S$ such that $t_1\gamma(c) = t_1g'(b')$. The S-torsion commutativity of the right square gives $t_2\mu \circ k(c) = t_2k' \circ \gamma(c)$ for some $t_2 \in S$. Then

$$\begin{aligned} t_1t_2\mu \circ k(c) &= t_2k'(t_1\gamma(c)) \\ &= t_2k'(t_1g'(b')) \\ &= t_1t_2k' \circ g'(b'). \end{aligned}$$

Since $g'(b') \in \mathrm{Im}\,(g') \subseteq \mathrm{Im}\,_S(g') = \mathrm{Ker}_S(k')$, there exists $t_3 \in S$ such that $t_3 k' \circ g'(b') = 0$, and so $t_1 t_2 t_3 \mu \circ k(c) = 0$. Therefore, $k(c) \in \mathrm{Ker}_S(\mu) = \mathrm{tor}_S(E)$. Consequently, there exists $t_4 \in S$ such that $t_4 k(c) = 0$, and hence $c \in \mathrm{Ker}_S(k) = \mathrm{Im}\,_S(g)$, that is, $t_5 c = t_5 g(b)$ for some $t_5 \in S$ and $b \in B$.

On the other hand, since $b \in B$, there exists $t_6 \in S$ such that $t_6 \gamma \circ g(b) = t_6 g' \circ \beta(b)$. Then

$$\begin{aligned} t_1 t_5 t_6 g'(b') &= t_1 t_5 t_6 \gamma(c) \\ &= t_1 t_6 \gamma(t_5 g(b)) \\ &= t_1 t_5 t_6 g' \circ \beta(b). \end{aligned}$$

Thus $t_1 t_5 t_6 g'(b' - \beta(b)) = 0$, and so $b' - \beta(b) \in \mathrm{Ker}_S(g') = \mathrm{Im}\,_S(f')$. Hence there exist $t_7 \in S$ and $a' \in A'$ such that $t_7(b' - \beta(b)) = t_7 f'(a')$. Since α is an S-torsion epimorphism, there exist $a \in A$ and $t_8 \in S$ such that $t_8 \alpha(a) = t_8 a'$. Hence

$$t_8 t_7 (b' - \beta(b)) = t_8 t_7 f'(a') = t_7 t_8 f' \circ \alpha(a).$$

Since $a \in A$, there exists $t_9 \in S$ such that $t_9 f' \circ \alpha(a) = t_9 \beta \circ f(a)$, and so

$$t_9 t_8 t_7 (b' - \beta(b)) = t_7 t_8 t_9 \beta \circ f(a).$$

Thus $tb' = t\beta(b + f(a))$ with $t := t_7 t_8 t_9 \in S$. Consequently, β is an S-torsion epimorphism. □

Let M be an R-module. Define $\psi : M \to M_S$ by $\psi(x) = \frac{x}{1}$ for every $x \in M$.

PROPOSITION 2.9. *Let $f : A \to B$ be an R-module homomorphism. Then $A/\mathrm{Ker}_S(f) \cong \psi(\mathrm{Im}\,(f))$.*

Proof. Let $x, y \in A$. Then we have:

$$\begin{aligned} \frac{f(x)}{1} = \frac{f(y)}{1} \in \psi(\mathrm{Im}\,(f)) &\Longleftrightarrow \exists s \in S : sf(x) = sf(y) \\ &\Longleftrightarrow \exists s \in S : sf(x - y) = 0 \\ &\Longleftrightarrow x - y \in \mathrm{Ker}_S(f) \\ &\Longleftrightarrow \overline{x} = \overline{y} \in A/\mathrm{Ker}_S(f). \end{aligned}$$

Hence the homomorphism:

$$g : A/\mathrm{Ker}_S(f) \to \psi(\mathrm{Im}\,(f)) \quad \text{defined by}$$

$$\overline{x} \mapsto g(\overline{x}) = \frac{f(x)}{1}$$

is an isomorphism. □

Let $\mathfrak{p}$ be a prime ideal of R. We say that an R-module homomorphism $f : M \to N$ is a $\mathfrak{p}$-*torsion epimorphism* (resp., $\mathfrak{p}$-*torsion monomorphism*, $\mathfrak{p}$-*torsion isomorphism*) if it is an $(R \setminus \mathfrak{p})$-torsion epimorphism (resp., $(R \setminus \mathfrak{p})$-torsion monomorphism, $(R \setminus \mathfrak{p})$-torsion isomorphism).

PROPOSITION 2.10. *Let $f : M \to N$ be an R-module homomorphism. Then the following statements are equivalent.*

1. *f is an epimorphism.*
2. *f is a $\mathfrak{p}$-torsion epimorphism for any prime ideal $\mathfrak{p}$ of R.*
3. *f is an $\mathfrak{m}$-torsion epimorphism for any maximal ideal $\mathfrak{m}$ of R.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) These are straightforward.

(3) ⇒ (1) Let $y \in N$. Since f is an $\mathfrak{m}$-torsion epimorphism for every maximal ideal $\mathfrak{m}$ of R, there exist $s_{\mathfrak{m}} \in R \setminus \mathfrak{m}$ and $x_{\mathfrak{m}} \in M$ such that $s_{\mathfrak{m}} f(x_{\mathfrak{m}}) = s_{\mathfrak{m}} y$. Since the ideal generated by all $s_{\mathfrak{m}}$ is equal to R, there exist maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_k$ and $\alpha_1, \dots, \alpha_k \in R$ such that $\alpha_1 s_{\mathfrak{m}_1} + \cdots + \alpha_k s_{\mathfrak{m}_k} = 1$. Thus

$$y = (\alpha_1 s_{\mathfrak{m}_1} + \cdots + \alpha_k s_{\mathfrak{m}_k}) y = \alpha_1 s_{\mathfrak{m}_1} f(x_{\mathfrak{m}_1}) + \cdots + \alpha_k s_{\mathfrak{m}_k} f(x_{\mathfrak{m}_k}) \in \mathrm{Im}\,(f).$$

So f is an epimorphism. □

Proposition 2.11. *Let $f : M \to N$ be an R-module homomorphism. Then the following statements are equivalent.*

1. *f is a monomorphism.*
2. *f is a $\mathfrak{p}$-torsion monomorphism for any prime ideal $\mathfrak{p}$ of R.*
3. *f is an $\mathfrak{m}$-torsion monomorphism for any maximal ideal $\mathfrak{m}$ of R.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) These are straightforward.

(3) ⇒ (1) Let $x \in \mathrm{Ker}(f)$ and set $S_{\mathfrak{m}} := R \setminus \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of R. Then $x \in \mathrm{Ker}_{S_{\mathfrak{m}}}(f) = \mathrm{tor}_{S_{\mathfrak{m}}}(M)$ for any maximal ideal $\mathfrak{m}$ of R. Hence there exists $s_{\mathfrak{m}} \in S_{\mathfrak{m}}$ such that $s_{\mathfrak{m}} x = 0$ for every maximal ideal $\mathfrak{m}$ of R. Since the ideal generated by all $s_{\mathfrak{m}}$ is equal to R, there exist maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_k$ and $\alpha_1, \dots, \alpha_k \in R$ such that $\alpha_1 s_{\mathfrak{m}_1} + \cdots + \alpha_k s_{\mathfrak{m}_k} = 1$. Thus

$$x = (\alpha_1 s_{\mathfrak{m}_1} + \cdots + \alpha_k s_{\mathfrak{m}_k}) x = \alpha_1 s_{\mathfrak{m}_1} x + \cdots + \alpha_k s_{\mathfrak{m}_k} x = 0.$$

Consequently, f is a monomorphism. □

Corollary 2.12. *Let $f : M \to N$ be an R-module homomorphism. Then the following statements are equivalent.*

1. *f is an isomorphism.*
2. *f is a $\mathfrak{p}$-torsion isomorphism for any prime ideal $\mathfrak{p}$ of R.*
3. *f is a $\mathfrak{p}$-torsion monomorphism and $\mathfrak{p}$-torsion epimorphism for any prime ideal $\mathfrak{p}$ of R.*
4. *f is an $\mathfrak{m}$-torsion isomorphism for any maximal ideal $\mathfrak{m}$ of R.*
5. *f is an $\mathfrak{m}$-torsion monomorphism and $\mathfrak{m}$-torsion epimorphism for any maximal ideal $\mathfrak{m}$ of R.*

Proposition 2.13. *Let A, B, C, and D be R-modules, and $f : A \to B$, $g : B \to D$, $h : A \to C$, and $k : C \to D$ be homomorphisms of R-modules. Then the following diagram:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{k} & D \end{array}$$

is commutative if and only if it is $\mathfrak{p}$-torsion (resp., $\mathfrak{m}$-torsion) commutative for any prime (resp., maximal) ideal $\mathfrak{p}$ (resp., $\mathfrak{m}$) of R.

Proof. It is clear that every commutative diagram is $\mathfrak{p}$-torsion (resp., $\mathfrak{m}$-torsion) commutative for any prime (resp., maximal) ideal $\mathfrak{p}$ (resp., $\mathfrak{m}$) of R.

Conversely, suppose that the above diagram is $\mathfrak{m}$-torsion commutative for any maximal ideal $\mathfrak{m}$ of R. Then for every $a \in A$ and every maximal ideal $\mathfrak{m}$ of R, there exists $s_{\mathfrak{m}} \in R \setminus \mathfrak{m}$ such that $s_{\mathfrak{m}} gf(a) = s_{\mathfrak{m}} kh(a)$. Since the ideal generated by all $s_{\mathfrak{m}}$ is equal to R, there exist maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ and $\alpha_1, \ldots, \alpha_k \in R$ such that $\alpha_1 s_{\mathfrak{m}_1} + \cdots + \alpha_k s_{\mathfrak{m}_k} = 1$. Thus

$$gf(a) = (\alpha_1 s_{\mathfrak{m}_1} + \cdots + \alpha_k s_{\mathfrak{m}_k})gf(a) = \alpha_1 s_{\mathfrak{m}_1} kh(a) + \cdots + \alpha_k s_{\mathfrak{m}_k} kh(a) = kh(a).$$

Therefore, the above diagram is commutative. □

Proposition 2.14. *Let R be a ring and $(*)$ $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of R-modules and homomorphisms. Then the following statements are equivalent.*

1. *$(*)$ is a complex (resp., an exact sequence).*
2. *$(*)$ is a $\mathfrak{p}$-torsion complex (resp., $\mathfrak{p}$-torsion exact sequence) for any prime ideal $\mathfrak{p}$ of R.*
3. *$(*)$ is an $\mathfrak{m}$-torsion complex (resp., $\mathfrak{m}$-torsion exact sequence) for any maximal ideal $\mathfrak{m}$ of R.*

Proof. This is analogous to Proposition 2.13. □

We end this section with the following theorem, which characterizes when every S-torsion commutative diagram (resp., S-torsion exact sequence, S-torsion monomorphism, S-torsion epimorphism, S-torsion isomorphism) is commutative (resp., exact, monomorphism, epimorphism, isomorphism).

Proposition 2.15. *The following conditions are equivalent for a ring R.*

1. *Every S-torsion commutative diagram is commutative.*
2. *Every S-torsion exact sequence is exact.*
3. *Every S-torsion monomorphism is a monomorphism.*
4. *Every S-torsion epimorphism is an epimorphism.*
5. *Every S-torsion isomorphism is an isomorphism.*
6. *Every element of S is a unit.*

Proof. (1) $\Rightarrow$ (5), (2) $\Rightarrow$ (3) and (5), and (6) $\Rightarrow$ (1), (2), (3) and (4) are straightforward.

(3) $\Rightarrow$ (6) Let $a \in S$ and consider the homomorphism $f : R/Ra \to 0$. Since $\mathrm{tor}_S(R/Ra) = R/Ra$, it follows that $R/Ra = \mathrm{tor}_S(R/Ra) \subseteq \mathrm{Ker}_S(f) \subseteq R/Ra$, and so $\mathrm{Ker}_S(f) = \mathrm{tor}_S(R/Ra)$. Hence f is an S-torsion monomorphism, and so it is a monomorphism by (3). Then $R/Ra = \mathrm{Ker}(f) = 0$, and so a is a unit.

(4) $\Rightarrow$ (6) Let $a \in S$ and consider the homomorphism $f : 0 \to R/Ra$. Since $\mathrm{tor}_S(R/Ra) = R/Ra$, it follows that $\mathrm{Im}_S(f) = \mathrm{Im}(f) + \mathrm{tor}_S(R/Ra) = 0 + R/Ra = R/Ra$. Hence f is an S-torsion epimorphism, and so it is an epimorphism. Consequently, $0 = \mathrm{Im}(f) = R/Ra$, and so a is a unit.

(5) $\Rightarrow$ (6) Let $a \in S$. Since $a(R/Ra) = 0$, it is easy to verify that $R/Ra \overset{S}{\simeq} 0$ (see Lemma 3.3), and so $R/Ra = 0$ by (5). Therefore, a is a unit. □

3. Characterization of S_1-projective modules using projective modules

The S_1-projective module was studied in [11] using an S_1-free module, a right S-torsion split sequence, and an S-torsion projective basis. In particular, if an R-module P is S-torsion isomorphic to a projective module, then P is S_1-projective. However, they did not show that these two versions, the S_1-projective module and the module that is S-torsion isomorphic to a projective module, coincide, nor did they provide any examples to distinguish them. We will address this question in this section.

It is noteworthy that, in the same context, Pu, Wang, and Zhao introduced and studied new concepts of nonnil-commutative diagrams and nonnil-projective modules in [10]. Among other results, they proved that an R-module nonnil-isomorphic to a projective module is nonnil-projective and proposed the following problem: Is every nonnil-projective module nonnil-isomorphic to some projective module? This problem was resolved affirmatively by the authors in [5].

Similarly, Zhang and Qi introduced and studied the concept of u(niformly)-S-projective modules in [14], where S denotes a multiplicative subset of a ring. In particular, they proved that an R-module u-S-isomorphic to a projective module is u-S-projective. Furthermore, H. Kim et al. demonstrated that the converse holds if S is regular [8, Theorem 2.11].

These results establish a robust foundation for understanding projective modules, underscoring their significance within the broader framework of generalized projectivity.

The following theorem resolves this issue by stating that an R-module is S_1-projective if and only if it is S-torsion isomorphic to a projective module.

THEOREM 3.1. *Every S_1-projective module is S-torsion isomorphic to some projective module.*

We need the following lemmas to prove Theorem 3.1.

LEMMA 3.2. *If $A_1 \overset{S}{\simeq} B_1$ and $A_2 \overset{S}{\simeq} B_2$, then $A_1 \oplus A_2 \overset{S}{\simeq} B_1 \oplus B_2$.*

Proof. Let $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ be two S-torsion isomorphisms. Then there exist two homomorphisms $g_1 : B_1 \to A_1$ and $g_2 : B_2 \to A_2$ such that $\operatorname{Im}_S(\mathbf{1}_{A_1} - f_1 \circ g_1) = \operatorname{tor}_S(A_1)$, $\operatorname{Im}_S(\mathbf{1}_{B_1} - g_1 \circ f_1) = \operatorname{tor}_S(B_1)$, $\operatorname{Im}_S(\mathbf{1}_{A_2} - f_2 \circ g_2) = \operatorname{tor}_S(A_2)$, and $\operatorname{Im}_S(\mathbf{1}_{B_2} - g_2 \circ f_2) = \operatorname{tor}_S(B_2)$. Define

$$\begin{aligned} f : A_1 \oplus A_2 &\to B_1 \oplus B_2 \quad \text{by} \\ (x_1, x_2) &\mapsto f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \end{aligned}$$

and

$$\begin{aligned} g : B_1 \oplus B_2 &\to A_1 \oplus A_2 \quad \text{by} \\ (x_1, x_2) &\mapsto g(x_1, x_2) = (g_1(x_1), g_2(x_2)). \end{aligned}$$

Then it is easy to verify that:

$$\begin{aligned} \operatorname{Im}_S(\mathbf{1}_{A_1 \oplus A_2} - f \circ g) &= \operatorname{Im}_S(\mathbf{1}_{A_1} - f_1 \circ g_1) \oplus \operatorname{Im}_S(\mathbf{1}_{A_2} - f_2 \circ g_2) \\ &= \operatorname{tor}_S(A_1) \oplus \operatorname{tor}_S(A_2) \\ &= \operatorname{tor}_S(A_1 \oplus A_2) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Im}_S(\mathbf{1}_{B_1 \oplus B_2} - g \circ f) &= \operatorname{Im}_S(\mathbf{1}_{B_1} - g_1 \circ f_1) \oplus \operatorname{Im}_S(\mathbf{1}_{B_2} - g_2 \circ f_2) \\ &= \operatorname{tor}_S(B_1) \oplus \operatorname{tor}_S(B_2) \\ &= \operatorname{tor}_S(B_1 \oplus B_2). \end{aligned}$$

Hence $A_1 \oplus A_2 \overset{S}{\simeq} B_1 \oplus B_2$. □

Lemma 3.3. *Let M be an R-module. Then $M \overset{S}{\simeq} 0$ if and only if M is an S-torsion R-module.*

Proof. Let $f : M \to 0$ be an S-torsion isomorphism. Then $\mathrm{Im}_S(1_M - f \circ 0) = \mathrm{tor}_S(M)$. Since $\mathrm{Im}_S(1_M - f \circ 0) = \mathrm{Im}_S(1_M) = M$, we get $M = \mathrm{tor}_S(M)$.

Conversely, assume that $M = \mathrm{tor}_S(M)$. Then $f : M \to 0$ is an S-torsion isomorphism since $\mathrm{Im}_S(1_M) = M = \mathrm{tor}_S(M)$. □

For any submodule N of an R-module M and any multiplicative subset S of R, we define

$$S^M(N) := \{x \in M \mid sx \in N \text{ for some } s \in S\},$$

called the S-component of N in M. If there is no confusion, we will also write $S(N)$ instead of $S^M(N)$.

Lemma 3.4. *Let $f : A \to B$ be an S-torsion isomorphism and N be a submodule of A. Then $S(N) \overset{S}{\simeq} f(S(N))$.*

Proof. Let $g : B \to A$ be a homomorphism such that $\mathrm{Im}_S(\mathbf{1}_A - g \circ f) = \mathrm{tor}_S(A)$ and $\mathrm{Im}_S(\mathbf{1}_B - f \circ g) = \mathrm{tor}_S(B)$. Define $f_{S(N)} : S(N) \to f(S(N))$ as the restriction of f on $S(N)$. Let $y = f(n') \in f(S(N))$ with $n' \in N$. Then there exists $t_1 \in S$ such that $t_1 n' \in N$. On the other hand, since $\mathrm{Im}_S(\mathbf{1}_A - g \circ f) = \mathrm{tor}_S(A)$, we get $n' - (g \circ f)(n') \in \mathrm{tor}_S(A)$. Then $t_2 n' = t_2(f \circ g)(n')$ for some $t_2 \in S$, and hence $t_2 t_1 g(y) = t_2 t_1 n' \in N$. Therefore, $f(y) \in S(N)$, and it is easy to verify that $\mathrm{Im}_S(\mathbf{1}_{S(N)} - g_{f(S(N))} \circ f_{S(N)}) = \mathrm{tor}_S(S(N))$ and $\mathrm{Im}_S(\mathbf{1}_{f(S(N))} - f_{S(N)} \circ g_{f(S(N))}) = \mathrm{tor}_S(f(S(N)))$. Hence $S(N) \overset{S}{\simeq} f(S(N))$. □

Lemma 3.5. *If N is a direct summand of A, then $S(N) \overset{S}{\simeq} N$.*

Proof. Let $A = N \oplus L$ for some submodule L of A. Let $x = n + l \in S(N)$ with $n \in N$ and $l \in L$. Then $tx = tn + tl \in N$ for some $t \in S$. Thus, $tl = tx - tn \in N \cap L = 0$, and so $tl = 0$, i.e., $l \in \mathrm{tor}_S(L)$. Therefore, $S(N) \subseteq N \oplus \mathrm{tor}_S(L)$.

Conversely, let $x = n + l \in N \oplus \mathrm{tor}_S(L)$. Then $tl = 0$ for some $t \in S$. Hence $tx = tn \in N$, and so $x \in S(N)$. Consequently, $S(N) = N \oplus \mathrm{tor}_S(L)$. Since $\mathrm{tor}_S(L) \overset{S}{\simeq} 0$ by Lemma 3.3, we have $S(N) \overset{S}{\simeq} N$ by Lemma 3.2. □

Proof of Theorem 3.1

Let P be an S_1-projective module. Then by [10, Theorem 3.7], P is a direct summand of an S_1-free module. Hence there exists a free R-module F such that $A = P \oplus L$ is S-torsion isomorphic to F. Let $f : A \to F$ be an S-torsion isomorphism. We want to show that $F = f(P) \oplus f(L)$. For this, let $g : F \to A$ be a homomorphism such that $\mathrm{Im}_S(\mathbf{1}_A - g \circ f) = \mathrm{tor}_S(A)$ and $\mathrm{Im}_S(\mathbf{1}_F - f \circ g) = \mathrm{tor}_S(F)$. Since F is a free R-module, $\mathrm{tor}_S(F) = 0$, and hence $\mathrm{Im}\,(\mathbf{1}_F - f \circ g) \subseteq \mathrm{Im}_S(\mathbf{1}_F - f \circ g) = \mathrm{tor}_S(F) = 0$. Therefore, f is an epimorphism, i.e., $F = f(A)$. Consequently, $F = f(P) + f(L)$.

Let $y \in f(P) \cap f(L)$. Then there exist $x \in P$ and $l \in L$ such that $y = f(x) = f(l)$. Thus $f(x - l) = 0$, and so $x - l \in \mathrm{Ker}(f) \subseteq \mathrm{Ker}_S(f) = \mathrm{tor}_S(A)$. Then $tx = tl$ for some $t \in S$. Since $tx = tl \in P \cap L = 0$, it follows that $tx = 0$, so $ty = f(tx) = 0$. Then $y = 0$ since F is a free R-module. Thus $F = f(P) \oplus f(L)$. Therefore, $f(P)$ is a projective R-module. By Lemma 3.5, $P \overset{S}{\simeq} S(P)$, and then $P \overset{S}{\simeq} f(S(P))$ by Lemma 3.4. Note that $f(S(P)) = f(P \oplus \mathrm{tor}_S(L)) = f(P) + f(\mathrm{tor}_S(L))$. Since $f(\mathrm{tor}_S(L)) \subseteq \mathrm{tor}_S(F) = 0$, we get $f(S(P)) = f(P)$. So $P \overset{S}{\simeq} f(P)$ and $f(P)$ is a projective R-module. $\square$

Note that Lemma 3.2 can be used to provide another demonstration of [11, Corollary 3.9], as shown below.

Remark 3.6. If P is S-torsion isomorphic to a projective module, then P is S_1-projective.

Proof. Let K be a projective module such that $P \overset{S}{\simeq} K$. Since K is projective, it is a direct summand of a free module F, and so $F = K \oplus L$ for some L. Since $P \overset{S}{\simeq} K$, it follows from Lemma 3.2 that $P \oplus L \overset{S}{\simeq} K \oplus L = F$. Hence P is a direct summand of an S_1-free module. Thus, P is an S_1-projective module by [10, Theorem 3.7]. $\square$

Corollary 3.7. *Let P_1 and P_2 be S_1-projective R-modules. Then $P_1 \otimes P_2$ is S_1-projective.*

Proof. Let P_1' and P_2' be projective modules such that $P_1 \overset{S}{\simeq} P_1'$ and $P_2 \overset{S}{\simeq} P_2'$. Then it is easy to show that $P_1 \otimes P_2 \overset{S}{\simeq} P_1' \otimes P_2'$. Since P_1' and P_2' are projective modules, $P_1' \otimes P_2'$ is projective by [13, Theorem 2.3.8]. Hence $P_1 \otimes P_2$ is S_1-projective. $\square$

Corollary 3.8. *Let R be a local ring. Then every S_1-projective module is S_1-free.*

Proof. Let P be an S_1-projective R-module. Then there exists a projective R-module P_0 such that $P \overset{S}{\simeq} P_0$. Since R is a local ring, P is free by [13, Theorem 2.3.17]. Hence P is S_1-isomorphic to a free R-module, so P is S_1-free. $\square$

Let M be an R-module. Then M is said to be S-finitely generated if $\psi_S(M)$ is a finitely generated $\phi_S(R)$-module. It is easy to verify that an R-module M is S-finitely generated if and only if there exists a finite set $\{x_1, \dots, x_n\} \subseteq M$ such that for every element $x \in M$, $tx = t(\alpha_1 x_1 + \cdots + \alpha_n x_n)$ for some $t \in S$ and $\alpha_i \in R$.

Theorem 3.9. *Let R be a ring and I be an S_1-projective ideal of R such that $S \cap I \neq \emptyset$. Then I is S_1-finitely generated.*

Proof. Let I be an S_1-projective ideal of R such that $S \cap I \neq \emptyset$. Then by [11, Theorem 3.8], there exist elements $\{x_i \mid i \in \Gamma\} \subseteq I$ and R-homomorphisms $\{f_i \mid i \in \Gamma\} \subseteq \mathrm{Hom}_R(I, R)$ such that:

1. If $x \in I$, then almost all $f_i(x) = 0$.
2. If $x \in I$, then there exists an element $s \in S$ such that $sx = s\sum_i f_i(x)x_i$.

Let $a \in I \cap S$. Then there exists a finite subset K of Γ such that $f_i(a) = 0$ for all $i \in \Gamma \setminus K$. Now let $x \in I$. Then there exists an element $s \in S$ such that $sx = s\sum_i f_i(x)x_i$. Hence

$$asx = as\sum_i f_i(x)x_i = s\sum_i xf_i(a)x_i = s\sum_k xf_k(a)x_k = sa\sum_k f_k(x)x_k \in saF$$

with $F = \sum_k Rx_k$. Therefore, I is S_1-finitely generated. $\square$

Let M be an S-torsion-free R-module. Then M is S_1-finitely generated if and only if it is finitely generated, and M is S_1-projective if and only if it is projective by [11, Lemma 4.1]. In particular, if S is a regular multiplicative subset of R and I is an ideal of R, then I is S_1-finitely generated if and only if it is finitely generated, and I is S_1-projective if and only if it is projective. It is well known that every projective ideal is finitely generated in an integral domain. The following result is a generalization of this fact.

Corollary 3.10. *Let R be a ring, then every regular projective ideal of R is finitely generated.*

Recall from [17] that an element r of a ring R is said to be S-regular if $\phi_S(r)$ is a regular element of $\phi_S(R)$. An ideal I of a ring R is said to be S-regular if $\phi_S(I)$ is a regular ideal of $\phi_S(R)$. Every regular element of a ring R is S-regular for any multiplicative subset S of R. Furthermore, each element in S is S-regular. It is also clear that each regular ideal of a ring R is S-regular for any multiplicative subset S of R. Moreover, an ideal I of R that meets S, i.e., $I \cap S \neq \emptyset$, is S-regular for any multiplicative subset S of R. Note that if R is a domain, then any S-regular element of R is regular, and any S-regular ideal of R is regular for any multiplicative subset S of R.

An R-submodule A of R_S is called an S-fractional ideal if there exists an element d in S such that dA is a regular ideal of $\phi_S(R)$. We denote by $\mathcal{F}_S(R)$ the set of all S-fractional ideals of R. We have $\phi_S(R) \in \mathcal{F}_S(R)$ and $\phi_S(I) \in \mathcal{F}_S(R)$ for every S-regular ideal I of R. Also, an S-fractional ideal A of a ring R is said to be S-invertible if there exists an S-fractional ideal B of R such that $AB = \phi_S(R)$. An ideal I of a ring R is said to be S-invertible if there exists an S-fractional ideal C of R such that $\phi_S(I)C = \phi_S(R)$.

Let I be an ideal of a ring R. Set

$$I_S^{-1} := \{x \in R_S \mid Ix \subseteq \phi_S(R)\}.$$

Then I is S-invertible if and only if $II_S^{-1} = \phi_S(R)$. Note that if I is an S-regular ideal, then I_S^{-1} is also an S-fractional ideal of R.

Theorem 3.11. *Let R be a ring and I be an ideal of R generated by finitely many elements in S. Then I is S_2-projective if and only if I is S-invertible.*

Proof. Assume that I is generated by elements $r_1, r_2, \dots, r_n$ in S. Set

$$x_i := \phi_S(r_i) = \frac{r_i}{1} \in \phi_S(I), \quad 1 \leqslant i \leqslant n.$$

If I is S-invertible, then $II_S^{-1} = \phi_S(R)$. Thus we have

$$\frac{1}{1} = \sum_{i=1}^{n} r_i h_i = \sum_{i=1}^{n} x_i h_i \quad \text{for some } h_i \in I_S^{-1}.$$

Set

$$I_S^* := \operatorname{Hom}_{\phi_S(R)}\left(\phi_S(I), \phi_S(R)\right),$$

and consider the following homomorphism:

$$\varphi_S : I_S^{-1} \to I_S^*$$

such that $\varphi_S(x)(a) = ax$, where $x \in I_S^{-1}$ and $a \in \phi_S(I)$. Then φ_S is an isomorphism of R-modules according to [17, Theorem 2.5]. Set $f_i := \varphi_S(h_i)$. Then

$$\phi_S(x) = \frac{x}{1} = \sum_{i=1}^{n} x_i h_i \frac{x}{1} = \sum_{i=1}^{n} f_i\left(\phi_S(x)\right) x_i$$

for all $x \in I$. Therefore, $\phi_S(I)$ has a projective basis $\{x_i, f_i \mid 1 \leqslant i \leqslant n\}$.

Conversely, if I has an S-projective basis $\{x_i, f_i \mid i \in \Gamma\}$ such that $\phi_S(x) = \sum_{i\in\Gamma} f_i\left(\phi_S(x)\right) x_i$ is a finite sum for all $x \in I \cap S$, set

$$a_i := \frac{f_i\left(\frac{x}{1}\right)}{x} \in R_S.$$

Then $a_i x = f_i\left(\frac{x}{1}\right) \in \phi_S(R)$. If $a \in I$, then $a = \sum_{i=1}^n r_i t_i$ for some $t_i \in R$, $1 \leqslant i \leqslant n$, where all r_i are in $S \cap I$. Thus

$$aa_i = \sum_{i=1}^n \left(r_i a_i\right) t_i \in \phi_S(R),$$

and so $a_i I \subseteq \phi_S(R)$ and $a_i \phi_S(I) \subseteq \phi_S(R)$. Hence

$$a_i \in I_S^{-1} \quad \text{and} \quad \phi_S(x) = \frac{x}{1} = \sum_{i=1}^n a_i x_i \frac{x}{1}.$$

Therefore, $1 = \sum_{i=1}^n a_i x_i$ and I is an S-invertible ideal of R. □

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