Korean J. Math. **33** (2025), No. 1, pp. 7–19
https://dx.doi.org/10.11568/kjm.2025.33.1.7

GENERALIZED HILBERT OPERATOR ON BERGMAN SPACES

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Abstract. We consider the generalized Hilbert operator $\mathcal{H}_\beta$ for $\beta \geq 0$ and find the condition on the parameter β for which the operator $\mathcal{H}_\beta$ is bounded on the Bergman space A^p for $2 < p < \infty$. Also, we estimates the upper bound of the norm on A^p. Further shows that $\mathcal{H}_\beta$ is not bounded on A^2 for $0 \leq \beta < 1$.

1. Introduction

We denote $\mathbb{D}$ as the unit disc in the complex plane $\mathbb{C}$. We consider the generalized Hilbert's inequality, as stated in [7] gives that for $\beta \geq 0$, if $1 < p < \infty$ and $(a_k)_{k \in N_0} \in \ell^p$ then the following inequality holds:

$$\left(\sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{\Gamma(n+\beta+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+k+\beta+2)} a_k \right|^p \right)^{\frac{1}{p}} \leq \frac{\Gamma(\frac{1}{p})\Gamma(\frac{1}{q}+\beta)}{\Gamma(1+\beta)} \left(\sum_{k=0}^{\infty} |a_k|^p \right)^{\frac{1}{p}}.$$

We call this as generalized Hilbert inequality, since when $\beta = 0$, it is the well-known Hilbert inequality

$$\left(\sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right|^p \right)^{\frac{1}{p}} \leq \frac{\pi}{\sin(\pi/p)} \left(\sum_{k=0}^{\infty} |a_k|^p \right)^{\frac{1}{p}}.$$

The constant $\frac{\pi}{\sin(\pi/p)}$ is best possible (see [4]). Thus the so-called generalized Hilbert matrix for $\beta \geq 0$, $H_\beta = \left(\frac{\Gamma(n+\beta+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+k+\beta+2)} \right)$, $n, k = 0, 1, 2 \ldots$ can be viewed as an operator on spaces of analytic functions by its action on the Taylor coefficients. If f is analytic on $\mathbb{D}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then for $\beta \geq 0$, so-called generalized Hilbert operator denoted by $\mathcal{H}_\beta$ is defined by

$$\mathcal{H}_\beta f(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\Gamma(n+\beta+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+k+\beta+2)} a_k \right) z^n.$$

The operator $\mathcal{H}_\beta$ was introduce in [6]. In particular $\beta = 0$, gives the classical Hilbert operator $\mathcal{H}$. In [12], we proved that the operator $\mathcal{H}_\beta$ is bounded on Hardy space H^p

Received November 23, 2024. Revised February 11, 2025. Accepted February 15, 2025.
2010 Mathematics Subject Classification: 30E20, 30H20, 31B10, 33C05.
Key words and phrases: Hilbert Operator,Bergman Spaces, Hypergeometric Functions.
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for $2 \leq p < \infty$ and found the upper bound for its norm see ([12], Corollary 3) as given below:

$$||\mathcal{H}_\beta||_{H^p} \leq \frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}+\beta\right)}{\Gamma(1+\beta)}.$$

We also the proved the above inequality holds for analytic functions f on $\mathbb{D}$ for $1 < p < 2$. Also in [13] obtained that the operator $\mathcal{H}_\beta$ is bounded on Dirichlet spaces.

Motivated by [3], in this article, we proved that $\mathcal{H}_\beta$ is a bounded operator on the Bergman spaces A^p for $2 < p < \infty$, which consist of analytic functions f on $\mathbb{D}$ for which

$$\| f \|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dm(z) < \infty,$$

where $dm(z) = \frac{1}{\pi} dxdy$ is the normalized Lebesgue measure on $\mathbb{D}$. We also found norm estimates of $\mathcal{H}_\beta$ on A^p spaces for $2 < p < \infty$.

2. Preliminaries

In this section, we shall discuss few results which we will use to proof our main results. First, we will find an integral form of $\mathcal{H}_\beta$. We provide an representation of $\mathcal{H}_\beta$ involving weighted composition operators for which we estimate A^p space norm. This representation is similar to the one use to prove the boundedness of generalized Cesàro operators, Cesàro operator on the Hardy spacess, Bergman spaces and Dirichlet spaces (see [8, 9, 14, 15]). Further in [10, 11], this representation is used to proof the boundedness of generalized Cesàro operator on BMOA and Bloch type spaces.

Lemma 2.1. *Let $f \in A^p$ for $p > 2$. Then*

$$\mathcal{H}_\beta(f)(z) = \int_0^1 \frac{f(t)(1-t)^\beta}{(1-tz)^{\beta+1}} dt$$

and $\mathcal{H}_\beta f \in A^p$.

Proof. Let us consider the operator

$$\mathcal{S}_\beta(f)(z) = \int_0^1 \frac{f(t)(1-t)^\beta}{(1-tz)^{\beta+1}} dt.$$

The convergence of the above integral is ensured by the Fejèr-Riesz inequality.

Let $f \in A^p$ for $p > 2$. From ([16], Corollary, p.755), we have

$$|f(z)| \leq \left(\frac{1}{1-|z|^2}\right)^{\frac{2}{p}} \| f \|_{A^p} . \tag{1}$$

Since $\beta \geq 0$, we get

$$\begin{aligned} |\mathcal{S}_\beta(f)(z)| &\leq ||f||_{A^p} \int_0^1 \frac{(1-t)^\beta}{(1-t)^{\frac{2}{p}}|1-tz|^{\beta+1}} dt \\ &\leq ||f||_{A^p} \int_0^1 \frac{\frac{1}{(1-t)^{\frac{2}{p}}}dt}{1-|z|} < \infty, \end{aligned}$$

which shows that the operator $\mathcal{S}_\beta$ is well defined on A^p spaces. Now, given $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A^p$, let $f_N(z) = \sum_{n=0}^{N} a_n z^n$, we obtain

$$\begin{aligned} \mathcal{H}_\beta(f_N)(z) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{N} \frac{\Gamma(n+\beta+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+k+\beta+2)} a_k z^n \right) \\ &= \frac{1}{\Gamma(\beta+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \left(\sum_{k=0}^{N} \int_0^1 \left(t^{n+k}(1-t)^\beta dt \right) a_k z^n \right) \\ &= \frac{1}{\Gamma(\beta+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \left(\int_0^1 f_N(t)(tz)^n(1-t)^\beta \, dt \right) \\ &= \int_0^1 f_N(t)(1-t)^\beta \sum_{n=0}^{\infty} \frac{(\beta+1,n)(1,n)}{(1,n)(1,n)} (tz)^n \, dt \\ &= \int_0^1 f_N(t)(1-t)^\beta F(\beta+1,1;1;tz) \, dt \\ &= \int_0^1 \frac{f_N(t)(1-t)^\beta dt}{(1-tz)^{\beta+1}} \\ &= \mathcal{S}_\beta f_N(z). \end{aligned}$$

Therefore, $\mathcal{H}_\beta$ is well defined on polynomials. Also, for $z \in \mathbb{D}$ and $p > 2$, we see that

$$\left| \mathcal{S}_\beta(f)(z) - \mathcal{H}_\beta(f_N)(z) \right| \leq \int_0^1 \frac{|f(t) - f_N(t)|}{1-|z|} dt \leq ||f - f_N||_{A^p} \int_0^1 \frac{(1-t)^{\frac{-2}{p}}}{1-|z|} \, dt.$$

Thus the series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{N} \frac{\Gamma(n+\beta+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+k+\beta+1)} a_k z^n$$

converges as $N \to \infty$ and defines an analytic function given by

$$\mathcal{H}_\beta(f)(z) = \mathcal{S}_\beta(f)(z) = \int_0^1 \frac{f(t)(1-t)^\beta}{(1-tz)^{\beta+1}} dt,$$

which belongs to the Bergman Spaces A^P for $p > 2$. □

Next result demonstrates how $\mathcal{H}_\beta$ can be expressed as the average of certain weighted composition operator.

Every analytic function $\phi : \mathbb{D} \to \mathbb{D}$ defines a bounded composition operator $C_\phi : f \to f \circ \phi$ onA^p for $1 \leq p \leq \infty$. The norm of this operator satisfies the inequality (as shown in [2],p.127)

$$\| C_\phi \|_{A^p} \leq \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|} \right)^{\frac{2}{p}} \tag{2}$$

Furthermore, if w(z) is bounded analytic function, the corresponding weighted composition operator

$$C_{\omega,\phi}(f)(z) = \omega(z) f(\phi(z))$$

is bounded on each A^p. This is the only property of the operator that will be relevant for our purposes.

The relationship between the Hilbert matrix and composition operators is established as follows:

The operator $\mathcal{H}_\beta$ can be written in terms of composition operators.

Lemma 2.2. *Let f be analytic in the $\mathbb{D}$. Suppose $z \in \mathbb{D}$ and $0 < r < 1$. Then*

$$\mathcal{H}_\beta f(z) = \int_0^1 T_t(f)(z) dt,$$

where

$$T_t(f)(z) = \omega_t(z) f(\phi_t(z))(1-t)^\beta,$$

$$\omega_t(z) = \frac{1}{(t-1)z+1} \quad \text{and} \quad \phi_t(z) = \frac{t}{(t-1)z+1}.$$

Proof. We define

$$C_r(f)(z) = \int_0^r f(t) \frac{(1-t)^\beta}{(1-tz)^{\beta+1}} dt. \tag{3}$$

Then, we have

$$\mathcal{H}_\beta(f)(z) = \lim_{r \to 1} C_r(f)(z)$$

Given $z \in \mathbb{D}$, we choose the path of integration as follows:

$t(s) = t_z(s) = \dfrac{rs}{r(s-1)z+1} dt$, $0 \leq s \leq 1$. By changing variable in (3), we obtain

$$\begin{aligned} C_r(f)(z) &= \int_0^1 \frac{f(t(s))(1-t(s))^\beta}{(1-t(s)z)^{\beta+1}} t'(s) ds \\ &= \int_0^1 \frac{r(r(s-1)z+1-rs)^\beta f(t(s))\, ds}{(1-rz)^\beta (r(s-1)z+1)}. \end{aligned}$$

Now, suppose $f \in A^P$ where $p > 2$. For every $z \in \mathbb{D}$ and $0 \leq s \leq 1$, define

$$h_r(s) = \frac{r(r(s-1)z+1-rs)^\beta}{(1-rz)^\beta (r(s-1)z+1)} f\left(\frac{rs}{r(s-1)z+1} \right)$$

$$= \frac{r(r(s-1)z+1-rs)^{\beta}}{(1-rz)^{\beta}(r(s-1)z+1)} f\Big(\phi_{r,s}(z)\Big),$$

where $\phi_{r,s}(z) = \dfrac{rs}{r(s-1)z+1}$, which is an analytic self-map on $\mathbb{D}$.

Since, $|r(s-1)z+1| \geq 1-|z|$, for $0 \leq s, r \leq 1$, and $|r(s-1)z+1-rs|^{\beta} \leq 2^{\beta}$ as $\beta \geq 0$, we find

$$\frac{r|r(s-1)z+1-rs|^{\beta}}{|(r(s-1)z+1)(1-rz)|^{\beta}} \leq \frac{2^{2\beta+1}}{(1-|z|^2)^{\beta+1}}$$

Thus, from (1), we obtain

$$|f \circ \phi_{r,s}(z)| \leq \frac{1}{(1-|z|^2)^{\frac{2}{p}}} \parallel f \circ \phi_{r,s}(z) \parallel_{A^p}$$

and from equation (2), we obtain

$$\begin{aligned} \parallel f \circ \phi_{r,s}(z) \parallel_{A^p} &\leq \left(\tfrac{1+|\phi_{r,s}(0)|}{1-|\phi_{r,s}(0)|}\right)^{\frac{2}{p}} \parallel f \parallel_{A^p} \\ &\leq \left(\tfrac{1+s}{1-s}\right)^{\frac{2}{p}} \parallel f \parallel_{A^p} . \end{aligned}$$

The above estimates lead to

$$|h_r(s)| \leq \frac{2^{2\beta+1}}{(1-|z|^2)^{\beta+1+\frac{2}{p}}} \left(\frac{1+s}{1-s}\right)^{\frac{2}{p}} \parallel f \parallel_{A^p} .$$

For $p > 2$, the expression on the right hand side is an integrable function of s, so by applying Lebesgue's dominated convergence theorem, we can conclude that

$$\begin{aligned} \mathcal{H}_{\beta}(f)(z) &= \lim_{r \to 1} \int_0^r \frac{r(r(s-1)z+1-rs)^{\beta}}{(1-rz)^{\beta}(r(s-1)z+1)} f\left(\frac{rs}{r(s-1)z+1}\right) ds \\ &= \int_0^1 \frac{(1-s)^{\beta}}{(s-1)z+1} f\left(\frac{s}{(s-1)z+1}\right) ds. \end{aligned}$$

In other words, we can represent $\mathcal{H}_{\beta}$ as:

$$\mathcal{H}_{\beta} f(z) = \int_0^1 T_t(f)(z)\, dt$$

of the family of weighted composition operator

$$T_t(f)(z) = \omega_t(z) f(\phi_t(z))(1-t)^{\beta},$$

where $\omega_t(z), \phi_t(z)$ as defined in the hypothesis. □

Thus, it is easy to observe that ω_t is a bounded function for $0 < t < 1$ and that ϕ_t is a self map of unit disc $\mathbb{D}$. Consequently the operator $T_t : A^p \to A^p, 1 \leq p \leq \infty$, is bounded on A^p for every $0 < t < 1$.

LEMMA 2.3. *For every analytic function f,*

$$\int_{\mathbb{D}} |f(z)|^p dm(z) \le \big(\frac{p}{2}+1\big) \int_{\mathbb{D}} |zf(z)|^p dm(z).$$

3. Main Results

In this section we state and prove the main results of this article. We first find estimates for the norms of the weighted composition operators T_t and then we demonstrated that $\mathcal{H}_\beta$ is bounded on Bergman spaces A^p and we also give norm estimates.

PROPOSITION 3.1. *Suppose $2 < p < \infty$, then*
(i) If $4 \le p < \infty$ and $f \in A^p$, then

$$\| T_t(f) \|_{A^p} \le \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}-\frac{\beta}{p}}} \| f \|_{A^p} .$$

(ii) If $2 < p < 4$ and $f \in A^P$, then

$$\| T_t(f) \|_{A^p} \le \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p}\right)^{\frac{1}{p}} \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}-\frac{\beta}{p}}} \| f \|_{A^p} .$$

Proof. Let $f \in A^p$ where $p > 2$. A simple calculation gives

$$\omega_t(z)^2 = \frac{1}{t(1-t)}\phi_t'(z).$$

Using the above equation, since $p \ge 4$ and $|\omega_t(z)| \le \dfrac{1}{t}$ we get

$$\begin{aligned}
\| T_t(f) \|_{A^p}^p &= \int_{\mathbb{D}} |\omega_t(z)|^p |f(\phi_t(z))|^p (1-t)^\beta dm(z) \\
&= (1-t)^\beta \int_{\mathbb{D}} |\omega_t(z)|^p |f(\phi_t(z))|^p |\phi_t'(z)|^{-2} dm(\phi_t(z)) \\
&\le \frac{(1-t)^\beta}{t^{p-2}(1-t)^2} \int_{\phi_t(\mathbb{D})} |f(z)|^p dm(z) \\
&\le \frac{\| f \|_{A^p}^p}{t^{p-2}(1-t)^{2-\beta}}.
\end{aligned}$$

We compute the inverse of ϕ_t:

$$\phi_t^{-1}(z) = \frac{z-t}{(1-t)z}$$

and

$$\omega_t(\phi_t^{-1}(z) = \frac{1}{(t-1)\phi_t^{-1}(z)+1} = \frac{z}{t}.$$

For the case $2 < p < 4$. We have

$$\begin{aligned}
\| T_t(f) \|_{A^p}^p &= \frac{(1-t)^\beta}{t^2(1-t)^2} \int\limits_{\phi_t(\mathbb{D})} \left|\omega_t\big(\phi_t^{-1}(\omega)\big)\right|^{p-4} |f(\omega)|^p dm(\omega) \\
&= \frac{(1-t)^\beta}{t^2(1-t)^2} \int\limits_{\phi_t(\mathbb{D})} \left|\frac{\omega}{t}\right|^{p-4} |f(\omega)|^p dm(\omega) \\
&\leq \frac{1}{t^{p-2}(1-t)^{2-\beta}} \int\limits_{\mathbb{D}} |\omega|^{p-4} |f(\omega)|^p dm(\omega).
\end{aligned}$$

Now, if we proceed similar to the proof of the Lemma 2(ii) in [3]. We conclude that for $2 < p < 4$,

$$I \leq \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p}\right) \frac{t^{2-p}}{(1-t)^{2-\beta}} \| f \|_{A^p}^p .$$

Thus, we have the estimate. This complete the proof. □

Now we can state and proof the main results.

THEOREM 3.2. *The operator $\mathcal{H}_\beta$ is bounded on Bergman spaces $A^p, 2 < p < \infty$. Further, if $\beta \geq 0$ satisfies the following inequalities:*

(i) If $4 \leq p < \infty$ and $f \in A^p$, then

$$\| \mathcal{H}_\beta(f) \|_{A^p} \leq \frac{\Gamma(\frac{2}{p})\Gamma(\frac{\beta-2}{p}+1)}{\Gamma(\frac{\beta}{p}+1)} \| f \|_{A^P} .$$

(ii) If $2 < p < \infty$ and $f \in A^p$, then

$$\| \mathcal{H}_\beta(f) \|_{A^p} \leq \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p}\right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{2}{p}\right)\Gamma\left(\frac{\beta-2}{p}+1\right)}{\Gamma\left(\frac{\beta}{p}+1\right)} \| f \|_{A^p} .$$

(iii) If $2 < p < 4$ and $f \in A^p$ with $f(0) = 0$, then

$$\| \mathcal{H}_\beta(f) \|_{A^p} \leq \left(\frac{p}{2}+1\right)^{\frac{1}{p}} \frac{\Gamma(\frac{2}{p})\Gamma(\beta+1-\frac{2}{p})}{\Gamma(\beta+1)} \| f \|_{A^p} .$$

Proof. Let $f \in A^P$. By applying Minkowski's inequality, we have

$$\begin{aligned}
\| \mathcal{H}_\beta(f) \|_{A^p} &= \left(\int_{\mathbb{D}} \left|\mathcal{H}_\beta(f)(z)\right|^p dm(z) \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{D}} \left| \int_0^1 T_t(f)(z) \right|^p dm(z) \right)^{\frac{1}{p}} \\
&\leq \int_0^1 \left(\int_{\mathbb{D}} \left| T_t(f)(z) \right|^p dm(z) \right)^{\frac{1}{p}} dt \\
&= \int_0^1 \| T_t(f) \|_{A^p} \, dt.
\end{aligned}$$

Using Lemma 2 for $p \geq 4$ we conclude

$$\begin{aligned}
\| \mathcal{H}_\beta(f) \|_{A^p} &\leq \int_0^1 t^{\frac{2}{p}-1}(1-t)^{\frac{\beta}{p}-\frac{2}{p}} dt \, \| f \|_{A^P} \\
&= \beta\left(\frac{2}{p}, \frac{\beta-2}{p}+1\right) \| f \|_{A^P} \\
&= \frac{\Gamma(\frac{2}{p})\Gamma(\frac{\beta-2}{p}+1)}{\Gamma(\frac{\beta}{p}+1)} \| f \|_{A^P} \, .
\end{aligned}$$

Similarly, for $2 < p < 4$ and $f \in A^p$ we have

$$\begin{aligned}
\| \mathcal{H}_\beta(f) \|_{A^p} &\leq \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p} \right)^{\frac{1}{p}} \int_0^1 \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}-\frac{\beta}{p}}} \| f \|_{A^p} \\
&= \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p} \right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{2}{p}\right)\Gamma\left(\frac{\beta-2}{p}+1\right)}{\Gamma\left(\frac{\beta}{p}+1\right)} \| f \|_{A^p} \, .
\end{aligned}$$

For the proof of case(iii), let $f \in A^p$ where $2 < p < 4$ with $f(0) = 0$, and express f(z)=$zf_0(z)$. Using Lemma 3 we get function f_0 belongs to the Bergman space and satisfies the inequality

$$\| f_0 \|_{A^p} \leq \left(\frac{p}{2} + 1 \right)^{\frac{1}{p}} \| f \|_{A^p} \, .$$

Indeed, this estimate is a special case of a results concerning A^p inner functions ([5], Corollary 3.23).

Now we compute

$$\begin{aligned}\mathcal{H}_\beta(f)(z) &= \int_0^1 \frac{(1-t)^\beta}{(t-1)z+1} f\left(\frac{t}{(t-1)z+1}\right)dt \\ &= \int_0^1 \frac{(1-t)^\beta t}{((t-1)z+1)^2} f_0\left(\frac{t}{(t-1)z+1}\right)dt \\ &= \int_0^1 \frac{(1-t)^\beta}{t} \phi_t(z)^2 f_0(\phi_t(z))dt \\ &= \int_0^1 \mathcal{S}_t(f_0)(z)dt,\end{aligned}$$

where $\mathcal{S}_t(g)(z) = \frac{(1-t)^\beta}{t}\ \phi_t(z)^2\ g(\phi_t(z)),\ g \in A^p,\ and\ \phi_t(z) = \frac{t}{(t-1)z+1}$.

An easy calculation shows that

$$\phi_t(z)^2 = \frac{t}{1-t}\phi_t'(z),\ \ z \in \mathbb{D}, 0 < t < 1.$$

It follows that

$$\begin{aligned}\| \mathcal{S}_t(g) \|_{A^P}^p &= \frac{(1-t)^{\beta p}}{t^p} \int_{\mathbb{D}} |\phi_t(z)|^{2p} |g(\phi_t(z))|^p dm(z) \\ &\le (1-t)^{\beta p-2} t^{2-p} \int_{\mathbb{D}} |\phi_t(z)|^{2p} |g(\phi_t(z)|^p |\phi_t'(z)|^{-2} dm(\phi_t(z)) \\ &= \frac{t^{2-p}}{(1-t)^{2-\beta p}} \int_{\mathbb{D}} |\phi_t(z)|^{2p-4} |g(\phi_t(z))|^p dm(\phi_t(z)) \\ &= \frac{t^{2-p}}{(1-t)^{2-\beta p}} \| g \|_{A^p}^p .\end{aligned}$$

Hence,

$$\| \mathcal{S}_t(g) \|_{A^p} \le \frac{t^{\frac{2}{p}}-1}{(1-t)^{\frac{2}{p}-\beta}} \| g \|_{A^p} .$$

To calculate the norm of $\mathcal{H}$, we proceed as follows

$$\begin{aligned}\| \mathcal{H}_\beta(f) \|_{A^p} &\le \left(\int_0^1 \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}-\beta}} dt\right) \| f_0 \|_{A^p} \\ &= \beta\left(\frac{2}{p}, 1+\beta-\frac{2}{p}\right) \| f_0 \|_{A^p} \\ &= \frac{\Gamma(\frac{2}{p})\Gamma(\beta+1-\frac{2}{p})}{\Gamma(\beta+1)} \| f_0 \|_{A^p} \\ &\le \left(\frac{p}{2}+1\right)^{\frac{1}{p}} \frac{\Gamma(\frac{2}{p})\Gamma(\beta+1-\frac{2}{p})}{\Gamma(\beta+1)} \| f \|_{A^p},\end{aligned}$$

which is the desired result. This completes the proof. □

For the proof of the next result we need Gaussian hypergeometric functions and some of its properties see, for example, [1]. The classical/Gaussian hypergeometric series is defined by the power series expansion

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)z^n}{(c,n)(1,n)n!} \quad (|z|<1).$$

Here a,b, c are complex numbers such that $c \neq -m$, m= 0,1,2,3... and (a,n) is the pochhamer's symbol which is defined as

$$(a,n) := a(a+1)...(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, n \in \mathbb{N}$$

and $(a,0) = 1$ for $a \neq 0$.

Let $\mathcal{D}$ be the usual Dirichlet space of analytic functions on $\mathbb{D}$ with square summable derivative. The following result is well known (see [3]).

Lemma 3.3. *Each bounded linear functional on the Bergman space A^2 can be associated to a function $g \in \mathcal{D}$ (by the pairing $< f, g >= \sum_{n=0}^{\infty} a_n b_n$) and the association is an isometric isomorphism of the spaces.*

We will use the above Lemma to prove the following result:

Proposition 3.4. *Let $0 \leq \beta < 1$. There is no bounded linear operator $T : A^2 \to A^2$ satisfying :*

$$T(\xi_n)(0) = \frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(n+\beta+2)}, \quad n = 0, 1, 2, \dots,$$

where $\mathcal{E}_n(z) = z^n$.

Proof. Assume, to the contrary, that there exists an operator T. Using the pairing that defines an isometric isomorphism between A^{2^*} and $\mathcal{D}$, we find that the adjoint operator $T^* : \mathcal{D} \to \mathcal{D}$ is bounded and satisfies the following relation.

$$< T(f), g >=< f, T^*(g) >, \tag{4}$$

for all $f \in A^2$ and $g \in \mathcal{D}$. Now, let's choose $g \equiv 1$ and express $T^*(1)(z)$ as the Taylor series: $T^*(1)(z) = \sum_{n=0}^{\infty} c_n z^n$, where $T^*(1) \in \mathcal{D}$. Using (4) for $f = \xi_n$ and $g \equiv 1$, we have

$$\begin{aligned} \frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(n+\beta+2)} &= T(\xi_n)(0) \\ &= \langle T(\xi_n), 1 \rangle \\ &= \langle \xi_n, T*(1) \rangle \\ &= c_n, \end{aligned}$$

for every $n = 0, 1, 2, \ldots$, hence

$$\begin{aligned} T^*(1)(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(n+\beta+2)} z^n. \\ &= \frac{1}{\beta+1} \sum_{n=0}^{\infty} \frac{(1,n)(1,n)}{(\beta+2,n)(1,n)} z^n \\ &= \frac{1}{(\beta+1)} F(1,1;\beta+2;z). \end{aligned}$$

In view of following well-known Gauss identity

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z),$$

and using the derivative formula for Gaussian hypergeometric function $F'(a,b;c;z) = \frac{ab}{c} F(a+1, b+1; c+1; z)$ (see [1]), we find

$$\begin{aligned} \int_D |(T^*(1)(z))'|^2 dm(z) &= \frac{1}{\pi(\beta+1)} \int_0^{2\pi} \int_0^1 |F(2,2;\beta+3;re^{i\theta})|^2 \, dr \, d\theta \\ &= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{\left|1-re^{i\theta}\right|^{2(\beta-1)}}{(\beta+1)} \left|F(\beta+1,\beta+1;\beta+3,re^{i\theta})\right|^2 \, dr \, d\theta. \end{aligned}$$

Now

$$\begin{aligned} \int_0^{2\pi} \int_0^1 \left|1-re^{i\theta}\right|^{2(\beta-1)} \, dr \, d\theta &\geq \int_0^{2\pi} \int_0^1 (1-r)^{2\beta} \left|1-re^{i\theta}\right|^{-2} \, dr \, d\theta \\ &\geq \int_0^{2\pi} \int_0^1 (1-r)^{2\beta} \left|2(1-re^{i\theta})\right|^{-1} \, dr \, d\theta. \end{aligned}$$

Since $|F(\beta+1, \beta+1; \beta+3, re^{i\theta})| \leq M$ if $0 \leq \beta < 1$ and the integral $\int_0^{2\pi} \left|2(1-re^{i\theta})\right|^{-1} d\theta$ is not finite, we get the function $T^*(1)$ is not in $\mathcal{D}$. □

From the above Proposition, we obtain the following:

Corollary 3.5. *The operator $\mathcal{H}_\beta$, for $0 \leq \beta < 1$ is not bounded on A^2.*

Proof. Using the integral form of $\mathcal{H}_\beta$ given in Lemma 1, we have

$$\mathcal{H}_\beta(\xi_n)(0) = \frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(n+\beta+2)}, \quad n = 0, 1, 2, \ldots.$$

The desired result follows from the Proposition 7. □

4. Competing Interests

The author declares that there is no conflict of interest regarding the publication of this article.

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