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ON RIGIDITY OF GRADIENT CONFORMAL RICCI SOLITONS

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Abstract. A soliton is a self similar solution of a non-linear PDE. Here we are associated with self similar solutions of the conformal Ricci flow which is a heat type pseudo parabolic partial differential equation in the perspective of Riemannian manifolds. The goal of the present article is to find some rigidity results on gradient conformal Ricci solitons. Some characterizations of conformal gradient Ricci solitons have been provided in terms of scalar curvature satisfying the Poisson equation.

1. Introduction

A Ricci soliton is a self similar solution of Hamilton's Ricci-flow [12] which is a pseudo-parabolic heat type partial differential equation. Theory of Ricci soliton became a ground of intensive study after the work of Perelman [15] to solve the famous Poincare conjecture.

A Riemannian manifold (M^n, g) ia a Ricci soliton if there exists a vector field X that satisfies

$$Ric + \frac{1}{2}\pounds_X g = \lambda g,$$

where Ric and $\pounds$ stand, respectively, for the Ricci tensor and Lie derivative. It is called expanding, steady or shrinking, if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$.

The theory has been further developed by several authors [1,3,5,6,13,14,16,17,20]. A Ricci soliton is known as gradient Ricci soliton [10] if the potential vector field is gradient of a function f; i.e., $X = \nabla f$. In that case the preceding equation turns out to be

$$Ric + Hessf = \lambda g,$$

where Hess$f$ stands for the $\nabla^2 f$.

In $(1, 1)$ tensor form, the Ricci soliton equation can be written as

$$Q + \nabla\nabla f = \lambda I,$$

or, in condensed form

$$Q + S = \lambda I,$$
$$S = \nabla\nabla f,$$

where Q is the Ricci operator defined by $g(QX, Y) = Ric(X, Y)$.

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Fischer [9] developed the theory of conformal Ricci flow in order to treat some aspects of relativistic mechanics and gravitation. Conformal Ricci flow is a variation of classical Ricci flow that replaces the unit volume constraint of the evolution equation to a scalar curvature constraint. Such a flow equation is analogous to Navier-Stokes equation of fluid mechanics. A conformal Ricci soliton is a self similar solution of conformal Ricci flow, upto diffeomorphisms and scalings. Ricci almost solitons and conformal Ricci solitons have been analyzed by the first author of the present paper in [10, 18, 19].

Concrete examples of conformal Ricci solitons can be foumd in [9] and [8]. According to [8], conformal Ricci solitons can be interpreted as a kinematic solution of conformal Ricci flow, whose profile yields a characterization of spaces of constant sectional curvature along with the locally symmetric spaces. In addition, geometric phenomenon of conformal Ricci solitons can evolve an interlink between a sectional curvature inheritance symmetry of space time and class of Ricci solitons. Conformal Ricci soliton is important since it helps in explaining the concepts of energy or entropy in general relativity.

If f is trivial, then the soliton represents Einstein metric. If $f = \frac{\lambda}{2}|x|^2$ on $\mathbb{R}^k$, then $Hessf = \lambda g$, consequently, the soliton is a flat gradient soliton. Combining the two cases, if we construct the product $N \times \mathbb{R}^k$ with N being Einstein, we get a new gradient soliton. Now, consider the quotient space $N \times_\Gamma \mathbb{R}^k$, where Γ acts freely on N and by orthogonal transformations without translation components on $\mathbb{R}^k$. The quotient space provides a flat vector bundle over a base that is Einstein. The space is a gradient soliton with $f = \frac{\lambda}{2}d^2$, where d is the distance in the flat fibers to the base. Any gradient soliton of the form $N \times_\Gamma \mathbb{R}^k$ is called a rigid soliton [16].

As for examples of rigid solitons it can be mentioned that in dimensions two or three all compact solitons are rigid. In higher dimensions if a soliton is steady or expanding, in addition with being compact, then it is also rigid [12–14]. On the other hand Perelman proved that if a three-dimensional shrinking gradient Ricci soliton with non-negative sectional curvature is rigid [15].

It is known that [16], if a gradient Ricci soliton is Rigid, then its radial curvature vanishes and its scalar curvature is constant. This is why a gradient Ricci soliton is important in view of geometric perspective as the scalar curvature of a manifold primarily represent the geometry of the manifold. In essence, a rigid Ricci soliton is one that possesses a nice unchanging structure, characterized by constant scalar curvature and radial flatness and can be considered as a flat bundle.

The Rigidity conditions for a gradient Ricci soliton was established by Petersen and Wylie [16]. Motivated by this work, in the present article, we find out the rigidity conditions for a gradient conformal Ricci soliton which will be a nice generalization of the work of Petersen and Wylie.

The Poisson equation [11] $\Delta\phi = \sigma$ on a Riemannian manifold provides a unique solution upto constant, whenever $\int \sigma = 0$ on the manifold. In order to study geometry of compact Ricci solitons, Poisson equation is an important tool [4].

Chen [4] established some beautiful geometric properties of compact shrinking Ricci solitons in terms of scalar curvature satisfying Poisson equation. By natural intuition one feels urge to characterize geometry of gradient conformal Ricci solitons in terms of scalar curvature satisfying the Poisson equation. Thus, one of our goals is also to analyze scalar curvature of a gradient conformal Ricci solitons satisfying Poisson equation.

The present article is constitututed as follows:

After the introduction in Section 1, some basic and preliminary results have been assembled and established in Section 2. Rigidity conditions for conformal Ricci solitons have been established in Section 3. Section 4 deals with some characterizations of conformal Ricci solitons in terms of scalar curvature satisfying Poisson equation.

2. Basic notations and results

On a closed connected orientable differentiable manifold (M^n, g) the conformal Ricci flow equation is of the form

$$\frac{\partial g}{\partial t} + 2(Ric + \frac{g}{n}) = -Pg,$$

$$s(g) = -1,$$

where P represents a time dependent non-dynamical scalar field and $s(g)$ being the scalar curvature of the manifold. Here the action of $-Pg$ maintains the scalar curvature constraint. In this manner, the conformal Ricci flow equation is analogous to classical Navier-Stokes equation of fluid flow. From this point of view P is also termed as conformal pressure. A conformal Ricci soliton is a self similar solution of conformal Ricci flow and it is assumed as a kind of generalization of the classical Ricci soliton.

For a constant λ and a vector field X, a Riemannian manifold M of dimension n with a metric g is a conformal Ricci soliton if [2, 7]

$$Ric + \frac{1}{2}\pounds_X g = (\lambda - (\frac{P}{2} + \frac{1}{n}))g,$$

where P is the conformal pressure.

When the vector field X is gradient of a smooth function defined on the manifold, the soliton is called gradient conformal Ricci soliton. Such a soliton is represented by

$$Ric + \nabla\nabla f = (\lambda - (\frac{P}{2} + \frac{1}{n}))g,$$

where gradient of f is the vector field X.

As a property of a rigid soliton, the following is well known:

$$R(., \nabla f)\nabla f = 0.$$

Also both the scalar curvature and conformal pressure are constants. In the converse situation, constant scalar curvature, constant conformal pressure and Ricci flatness $R(., \nabla f)\nabla f = 0$, each implies rigidity of compact solitons. For the aspects of non-compact case, one may follow [16]. Thus the discussion can be formulated as

Theorem 2.1. *A shrinking (expanding) gradient conformal Ricci soliton*

$$Ric + Hessf = (\lambda - (\frac{P}{2} + \frac{1}{n}))g$$

is rigid if and only if it has constant scalar curvature, constant conformal pressure and is radially flat.

If φ be a smooth function defined on a compact Riemannian manifold M its average φ_{av}, is

$$\varphi_{av} = \frac{1}{Vol(M)} \int_M \varphi.$$

The Poisson equation on a Riemannian manifold (M, g) is

$$\Delta\varphi = \sigma,$$

where Δ is the Laplace operator, σ is a given function, and φ is the solution to be determined.

Let us establish some results to be used in the sequel.

Lemma 2.2. *For an n-dimensional Riemannian manifold with metric g, one has*

$$div(\pounds_X g) = \frac{1}{2}\Delta|X|^2 - |\nabla X|^2 + Ric(X,X) + D_X divX.$$

For $X = \nabla f$,

$$(div\pounds_X g)(Z) = 2Ric(Z,X) + 2D_Z divX$$

or,

$$div\nabla\nabla f = Q(\nabla f) + \nabla\Delta f,$$

where Q indicates the Ricci operator.

Proof. By a routine calculation, one can infer

$$div(\pounds_X g)(X) = \Delta\frac{1}{2}|X|^2 - |\nabla X|^2 + Ric(X,X) + D_X divX.$$

If $Z \to \nabla_Z X$ is self-adjoint, then

$$(div\pounds_X g)(Z) = 2Ric(Z,X) + 2D_Z divX.$$

□

Corollary 2.3. *If X is a Killing vector field, then,*

$$\Delta\frac{1}{2}|X|^2 = |\nabla X|^2 - Ric(X,X).$$

Proof. $\pounds_X g = 0 = divX$ gives the above. □

Now we focus to gradient conformal Ricci solitons. For this case we use $(1,1)$ tensors and write the soliton equation as

$$Q + \nabla\nabla f = (\lambda - (\frac{P}{2} + \frac{1}{n}))I,$$

or, in condensed form

$$Q + S = (\lambda - (\frac{P}{2} + \frac{1}{n}))I,$$
$$S = \nabla\nabla f.$$

Lemma 2.4. *On a gradient conformal Ricci soliton, the following relations hold*:

(i) $\nabla s = 2Q(\nabla f) - n\nabla P$,

(ii) $\nabla_{\nabla f} S + S \circ S - (\lambda - \frac{P}{2} - \frac{1}{n})I = -R(.,\nabla f)\nabla f - \frac{1}{2}\nabla.\nabla s - \frac{n}{2}\nabla.\nabla P$,

(iii) $\nabla_{\nabla f} Q + Q \circ (\lambda - \frac{P}{2} + \frac{1}{n})I - Q = R(.,\nabla f)\nabla f + \frac{1}{2}\nabla.\nabla s + \frac{n}{2}\nabla.\nabla P$,

(iv) $\frac{1}{2}\Delta_f s + \frac{n}{2}\Delta_f P - n\nabla_{\nabla f}P = tr(Q \circ ((\lambda - \frac{P}{2} - \frac{1}{n})I - Q))$.

Proof. As a consequence of Bochner formula we infer

$$div(\nabla\nabla f) = Q(\nabla f) + \nabla\Delta f.$$

By tracing the soliton equation one has

$$s + \Delta f = n(\lambda - (\frac{P}{2} + \frac{1}{n})),$$
$$\nabla s + \nabla\Delta f = -\frac{n}{2}\nabla P.$$

Further, divergence of solition equation yields

$$divQ + div(\nabla\nabla f) = 0.$$

Combining the above

$$\nabla s = 2divQ = -2Q(\nabla f) + 2\nabla s + n\nabla P$$

and hence (i) is realized.

From the definition of curvature tensor

$$\begin{aligned} R(E, \nabla f)\nabla f &= \nabla_E\nabla_{\nabla f}\nabla f - \nabla_{\nabla f}\nabla_E\nabla f - \nabla_{[E,\nabla f]}\nabla f \\ &= \nabla^2_{E,\nabla f}\nabla f - \nabla^2_{\nabla f,E}\nabla f. \end{aligned}$$

Now

$$\nabla^2_{\nabla f,E}\nabla f = (\nabla_{\nabla f}S)(E) = -(\nabla_{\nabla f}Q)(E).$$

Also

$$\begin{aligned} \nabla^2_{E,\nabla f}\nabla f &= -(\nabla_E Q)(\nabla f) \\ &= -\frac{1}{2}\nabla_E\nabla s - \frac{n}{2}\nabla_E\nabla P + Q \circ ((\lambda - \frac{P}{2} - \frac{1}{n})I - Q). \end{aligned}$$

Therefore (ii) is reached.

Tracing (ii), we obtain (iii). Further tracing (iii), we infer

$$\nabla_{\nabla f}s + tr(Q \circ ((\lambda - \frac{P}{2} - \frac{1}{n})I - Q)) = Ric(\nabla f, \nabla f) + \frac{1}{2}\Delta s + \frac{n}{2}\Delta P,$$

since

$$Ric(\nabla f, \nabla f) = \frac{1}{2}D_{\nabla f}s + \frac{n}{2}D_{\nabla f}P.$$

Hence, (iv) follows. □

Remark 2.5. Considering λ_i as the eigenvalues of the Ricci operator, we have different version of the result (iv) as follows.

$$\begin{aligned} \frac{1}{2}\Delta_f s + \frac{n}{2}\Delta_f P - n\nabla_{\nabla f} &= tr(Q \circ ((\lambda - \frac{P}{2} - \frac{1}{n})I - Q)) \\ &= \sum \lambda_i(\lambda - \frac{P}{2} - \frac{1}{n} - \lambda_i) \\ &= -|Q^2| + (\lambda - \frac{P}{2} - \frac{1}{n})s \\ &= -|Q - \frac{1}{n}sg|^2 + s(\lambda - \frac{P}{2} - \frac{1}{n} - \frac{1}{n}s). \end{aligned}$$

Proposition 2.6. *On a compact gradient conformal Ricci soliton M of dimension n*

$$\int_M (n(\lambda - (\frac{P}{2} + \frac{1}{n})) - s) = 0.$$

Proof. Suppose (M, g, f, λ) be an n-dimensional compact gradient conformal Ricci soliton . Also consider $\chi(M)$ as the Lie algebra of smooth vector fields on M. So, we infer

$$H_f(X,Y) + Ric(X,Y) = (\lambda - (\frac{P}{2} + \frac{1}{n}))g(X,Y), \quad X, Y \in \chi(M), \tag{1}$$

where $H_f(X,Y) = g(\nabla_X \nabla f, Y)$ is the Hessian and ∇f is the gradient of the potential function f.

By virtue of (1) one can conclude

$$\Delta f = n(\lambda - (\frac{P}{2} + \frac{1}{n})) - s, \tag{2}$$

where $\Delta f = Trace(H_f)$ is the Laplacian of f. The Ricci operator Q agrees with the equation

$$Ric(X,Y) = g(QX,Y), \quad X, Y \in \chi(M). \tag{3}$$

Hence,

$$\sum_i (\nabla Q)(e_i, e_i) = \frac{1}{2}\nabla s, \tag{4}$$

where $\{e_1, ..., e_n\}$ is a local orthogonal frame and ∇Q is the covarient derivative of Q defined by

$$(\nabla Q)(X,Y) = \nabla_X(QY) - Q(\nabla_X Y).$$

Consider the symmetric operator A_f given by

$$H_f(X,Y) = g(A_f X, Y), \quad X, Y \in \chi(M).$$

Now it follows that

$$(\nabla A_f)(X,Y) - (\nabla A_f)(Y,X) = R(X,Y)\nabla f.$$

Using the above equation, $\Delta f = Trace(A_f)$, and the symmetry of A_f,one obtains

$$\begin{aligned} X(\Delta f) &= \sum_i g((\nabla A_f)(X, e_i), e_i) \\ &= \sum_i g((\nabla A_f)(e_i, X) + R(X, e_i)\nabla f, e_i) \\ &= -Ric(X, \nabla f) + \sum_i g((\nabla A_f)(e_i, e_i), X) \end{aligned} \tag{5}$$

for $X \in \chi(M)$. Again by virtue of (1)

$$(\nabla A_f)(X,Y) = -(\nabla Q)(X,Y).$$

In view of (2), (4), (5) and the above equation we infer

$$-X(s) = -Ric(X, \nabla f) - \frac{1}{2}X(s),$$

which yields

$$Q(\nabla f) = \frac{1}{2}\nabla s. \tag{6}$$

If (M, g, f, λ), is connected, in view of (1) and (6) one obtains

$$\frac{1}{2}X(||\nabla f||^2 + s) = H_f(X, \nabla f) + Ric(X, \nabla f) = (\lambda - (\frac{P}{2} + \frac{1}{n}))g(X, \nabla f),$$

that is

$$X(||X||^2 + s - 2(\lambda - (\frac{P}{2} + \frac{1}{n}))f) = 0, \quad X \in \chi(M).$$

This gives

$$\frac{1}{2}(||\nabla f||^2 + s) - (\lambda - (\frac{P}{2} + \frac{1}{n}))f = c$$

for a constant c. Now, after replacing the potential function f of the connected gradient conformal Ricci soliton (M, g, f, λ) by $f - \frac{c}{(\lambda - (\frac{P}{2} + \frac{1}{n}))}$ and assuming $\lambda > (\frac{P}{2} + \frac{1}{n})$, we see that the gradient conformal Ricci soliton (M, g, f, λ) satisfies

$$2(\lambda - (\frac{P}{2} + \frac{1}{n})f = ||\nabla f||^2 + s. \tag{7}$$

Also, equation (2) gives

$$\int_M (n(\lambda - (\frac{P}{2} + \frac{1}{n})) - s) = 0. \tag{8}$$

□

3. Rigidity characterizations of gradient conformal Ricci solitons

Theorem 3.1. *A compact conformal Ricci soliton with*

$$Ric(X, X) \leq 0$$

is Einstein. In particular, compact conformal gradient solitons with constant scalar curvature and constant conformal pressure are Einstein.

Proof. By definition

$$Ric + \pounds_X g = (\lambda - (\frac{P}{2} + \frac{1}{n}))g.$$

Then Laplacian of X satisfies

$$\Delta\frac{1}{2}|X|^2 = |\nabla X|^2 - Ric(X, X) \geq 0.$$

By divergence theorem, one has $\nabla X = 0$. In particular $\pounds_X g = 0$.

The second part follows for $X = \nabla f$ and the equation

$$D_{\nabla f}s = 2Ric(\nabla f, \nabla f) - nD_{\nabla f}P.$$

□

Proposition 3.2. *A gradient conformal Ricci soliton with nonnegative(or nonpositive) Ricci curvature has constant scalar curvature and constant conformal pressure if and only if $Ric(\nabla f, \nabla f) = 0$.*

Proof. For a self adjoint operator T which is non-negative or non-positive, one has

$$< Tv, v >= 0 \implies Tv = 0.$$

Replacing T by Ricci tensor and using $\nabla s = 2Q(\nabla f) - n\nabla P$ the result is reached.

Steady solitons are also easy to deal with. □

Proposition 3.3. *A steady gradient conformal Ricci soliton with constant scalar curvature and constant conformal pressure is a Ricci flat. Moreover, if f is not constant then it is a product of a Ricci flat manifold with $\mathbb{R}$.*

Proof. It is seen that

$$\begin{aligned} 0 &= \frac{1}{2}\Delta_f s + \frac{n}{2}\Delta_f P - n\nabla_{\nabla f}P \\ &= -|Q - \frac{1}{n}sg|^2 + s((\lambda - \frac{P}{2} - \frac{1}{n}) - \frac{1}{n}s) \\ &= -|Q - \frac{1}{n}sg|^2 - (\frac{P}{2} + \frac{1}{n})s - \frac{1}{n}s^2 \\ &\leq 0 \end{aligned}$$

Consequently, $s = 0$, $P = 0$ and $Q = 0$. Thus, Hessf $= 0$. Therefore, either f is constant or M splits along the gradient of f.

Now, we prove the following: □

Proposition 3.4. *Consider a gradient conformal Ricci soliton*

$$Ric + Hessf = (\lambda - (\frac{P}{2} + \frac{1}{n}))g$$

with constant scalar curvature, constant conformal pressure and $\lambda \neq 0$. When $\lambda > 0$ we have $0 \leq s \leq (\lambda - (\frac{P}{2} + \frac{1}{n}))n$. When $\lambda < 0$ we have $(\lambda - (\frac{P}{2} + \frac{1}{n}))n \leq s \leq 0$. In either case the metric is Einstein when the scalar curvature equals either of the extreme values.

Proof. As

$$\begin{aligned} 0 &= \frac{1}{2}\Delta_f s + \frac{n}{2}\Delta_f P - n\nabla_{\nabla f}P \\ &= -(Q - \frac{1}{n}sg)^2 + s((\lambda - \frac{P}{2} - \frac{1}{n}) - \frac{1}{n}s), \end{aligned}$$

one can write

$$0 \leq (Q - \frac{1}{n}sg)^2 = s((\lambda - \frac{P}{2} - \frac{1}{n}) - \frac{1}{n}s).$$

So, s$\in$ $[0, (\lambda - (\frac{P}{2} + \frac{1}{n}))n]$ if $\lambda > (\frac{P}{2} + \frac{1}{n})$ and the metric is Einstein if the scalar curvature takes on either of the boundary values. For expanding case, the similar result follows. □

Proposition 3.5. *For a shrinking gradient conformal Ricci soliton*

$$Ric + Hessf = (\lambda - (\frac{P}{2} + \frac{1}{n}))g$$

each of the following conditions implies that it is radially flat.
(1) *The scalar curvature and conformal pressure are constant and $sec(E, \nabla f) \geq 0$.*
(2) *The scalar curvature and conformal pressure are constant and $0 \leq Q \leq (\lambda - \frac{P}{2} - \frac{1}{n})g$.*
(3) *The curvature tensor is harmonic.*

Proof. (1) By

$$0 = \frac{1}{2}\nabla_{\nabla f}s + \frac{n}{2}\nabla_{\nabla f}P = Ric(\nabla f, \nabla f) = \Sigma g(R(E_i, \nabla f)\nabla f, E_i)$$

we see that $g(R(E_i,\nabla f)\nabla f, E_i) = 0$ if the radial curvatures are always nonnegative (nonpositive). Which is not possible. Hence $R(E_i,\nabla f)\nabla f = 0$ and the result follows.

(2) First observe that

$$0 = \frac{1}{2}\Delta_f s + \frac{n}{2}\Delta_f P - n\nabla_{\nabla f} P = tr(Q \circ ((\lambda - \frac{P}{2} - \frac{1}{n})I - Q))$$

We note that the only possible eigenvalues for Q and $\nabla\nabla f$ are 0 and $(\lambda - \frac{P}{2} - \frac{1}{n})$. To exhibit radial flatness we utilize the formula

$$\nabla_{\nabla f} Q + Q \circ ((\lambda - \frac{P}{2} - \frac{1}{n})I - Q) = R(.,\nabla f)\nabla f + \frac{1}{2}\nabla.\nabla s + \frac{n}{2}\nabla.\nabla P,$$

which takes the form

$$R(.,\nabla f)\nabla f = \nabla_{\nabla f} Q = -\nabla^2_{\nabla f,.}\nabla f.$$

Next, let E be a vector field such that $\nabla_E \nabla f = 0$, then

$$g(\nabla^2_{\nabla f,E}\nabla f, E) = 0$$

and also for $\nabla_E \nabla f = (\lambda - \frac{P}{2} - \frac{1}{n})E$,

$$g(\nabla^2_{\nabla f,E}\nabla f, E) = 0.$$

Thus $g(R(E,\nabla f)\nabla f, E) = 0$ for all eigenfields. So the metric is radially flat.

(3) In view of the soliton equation , one obtains

$$(\nabla_X Q)(Y) - (\nabla_Y Q)(X) = -R(X,Y)\nabla f$$

$$(\nabla_X Q)(Y,Z) - (\nabla_Y Q)(X,Z) = -g(R(X,Y)\nabla f, Z).$$

By the second Bianchi identity one infer

$$(\nabla_X Q)(Y,Z) - (\nabla_Y Q)(X,Z) = divR(X,Y,Z) = 0,$$

as the curvature is harmonic. Thus $R(X,Y)\nabla f = 0$. In particular $\sec(E,\nabla f) = 0$. The expanding situation is similar. □

Proposition 3.6. *Consider a gradient conformal Ricci soliton*

$$Ric + Hessf = (\lambda - (\frac{P}{2} - \frac{1}{n}))g$$

with constant scalar curvature, constant conformal pressure, $\lambda \neq 0$ and a nontrivial f. For a suitable constant α

$$f + \alpha = \frac{1}{2}(\lambda - \frac{P}{2} - \frac{1}{n})r^2,$$

where r is a smooth function whwnever $\nabla f \neq 0$ and satisfies

$$|\nabla r| = 1.$$

Proof. We note that

$$\frac{1}{2}(s + |\nabla f|^2) = (\lambda - \frac{P}{2} - \frac{1}{n})\nabla f.$$

The above implies

$$s + |\nabla f|^2 - 2(\lambda - \frac{P}{2} - \frac{1}{n})\nabla f = \text{constant}$$

By addition of suitable constant to f we see that

$$|\nabla f|^2 = 2(\lambda - \frac{P}{2} - \frac{1}{n})f.$$

Therefore, f and λ have the same sign and f and Df have same zeros. Let us define r such that

$$f = \frac{1}{2}(\lambda - \frac{P}{2} - \frac{1}{n})r^2,$$

then

$$\nabla f = (\lambda - \frac{P}{2} - \frac{1}{n})r\nabla r$$

and

$$2(\lambda - \frac{P}{2} - \frac{1}{n})f = |\nabla f|^2 = 2(\lambda - \frac{P}{2} - \frac{1}{n})f|\nabla r|^2$$

This provides a characterization of gradient conformal Ricci soliton. □

Theorem 3.7. *A gradient conformal Ricci soliton*

$$Ric + Hessf = (\lambda - (\frac{P}{2} + \frac{1}{n}))g$$

is rigid if it is radially flat and has constant scalar curvature and constant conformal pressure.

Proof. Consider the cases for which λ is positive, negative, zero. For the first case, i.e., $\lambda > 0$ from the soliton equation

$$Q + S = (\lambda - \frac{P}{2} - \frac{1}{n})I,$$
$$S = \nabla\nabla f,$$

we have

$$\nabla_{\nabla f} S + S \circ (S - (\lambda - \frac{P}{2} - \frac{1}{n})I) = 0,$$
$$\nabla_{\nabla f} Q + Q \circ ((\lambda - \frac{P}{2} - \frac{1}{n})I - Q) = 0.$$

Consoder that $f = \frac{1}{2}(\lambda - \frac{P}{2} - \frac{1}{n})r^2$ where r is a nonnegative distance function. The minimum set for f

$$N = \{x : f(x) = 0\}$$

is also written as

$$N = \{x \in M : \nabla f(x) = 0\}.$$

This expresses that $S \circ (S - (\lambda - \frac{P}{2} - \frac{1}{n})I) = 0$ on N.

If $r > 0$ we note that the smallest eigenvalue for S is absolutely continuous and therefore satisfies the differential equation.

$$D_{\nabla f}\nu_{min} = \nu_{min}(\lambda - \frac{P}{2} - \frac{1}{n} - \nu_{min}).$$

We claim that $\nu_{min} \geq 0$. Using $r > 0$ as an independent coordinate and $\nabla f = (\lambda - \frac{P}{2} - \frac{1}{n})r\nabla r$ yields

$$\delta_r \nu_{min} = \frac{1}{\lambda r}\nu_{min}((\lambda - \frac{P}{2} - \frac{1}{n}) - \nu_{min}).$$

This equation is solvable by separation of variables. In particular, $\nu_{min} \to -\infty$ in finite time provided $\nu_{min} < 0$ somewhere. The above is not agreeing with smoothness of f. Thus we infer that $\nu_{min} \geq 0$ and consequently f is convex.

As we know f is convex, the minimum set N must be totally convex. It is also known that on N the eigenvalues of $\nabla\nabla f$ are 0 and $(\lambda - \frac{P}{2} - \frac{1}{n})$ with constant multiplicities. Observing that the rank of $\nabla\nabla f$ is constant, we have N is a submanifold whose tangent space is $\mathrm{Ker}(\nabla\nabla f)$. In other words it shows that N is a totally geodesic submanifold.

When $\lambda > 0$ the minimum set N is in fact compact as it must be an Einstein manifold.

The normal exponential map

$$exp : v(N) \to M$$

follows the path along the integral curves for ∇f or ∇r and therefore is a diffeomorphism.

By the fundamental equations (see [17]) the metric is completely determined as it is radially flat and that N is totally geodesic. From this it follows that the bundle is flat and hence of the type $N \times_\Gamma \mathbb{R}$.

Proof for other values of λ is similar.

□

4. Scalar curvature of gradient conformal Ricci soliton satisfying Poisson equation

Theorem 4.1. *A compact connected gradient conformal Ricci soliton* (M^n, g, f, λ) *with* $\lambda > (\frac{P}{2} + \frac{1}{n})$ *and normalized potential function is trivial if and only if*

$$(fs)_{av} \leq \frac{1}{2}n^2(\lambda - (\frac{P}{2} + \frac{1}{n})),$$

and conformal pressure is constant, where s *is the scalar curvature of* (M, g).

Proof. Consider an n-dimensional compact and connected gradient conformal Ricci soliton (M, g, f, λ) with $\lambda > (\frac{P}{2} + \frac{1}{n})$. By virtue of the equations (2) and (7) we get

$$\frac{1}{2}\Delta f^2 = f\Delta f + ||\nabla f||^2 = (n+2)(\lambda - (\frac{P}{2} + \frac{1}{n}))f - fs - s, \tag{9}$$

which in view of (8) gives

$$\int_M fs = (\lambda - (\frac{P}{2} + \frac{1}{n}))(n+2)\int_M (f - \frac{n}{(n+2)}). \tag{10}$$

The equations (7) and (8) imply

$$\int_M (f - \frac{n}{2}) = \frac{1}{2(\lambda - (\frac{P}{2} + \frac{1}{n}))}\int_M ||\nabla f||^2,$$

which together with equation (9) yields

$$\int_M fs = \frac{1}{2}n^2(\lambda - (\frac{P}{2} + \frac{1}{n}))Vol(M) + \frac{n+2}{n}\int_M ||\nabla f||^2. \tag{11}$$

For $(fs)_{av} \leq \frac{1}{2}n^2(\lambda - (\frac{P}{2} + \frac{1}{n}))$ ine has

$$\int_M fs \leq \frac{1}{2}n^2(\lambda - (\frac{P}{2} + \frac{1}{n}))Vol(M). \tag{12}$$

By (11) and (12), we infer $\int_M ||\nabla f||^2 = 0$, which shows that the potential function f is a constant. As a result, it follows from (1) that M is Einstein. Hence, the soliton is trivial.

Conversely, suppose the solitonn with $\lambda > (\frac{P}{2}+\frac{1}{n})$ is trivial, then $s = n(\lambda-(\frac{P}{2}+\frac{1}{n}))$ and f is a constant. By (7) we obtain $f = \frac{s}{2(\lambda-(\frac{P}{2}+\frac{1}{n}))}$. So, we have $(fs)_{av} = \frac{1}{2}n^2(\lambda - (\frac{P}{2} + \frac{1}{n}))$. Hence, the result follows. □

Theorem 4.2. *Suppose (M, g, f, λ) is an n-dimensional compact connected gradient conformfal Ricci soliton having $\lambda > (\frac{P}{2}+\frac{1}{n})$ and let $\sigma = (\lambda-(\frac{P}{2}+\frac{1}{n}))(n(\lambda-(\frac{P}{2}+\frac{1}{n}))-s)$. If the scalar curvature s satisfies the Poisson equation*

$$\Delta\varphi = \sigma,$$

then either M is trivial or the first nonzero eigenvalue λ_1 of the Laplace operator Δ of M satisfies $\lambda_1 \leq (\lambda - (\frac{P}{2} + \frac{1}{n}))$, provided the conformal pressure is constant.

Proof. Suppose (M, g, f, λ) is an n-dimensional compact and connected gradient conformal Ricci soliton having $\lambda > (\frac{P}{2} + \frac{1}{n})$. Consider the scalar curvature s satisfies the Poisson equation

$$\Delta\varphi = \sigma, \tag{13}$$

with $\varphi = (\lambda-(\frac{P}{2}+\frac{1}{n}))(n(\lambda-(\frac{P}{2}+\frac{1}{n}))-s)$. It is seen the the function $\psi = \frac{1}{2}(||\nabla f||^2+s)$ agrees with

$$\psi = (\lambda - (\frac{P}{2} + \frac{1}{n}))f \tag{14}$$

due to equation (7). The above equation with (2) yields

$$\Delta\psi = (\lambda - (\frac{P}{2} + \frac{1}{n}))(n(\lambda - (\frac{P}{2} + \frac{1}{n})) - s) = \sigma.$$

Therefore, both s and ψ are the solutions of the Poisson equation (13). So, we get $s = \psi + c$ for some constant c. As a result, we obtain

$$\nabla s = \nabla\psi = (\lambda - (\frac{P}{2} + \frac{1}{n}))\nabla f - \frac{1}{2}\nabla Pf.$$

As the conformal pressure is constant, we obtain

$$\nabla s = (\lambda - (\frac{P}{2} + \frac{1}{n}))\nabla f. \tag{15}$$

In view of the minimum principle of λ_1 and the equation (8), we infer

$$\int_M ||\nabla s||^2 \geq \lambda_1 \int_M (n(\lambda - (\frac{P}{2} + \frac{1}{n})) - s)^2. \tag{16}$$

Again, from equation (8) it is seen that

$$\int_M (n(\lambda - (\frac{P}{2} + \frac{1}{n})) - s)^2 = \int_M (s^2 - n^2(\lambda - (\frac{P}{2} + \frac{1}{n}))^2).$$

As a result, the inequality (16) takes the form

$$\int_M ||\nabla s||^2 \geq \lambda_1 \int_M (s^2 - n^2(\lambda - (\frac{P}{2} + \frac{1}{n}))^2). \tag{17}$$

Since the scalar curvature s agrees with (13) with $\sigma = (\lambda-(\frac{P}{2}+\frac{1}{n}))(n(\lambda-(\frac{P}{2}+\frac{1}{n}))-s)$, we have

$$s\Delta s = (\lambda - (\frac{P}{2} + \frac{1}{n}))(n(\lambda - (\frac{P}{2} + \frac{1}{n}))s - s^2). \tag{18}$$

Integrating both sides of the above and using (8), we have

$$\int_M ||\nabla s||^2 = (\lambda - (\frac{P}{2} + \frac{1}{n})) \int_M (s^2 - n^2(\lambda - (\frac{P}{2} + \frac{1}{n}))^2),$$

which together with the inequality (17) yields

$$(\lambda_1 - (\lambda - (\frac{P}{2} + \frac{1}{n}))) \int_M (n^2(\lambda - (\frac{P}{2} + \frac{1}{n}))^2 - s^2) \geq 0.$$

It is seen that by (2), one has

$$n^2(\lambda - (\frac{P}{2} + \frac{1}{n}))^2 - s^2 = (n(\lambda - (\frac{P}{2} + \frac{1}{n})) + s)\Delta f = n(\lambda - (\frac{P}{2} + \frac{1}{n}))\Delta f + s\Delta f,$$

which on insertion in the above inequality gives

$$(\lambda_1 - (\lambda - (\frac{P}{2} + \frac{1}{n})) \int_M (s - \Delta f) \geq 0. \tag{19}$$

From (2), (19), and (18) we have

$$\begin{aligned} 0 &\leq (\lambda_1 - (\lambda - (\frac{P}{2} + \frac{1}{n}))) \int_M (s\Delta f) \\ &= (\lambda_1 - (\lambda - (\frac{P}{2} + \frac{1}{n}))) \int_M s(n(\lambda - (\frac{P}{2} + \frac{1}{n})) - s) \\ &= \frac{\lambda_1 - (\lambda - (\frac{P}{2} + \frac{1}{n}))}{(\lambda - (\frac{P}{2} + \frac{1}{n}))} \int_M ||\nabla s||^2. \end{aligned}$$

By combining the above equation with (15), we infer

$$(\lambda - (\frac{P}{2} + \frac{1}{n}))(\lambda_1 - (\lambda - (\frac{P}{2} + \frac{1}{n}))) \int_M ||\nabla f||^2 \leq 0,$$

which implies that either $\lambda_1 \leq (\lambda - (\frac{P}{2} + \frac{1}{n}))$ or (M, g, f, λ) is trivial.

This completes the proof.

□

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References

[1] C. Aquino, A. Barros, and E. Ribeiro, *Some applications of the Hodge-de Rham decomposition to Ricci solitons*, Results Math. **60** (1) (2011), 245–254.
https://doi.org/10.1007/s00025-011-0166-1

[2] N. Basu and A. Bhattacharyya, *Conformal Ricci solitons in Kenmotsu manifolds*, Global J. Adv. Res. Clas. Mod. Geom. **4** (2015), 15–21.

[3] H. D. Cao, *Limits of solutions to the Kähler-Ricci flow*, J. Differential Geom. **45** (1997), 257–272.

[4] B. Y. Chen and S. Deshmukh, *Geometry of compact shrinking Ricci solitons*, Balkan J. Geom. Appl. **19** (2014), 13–21.

[5] G. Catino, C. Mantegazza, and L. Mazzieri, *On the global structure of conformal gradient solitons with nonnegative Ricci tensor*, Commun. Contemp. Math. **14** (2012), 1250045.
https://doi.org/10.1142/S0219199712500459

[6] B. Chow, P. Lu, and L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, AMS, Providence, RI, 2006.
https://doi.org/10.1090/gsm/077

[7] D. Dey and P. Majhi, *Almost Kenmotsu metric as a conformal Ricci soliton*, Conf. Geom. Dyn. **23** (2019), 105–116.
https://doi.org/10.1090/ecgd/335

[8] S. Dey, *A study on conformal Ricci solitons and conformal Ricci almost solitons within the framework of almost contact geometry*, Carpathian Math. Publ. **18** (2023), 31–42.
https://doi.org/10.15330/cmp.15.1.31-42

[9] A. E. Fischer, *An introduction to conformal Ricci flow*, Class. Quantum Gravity **21** (2004), 3171–3228.
https://doi.org/10.1088/0264-9381/21/3/011

[10] M. Falcitelli, A. Sarkar, and S. Halder, *Conformal vector fields and conformal Ricci solitons on α-Kenmotsu manifolds*, Mediterr. J. Math. **20** (3) (2023), Article No. 127, 18 pp.
https://doi.org/10.1007/s00009-023-02339-9

[11] W. O. C. Filho, *The Poisson equation on compact Ricci solitons and Ricci-harmonic solitons*, J. Geom. **114** (2023), Article No. 14.
https://doi.org/10.1007/s00022-023-00675-5

[12] R. Hamilton, *The Ricci flow on surfaces*, in: Mathematics and General Relativity, Contemp. Math. **71**, AMS, Providence, RI, 1988, 237–262.

[13] T. Ivey, *Ricci solitons on compact three-manifolds*, Differential Geom. Appl. **3** (1993), 301–307.
https://doi.org/10.1016/0926-2245(93)90008-O

[14] T. Ivey, *New examples of compact Ricci solitons*, Proc. Amer. Math. Soc. **122** (1994), 241–245.

[15] G. Y. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159.
https://doi.org/10.48550/arXiv.math/0211159

[16] P. Petersen and W. Wylie, *Rigidity of gradient Ricci solitons*, Pacific J. Math. **242** (2009), 329–345.
https://doi.org/10.2140/pjm.2009.241.329

[17] P. Petersen, *Riemannian Geometry*, Graduate Texts in Mathematics **171**, Springer-Verlag, New York, 1998.

[18] A. Sarkar and S. Halder, *Conformal Ricci almost solitons with certain soliton vector fields*, Indian J. Pure Appl. Math. (2024).
https://doi.org/10.1007/s13226-024-00550-2

[19] A. Sarkar, A. Sil, and A. K. Paul, *Ricci almost solitons on three-dimensional quasi-Sasakian manifolds*, Proc. Natl. Acad. Sci. India Sect. A Phys. Sci. **89** (2019), 705–710.
https://doi.org/10.1007/s40010-018-0504-8

[20] W. Tokura, E. Batista, and P. Kai, *Triviality results for quasi κ-Yamabe solitons*, J. Math. Anal. Appl. **502** (2) (2021), 125274.
https://doi.org/10.1016/j.jmaa.2021.125274

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