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THE TENSOR PRODUCT OF TOPOLOGICAL MODULES OVER A RING

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Abstract. Tensor products provide an essential tool in the theory of rings and modules, but its topological structure has been rarely studied. In the present article, we give a foundational description of a natural topology on the (algebraic) tensor product of topological modules over a commutative topological ring. Our approach is to give a topology directly to the algebraic tensor product instead of introducing an universal object in the category of topological modules with respect to continuous bilinear maps.

1. Introduction

Suppose that R is a commutative ring with unity. For modules M, N, P over a ring R, a mapping $f : M \times N \to P$ is R-bilinear if it satisfies the following two conditions:
(1) $f(ax_1 + bx_2, y) = af(x_1, y) + bf(x_2, y)$ for every $a, b \in R$, $x_1, x_2 \in M$ and $y \in N$
(2) $f(x, ay_1 + by_2) = af(x, y_1) + bf(x, y_2)$ for every $a, b \in R$, $x \in M$ and $y_1, y_2 \in N$.

In other words, f is bilinear if $f(\_, y)$ is a R-linear map from M into P for every $y \in N$ and $f(x, \_)$ is a R-linear map from N into P for every $x \in M$.

For two modules M, N over a ring R, the tensor product $M \otimes_R N$ is an R-module T together with an R-bilinear map $\varphi : M \times N \to T$ with the following universal property (c.f., [1]):
Given any R-module P and any R-bilinear map $f : M \times N \to P$, there exists a unique R-linear map $f' : T \to P$ such that $f = f' \circ \varphi$.

In case R is not commutative, M is a right R-module and N is a left R-module, a tensor product $M \otimes_R N$ is defined to be an abelian group T with a R-bilinear map $\varphi : M \times N \to T$ such that, for any abelian group P and any R-bilinear map $f : M \times N \to P$, there exists a unique group homomorphism $f' : T \to P$ satisfying $f = f' \circ \varphi$ (c.f., [5]). In this case, an R-bilinear map f is defined to be a bi-additive map, i.e., $f(x_1+x_2, y) = f(x_1, y)+f(x_2, y)$ and $f(x, y_1+y_2) = f(x, y_1)+f(x, y_2)$ such that $f(xa, y) = f(x, ay)$ for every $a \in R, x \in M, y \in N$. As with any definition using some universal property, the uniqueness of $M \otimes_R N$ defined via universal property can be proved quite easily.

Equivalently, the tensor product may be defined more explicitly as generators and relations. Let C denote the free R-module with the set of free generators $M \times N$, i.e.,

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$C = R^{M\times N}$. The elements of C are expressions of the form $\sum_{i=1}^{n} a_i(x_i, y_i)$ with $a_i \in R$, $x_i \in M$, $y_i \in N$. Let D be the submodule of C generated by all elements of C of the following types:
(1) $(x_1 + x_2, y) - (x_1, y) - (x_2, y)$,
(2) $(x, y_1 + y_2) - (x, y_1) - (x, y_2)$,
(3) $(ax, y) - a(x, y)$,
(4) $(x, ay) - a(x, y)$,
for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $a \in R$. Then the tensor product $M \otimes_R N$ is defined to be the quotient module C/D. We will write $x \otimes y$ to denote the image of $(x, y) \in C$ in C/D. Then, by the definition, we have the following obvious relations:
(1') $(x_1 + x_2) \otimes y - x_1 \otimes y - x_2 \otimes y$,
(2') $x \otimes (y_1 + y_2) - x \otimes y_1 - x \otimes y_2$,
(3') $(ax) \otimes y - a(x \otimes y)$,
(4') $x \otimes (ay) - a(x \otimes y)$,
for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $a \in R$.

To show that these two definitions of a tensor product $M \otimes_R N$ are in fact isomorphic, we need to prove that the tensor product defined via the explicit construction satisfies the universal property. Let $T = C/D$ in the above definition and let us prove first that $\varphi : M \times N \to T$ where $\varphi(x, y) = x \otimes y$ is R-biliear. But, this follows directly from the relations we listed before. On the other hand, let $f : M \times N \to P$ be any R-bilinear map. We can extend f to an R-linear map $\overline{f} : C \to P$ since C is the free module with the set $M \times N$ of free generators. Clearly, $\overline{f}$ vanishes on the generating elements of the submodule D in the definition of tensor product via a concrete construction. Therefore, $\overline{f}$ induces an R-linear map $f' : T \to P$ such that $f'(x \otimes y) = f(x, y)$. The mapping f' is uniquely determined by this condition and thus (T, φ) satisfies the universal property.

Now suppose that R is a topological commutative ring with unity, which means that the addition and multiplication $R\times R \to R$ are continuous as maps. Also suppose that M, N are topological R-modules, that is, R-modules where scalar multiplication $R \times M \to M$ and the addition $M \times M \to M$ are continuous as maps. We define a topology on $M \otimes_R N$ as a quotient topology of $(M \times N)^\infty$ which is introduced in Section 2.

In Section 3, basic properties of the topology on the tensor product are given. Especially, we look into the tensor product of locally compact topological abelian groups like $\mathbb{R}$. It turns out that a tensor product of $\mathbb{R}$ with itself is not locally compact although it is compactly generated.

Our study is to give a topology directly to the algebraic tensor product, not as an object in the category of topological modules over R with an obvious universal property with respect to the continuous bilinear maps. The latter approach would be similar to the one adopted in [2]. The reason behind this is that dealing directly with algebraic tensor product enables us to apply the topological method to the algebraically defined object like Milnor's K-theory of a field, which happens to be a topological field (c.f., [7]).

2. The Topology on the Tensor Product

Let R be a topological commutative ring with unity and M be a topological R-module. All topological rings and modules are assumed to be Hausdorff. Let $M^n = M \times M \times \cdots \times M$ be the product of n-copies of M. M^n is given the product topology. Then M^n can be regarded as a subspace of M^{n+1} via the injective map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. Now the product space $M \times N$ can be given the obvious R-module direct sum structure $M \oplus N$ and we will switch back and forth between these two notations.

If N is another topological R-module, let $(M \times N)^\infty$ be the union $\bigcup_{n \to \infty} M^n \times N^n$ of the topological spaces $M^n \times N^n$ under the obvious injective maps $M^n \times N^n \to M^{n+1} \times N^{n+1}$, which sends the pair $((x_1, \ldots, x_n), (y_1, \ldots, y_n))$ to the pair $((x_1, \ldots, x_n, 0), (y_1, \ldots, y_n, 0))$. The set $(M \times N)^\infty$ is endowed with the direct limit topology (c.f., [11]), which is also called the inductive limit topology. In other words, a subset U of $(M \times N)^\infty$ is defined to be open if $U \cap M^n \times N^n$ is open in $M^n \times N^n$ for every positive integer n.

Now we define the topology on $M \otimes_R N$.

Definition 2.1. For a topological commutative ring R with unity and two topological R-modules M, N, let $p : (M \times N)^\infty \to M \otimes_R N$ be the map which sends an element $((x_1, \ldots, x_n), (y_1, \ldots, y_n))$ of $M^n \times N^n$ to $\sum_{i=1}^n (x_i \otimes y_i)$. We give $M \otimes_R N$ the quotient topology with respect to this map p.

In this definition, p is obviously surjective and the quotient topology is the finest topology which makes p a continuous map.

The main technical difficulty with this definition stems from the fact that the product of quotient maps is in general no longer a quotient map. So, before we proceed, we recall some definitions in topology. A continuous map $f : C \to X$ from a compact Hausdorff space C is called a "test map". A space X is called "weakly Hausdorff" if the image of every test map is closed. Note that a Hausdorff space is weakly Hausdorff. A weakly Hausdorff space is a T_1-space. If X is weakly Hausdorff, then the image of every test map is Hausdorff.

A subset A of a space X is called "k-closed" if, for every test map $f : K \to X$, $f^{-1}(A)$ is closed in K. Similarly, a subset U of X is called "k-open" if $f^{-1}(U)$ is open in K for each test map $f : K \to X$. The k-open sets in X form a topology on X and the consequent topological space is denoted by kX (c.f., [11]). In particular, kX is same as X as a set, but the topology on kX is finer than X in general. For every compact Hausdorff space K, we have a natural bijection $TOP(K, kX) \simeq TOP(K, X)$, where TOP denotes the set of continuous maps. X is called a "k-space" (a.k.a. compactly generated space) if X and kX have the same topology. Every k-space is characterized as a quotient space of a locally compact space.

Since a product of k-spaces is not always a k-space, we use the notation $X \times_k Y$ for the product of two k-spaces X and Y in the category $k - TOP$ of k-spaces. If X and Y are k-spaces and two maps $f : X \to X'$ and $g : Y \to Y'$ are quotient maps, then the product map $f \times g : X \times_k Y \to X' \times_k Y'$ is a quotient map.

Let $\mathcal{C}$ be a cover of X. The space X is said to be determined by its cover $\mathcal{C}$ if a subset U of X is open in X if and only if $U \cap C$ is relatively open in C for every

$C \in \mathcal{C}$. (c.f., [9]). Using this terminology, a space is a k-space if it is determined by the cover of all compact subsets.

A topological space X is called a k_ω-space if it is determined by a cover $\mathcal{C}$ of countably many compact subsets (c.f., [9]).

Note that $(M \times N)^\infty$ with the direct limit topology is not a topological group in general, but it is a topological group if M and N are locally compact (c.f. [10]). The following theorem is a fundamental result on the topology of tensor products. Recall that a space is second-countable if it has a countable basis.

Theorem 2.2. *Let R be a locally compact second-countable topological commutative ring with unity. For locally compact second-countable topological modules M and N, $M \otimes_R N$ is a topological module over R.*

Proof. Let $f : (M \times N)^\infty \times (M \times N)^\infty \to (M \times N)^\infty$ be the map given by sending the pair of elements $((x_1, x_2, \dots), (y_1, y_2, \dots))$ and $((x'_1, x'_2, \dots), (y'_1, y'_2, \dots))$ to the element $((x_1, x'_1, x_2, x'_2, \dots), (y_1, y'_1, y_2, y'_2, \dots))$. Since f just permutes the coordinates, it is a homeomorphism. The map f gives rise to $(M \times N)^\infty$ a continuous (non-associative) binary operation. Note that $M^n \times N^n$ is also locally compact and second countable and thus is Lindelöf (c.f., [8]), for every positive integer n. Hence, $X_n = p(M^n \times N^n)$ under the map p in Definition 2.1 is a k_ω-space and so $X_n \times X_n$ is a k-space for every positive integer n by Corollary 10 of [9]. This shows that $p \times p : (M^n \times N^n) \times (M^n \times N^n) \to X_n \times X_n$ is also a quotient map by Proposition 5.8 in Appendix A of [6]. Now we have to apply the direct limit, but, in general, it is not necessarily true that $\varinjlim(X_n \times X_n) = \varinjlim(X_n) \times \varinjlim(X_n)$ (e.g., [4]). But, in our case, every X_n is a k_ω-space for every n, and so we have $\varinjlim(X_n \times X_n) = \varinjlim(X_n) \times \varinjlim(X_n)$ by Proposition 4.7 of [3]. Therefore, we see that $p \times p : (M \times N)^\infty \times (M \times N)^\infty \to (M \otimes_R N) \times (M \otimes_R N)$ is a quotient map. Since $p \circ f = + \circ (p \times p)$, we have the following commutative diagram.

$$\begin{array}{ccc} (M \times N)^\infty \times (M \times N)^\infty & \xrightarrow{f} & (M \times N)^\infty \\ \downarrow{\scriptstyle p \times p} & & \downarrow{\scriptstyle p} \\ (M \otimes_R N) \times (M \otimes_R N) & \xrightarrow{+} & M \otimes_R N \end{array}$$

By the universal property of a quotient map $p \times p$, the addition map $+$ for $M \otimes_R N$ is continuous. Similarly, the scalar multiplication can be shown to be continuous. □

3. Properties of the Topology on the Tensor Product

Proposition 3.1. *Suppose that we have two R-linear maps $\alpha : M \to M'$ and $\beta : N \to N'$ of locally compact second-countable topological R-modules. Then the algebraically induced map $\alpha \otimes \beta : M \otimes_R N \to M' \otimes_R N'$ is continuous.*

Proof. A proof is given by the fact that $\alpha \times \beta$ induces a continuous map from $M^n \times N^n$ to $M'^n \times N'^n$ for every positive integer n. Pass to the direct limit as in the proof of Theorem 2.2. □

As for the tensor product of connected modules, we obtain a connected module.

Proposition 3.2. *Suppose that M and N are connected topological modules over R. Then $M \otimes_R N$ is also connected.*

Proof. $M^n \times N^n$ is connected for every positive integer n. Thus, the union $(M \times N)^\infty$ of these spaces is connected. Therefore, its image $M \otimes_R N$ is also connected. □

So, for example, if one takes $R = \mathbb{Z}$ with the discrete topology, we are dealing with topological abelian groups. In this case, we just write $\otimes$ to replace $\otimes_\mathbb{Z}$. By the previous Proposition, $\mathbb{R} \otimes \mathbb{R}$, $\mathbb{C} \otimes \mathbb{C}$, $\mathbb{C}^\times \otimes \mathbb{C}^\times$ are all connected.

As for $\mathbb{R}^\times \otimes \mathbb{R}^\times$, we have two connected components. Note that $(\mathbb{R}^\times)^n \times (\mathbb{R}^\times)^n$ has 2^{2n} connected components determined by signs of $2n$ coordinates. For every element $((x_1, \dots, x_n), (y_1, \dots, y_n))$ of $(\mathbb{R}^\times)^n \times (\mathbb{R}^\times)^n$, we can associate the product of the Hilbert symbols $(x_1, y_1)_R \times \cdots (x_n, y_n)_\mathbb{R}$, where the Hilbert symbol $(\ ,\ )_\mathbb{R}$ is 1 if at least one of the coordinates is positive and -1 if all coordinates are negative. Since the Hilbert symbol is continuous, $\mathbb{R}^\times \otimes \mathbb{R}^\times$ has at least two connected components. On the other hand, if one of the coordinates, say x_1 is positive, then (x_1, y_1) is in the same component as $(1, y_1)$. Now, the image of $(1, y_1)$ under the quotient map p in Definition 2.1 is equal to the image $(1, y_1^2)$ by the definition of the tensor product. Therefore, one sees that every element is in the component containing $1 \otimes 1$ or $-1 \otimes -1$ and so $\mathbb{R}^\times \otimes \mathbb{R}^\times$ has two connected components. This argument extends to a tensor power of the group of nonzero real numbers and one concludes that $\mathbb{R}^\times \otimes \cdots \otimes \mathbb{R}^\times$ has two connected components no matter how many copies of $\mathbb{R}^\times$ are tensored.

Example 3.3. $\mathbb{R} \otimes \mathbb{R}$ is not locally compact, but is a k-space. Since $\mathbb{R} \otimes \mathbb{R}$ is the topological quotient of a k-space $\mathbb{R}^\infty$, it is compactly generated. On the other hand, let $X_n = p((\mathbb{R} \times \mathbb{R})^n)$ where $p : (\mathbb{R} \times \mathbb{R})^\infty \to \mathbb{R} \otimes \mathbb{R}$ is the quotient map. Each X_n is closed in $\mathbb{R} \otimes \mathbb{R}$ and we have $\mathbb{R} \otimes \mathbb{R} = \bigcup_{n=1}^\infty X_n$. Now we let U be an arbitrary open neighborhood of 0 in $\mathbb{R} \otimes \mathbb{R}$ and let us prove that the closure of U cannot be compact. To see this, construct an infinite tower of distinct subfields $\mathbb{Q} < k_1 < k_2 < \cdots$ of $\mathbb{R}$ and since the tensor product by a field, which is flat as an additive abelian group, preserves an injective map, we have the strict inclusion $k_i \otimes k_i \subsetneq k_{i+1} \otimes k_{i+1}$ of subgroups of $\mathbb{R} \otimes \mathbb{R}$. Since $k_i \otimes k_i$ is dense in $\mathbb{R} \otimes \mathbb{R}$, one can find an element $z_n \in U$ which is also an element of $\in k_{i+1} \otimes -k_i \otimes k_i$. Now the set $Z = \{z_n | n = 2, 3, \dots\}$ is a discrete closed subset of $\mathbb{R} \otimes \mathbb{R}$ and has no limit point in $\mathbb{R} \otimes \mathbb{R}$. So, the closure of U cannot be compact in $\mathbb{R}$ by Theorem 28.1 of [8] and consequently $\mathbb{R} \otimes \mathbb{R}$ is not locally compact.

One notes that every compact subset Z of $\mathbb{R} \otimes \mathbb{R}$ is contained in X_n for some n with the same notation as in the previous example. To see this, one notes that $\mathbb{R} \otimes \mathbb{R}$ is the union of X_n's ($n = 1, 2, \dots$) and let $Y_n = X_n \cap Z$. If Z is not contained in some X_n, then one has a sequence w_n of points in $\mathbb{R} \otimes \mathbb{R}$ such that $w_n \in Y_n - Y_{n-1}$ ($n = 2, 3, \dots$). The infinite set $A = \{x_n | n = 2, 3, \dots\}$ is a closed subset of Z and this contradicts to the fact that Z is compact. Therefore, Z is in X_n for some n. Note also that X_n is the image of a simply connected space $(\mathbb{R} \times \mathbb{R})^n$.

Example 3.4. By a similar argument as in Example 3.3, one can see that $\mathbb{R}/\mathbb{Z} \otimes \mathbb{R}/\mathbb{Z}$ is not compact. Therefore, the tensor product of two compact modules is not necessarily compact.

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