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BI-BAZILEVIČ FUNCTIONS BASED ON HURWITZ-LERCH ZETA FUNCTION ASSOCIATED WITH EXPONENTIAL PARETO DISTRIBUTION

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Abstract. In this paper, we introduce and investigate new subclass of bi-univalent functions defined in the open unit disk, which are based on Hurwitz-Lerch Zeta function associated with exponential Pareto distribution , satisfying subordinate conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these new subclass. Several new consequences of the results are also pointed out.Additionally we discussed Fekete-Szegö inequality results

1. Introduction, Definitions and Preliminaries

In recent years, one of the most fascinating subjects has emerged: the study of the geometric behavior of analytic functions. Studying and describing the characteristics of analytic functions using geometrical and topological techniques is the primary goal of geometric function theory. One aims to link the analytical characteristics of functions with topological and geometrical insights, offering a more profound comprehension of the behavior of analytic functions. We now go over some fundamentals of geometric function theory as well as the analytic function subclasses that fall under this study's purview.

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\Delta_U = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$. Further denote by $\mathcal{S}$, the class of all functions in $\mathcal{A}$ which are univalent in Δ_U. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in Δ_U and the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha(0 \leq \alpha, 1)$ in Δ_U. (for more details see [4])

An analytic function f is subordinate to an analytic function g, written by $f(z) \prec g(z)$, provided there is an analytic function w defined on Δ_U with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [5] unified various subclasses of

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starlike and convex functions for which either of the quantity $\frac{z\,f'(z)}{f(z)}$ or $1+\frac{z\,f''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function Φ with positive real part in the unit disk $\Delta_U, \Phi(0)=1, \Phi'(0)>0$, and Φ maps Δ_U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\Phi(z) = 1 + \mathcal{B}_1 z + \mathcal{B}_2 z^2 + \mathcal{B}_3 z^3 + \cdots, \quad (\mathcal{B}_1 > 0). \tag{2}$$

The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\frac{z\,f'(z)}{f(z)} \prec \Phi(z)$. Similarly, the class of Ma-Minda convex functions of functions $f \in \mathcal{A}$ satisfying the subordination $1+\frac{z\,f''(z)}{f'(z)} \prec \Phi(z)$.

1.1. Exponential Pareto Distribution (EPD). In [7], Al-kadim and Bashi defined the cumulative probability function of the exponential Pareto distribution (EPD) as

$$G(x) = 1 - e^{-\hbar\left(\frac{x}{\kappa}\right)^{\sigma}} \quad x > 0 \tag{3}$$

where $\hbar, \sigma > 0$ are two shape parameters and $\kappa > 0$ is a scale parameter. The concept of the aforementioned distribution was first initiated by Gupta et al. [8] in 1998, with its probability density function (pdf) expressed as

$$G(x, \hbar, \sigma) = \hbar\theta(1+x)^{1-\hbar}[1-(1+x)^{-\hbar}]^{\theta-1}$$

where $\hbar, \theta$ are two shape parameters . The scaling parameter of the exponential Pareto distribution (EPD) is a significant benefit that can be used in a variety of real-world contexts. The distribution can detect patterns and trends in data and capture the long-tailed nature of many real-world data sets. Because the exponential Pareto distribution is crucial for analyzing lifetime data FC, the literature has examined a number of applications of the distribution. In [7], for instance, Al-Kadim and Boshi talked about exponential and Pareto distributions and gave some of their properties, such as the moment generated function, mean, mode, median, variance, the r-th moment about the mean, the r-th moment about the origin, reliability, hazard functions, and coefficients of variation, skweness, and kurtosis, and they estimated the parameter. The recovery rate of COVID-19 was modeled by Haj Ahmad et al. [9] using a unit exponential Pareto distribution; Idowu and Ajibode [10] examined the application of the exponential Pareto distribution to build control charts and enhance the quality of raw materials used in cement manufacturing. For more information, see, among others, [11, 12]. It is, however, observed that there are no known applications of the exponential and Pareto distribution in geometric function theory in the literature now. Hence, the authors intend to investigate some relevant connections of this distribution (EPD) in geometric function theory in this study. Consequently, we let $z = \hbar + \varepsilon i$ for $\varepsilon > 0$ such that $G(x)$ is now defined as

$$G(x) = 1 - e^{-z\left(\frac{x}{\kappa}\right)^{\sigma}} \quad x, \kappa, \sigma > 0; z \in \mathbb{C}. \tag{4}$$

The convergence of the series $\left(\frac{x}{\kappa}\right)^{-\sigma} G(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} z^n \left(\frac{x}{\kappa}\right)^{\sigma(n-1)}$ depends on the value of , as specified in the condition $|z| < 1$. This series has the form of a generalized power series in , multiplied by a function of . The factorial in the denominator grows faster than the powers in the numerator, so it ensures convergence for small values of .

REMARK 1.1. If we set $\varepsilon = 0$, then (4) reduces to the usual exponential Pareto distribution given by (3) .

Equation (4) can be normalized such that

$$(-1)^{-(n+1)} \left(\frac{x}{\kappa}\right)^{-\sigma} G(x) = z + \sum_{n=2}^{\infty} \left(\frac{x}{\kappa}\right)^{\sigma(n-1)} \frac{z^n}{n!}. \tag{5}$$

In view of (1) and (5), we can say that

$$F(z) = (-1)^{-(n+1)} \left(\frac{x}{\kappa}\right)^{-\sigma} G(x) * f(z).$$

That is,

$$F(z) = z + \sum_{n=2}^{\infty} \left(\frac{x}{\kappa}\right)^{\sigma(n-1)} \frac{a_n}{n!} z^n. \tag{6}$$

The factorial decay dominates all polynomial growth, meaning the series converges for all finite z.

1.2. Hurwitz-Lerch Zeta function (HLZ). We recall here a general Hurwitz-Lerch Zeta(HLZ) function $\Phi(z, s, a)$ defined by (cf., e.g., [[13],p. 121]).

$$\Phi(z, \wp, b) := \sum_{n=0}^{\infty} \frac{z^n}{(n+b)^{\wp}} \tag{7}$$

$$(b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \wp \in \mathbb{C}, \Re(\wp) > 1 \text{ and } |z| = 1)$$

where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$, $(\mathbb{Z} := \{\pm 1, \pm 2, \pm 3, ...\}); \mathbb{N} := \{1, 2, 3, ...\}$. the normalized form is for convenience,

$$\mathcal{H}_{\wp,b}(z) := (1+b)^{\wp}[\Phi(z, \wp, b) - b^{-\wp}] \quad (z \in \mathbb{U}). \tag{8}$$

Several interesting properties and characteristics of the $\Phi(z, \wp, b)$ can be found in Choi and Srivastava [14], Srivastava and Attiya [16] (see also Prajapat and Goyal [17] and references cited therein).

Using the normalized, form of PED and HLZ functions we define a new linear operator:

$$\mathfrak{M}(\wp, \ell; z) : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product by

$$\mathfrak{M}^{\sigma}_{\wp,b} f(z) = \mathcal{H}_{b,\wp} * F(z) \tag{9}$$

$$\mathfrak{M}^{\sigma}_{\wp,b} f(z) = z + \sum_{n \geq 2} \left(\frac{1+b}{n+b}\right)^{\wp} \left(\frac{x}{\kappa}\right)^{\sigma(n-1)} \frac{a_n}{n!} z^n, \tag{10}$$

$$= z + \sum_{n \geq 2} \mathfrak{y}_n(\sigma, \wp) a_n z^n, \tag{11}$$

where

$$\mathfrak{y}_n(\sigma, \wp) = \left| \frac{1}{n!} \left(\frac{1+b}{n+b}\right)^{\wp} \left(\frac{x}{\kappa}\right)^{\sigma(n-1)} \right| \tag{12}$$

and (throughout this paper unless otherwise mentioned) the parameters $\wp, b$ are constrained as $b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \wp \in \mathbb{C}$. By the ratio test of convergence, the series converges for $|\left(\frac{x}{\kappa}\right)^\sigma| < 1$ and diverges for $|\left(\frac{x}{\kappa}\right)^\sigma| > 1$. The test fails for $|\left(\frac{x}{\kappa}\right)^\sigma| = 1$. For the purpose of this study, therefore, we assume $\left(\frac{x}{\kappa}\right)^\sigma < 1$, since the series converges with the radius of convergence and the interval of convergence $-\left(\frac{x}{\kappa}\right)^\sigma < |z| < \left(\frac{x}{\kappa}\right)^\sigma$. We remark that since x, κ, σ are greater than zero or non-negative, the parameters is such that $\left(\frac{x}{\kappa}\right)^\sigma \in (0,1)$, the unit disc Δ_U.

1.3. Bi-Univalent functions Σ. One area of study that dates back to the early days of univalent function research is the study of coefficients of the functions in specific special classes. In 1916, Bieberbach solved an equivalent problem for the class $\mathcal{S}$ and his famous conjecture-which was only verified in 1984 [18], sparked the development of many techniques in the geometric theory of functions of a complex variable. The first two Taylor-Maclaurin coefficients are typically estimated in the study of bi-univalent functions, just as in the case of the classes examined by Gronwall and Bieberbach. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, [19] defined by

$$f^{-1}(f(z)) = z \qquad (z \in \Delta_U)$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{13}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ_U if both $f(z)$ and $f^{-1}(z)$ are univalent in Δ_U. Let Σ denote the class of bi-univalent functions in Δ_U given by (1). An interest in studying the class Σ has recently grown, and non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of Taylor coefficients in (1). However, each of the subsequent Taylor-Maclaurin coefficients has the following coefficient issue:Let

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1,2\};\ \mathbb{N} := \{1,2,3,\cdots\}$$

The problem of remains unresolved (see [20–24]). Srivastava et al. [25] have reignited the research of bi-univalent functions Σ, leading to numerous follow-up studies, on bi-univalent functions has a number of intriguing subclasses that have been introduced and studied (see [26–30, 32]and references cited therein). These researchers have discovered non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Subfamilies of Bazilevic functions of type $\aleph$ [33] have been examined by a number of writers (see also [34, 35])various perspective.They examined it from the perspective of convexity, inclusion theorem, radii of starlikeness, and convexity boundary rotational problem, subordination just to mention few. Most astonishingly, it is difficult to find any of these authors discussing about the coefficient bounds and inequality of these subfamilies of the Bazilevic function, particularly when the parameter $\aleph$ is bigger than one ($\aleph$ is real).

Since the inception of complex function research, the operators have been employed. They have made many known results simpler to use, and they may provide novel conclusions, particularly those pertaining to the convexity and starlikeness of particular functions. Most often, the study involving operators leads to the introduction of new classes of analytic functions. As evidenced by the results from papers [29, 30], the study of bi-univalent functions using operators is also a popular approach these days. According to the most recent paper [31], there is special interest in obtaining the Fekete–Szegő functional for the newly introduced special classes. Inspired by the Σ study (see [25–30, 32]), we present new subfamilies of bi-Bazilevic functions of the function class Σ, related to HZL and EPD, represented by $\mathfrak{M}^{\sigma}_{\wp,b}$, which are defined as in Definition 1.2. We obtain estimates for $f \in \mathfrak{M}^{\sigma}_{\wp,b}$. on the coefficients $|a_2|$ and $|a_3|$. Connections to new findings are described as corollaries, and a number of new related classes are also taken into consideration. Fekete-Szegö inequality results for $f \in \mathfrak{M}^{\sigma}_{\wp,b}$. were obtained using the estimations of a_2 and a_3.

Definition 1.2. Let $h : \Delta_U \to \mathbb{C}$ be a convex univalent function such that $h(0) = 1$ and $h(\overline{z}) = \overline{h(z)}$ for $z \in \Delta_U; \Re(h(z)) > 0$.Suppose also that the function $h(z)$ is given by

$$h(z) = 1 + \sum_{n=1}^{\infty} \mathcal{B}_n z^n \qquad (z \in \Delta_U). \tag{14}$$

A function $f \in \Sigma$ given by (1) is said to be in the class $\mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\varpi, \aleph, h)$ if the following conditions are satisfied:

$$e^{i\varpi}\left(\frac{z^{1-\aleph}(\mathfrak{M}^{\sigma}_{\wp,b}f(z))'}{[\mathfrak{M}^{\sigma}_{\wp,b}f(z)]^{1-\aleph}}\right) \prec h(z)cos\varpi + isin\varpi \tag{15}$$

and

$$e^{i\varpi}\left(\frac{w^{1-\aleph}(\mathfrak{M}^{\sigma}_{\wp,b}g(w))'}{[\mathfrak{M}^{\sigma}_{\wp,b}g(w)]^{1-\aleph}}\right) \prec h(w)cos\varpi + isin\varpi \tag{16}$$

where $\varpi \in (\frac{-\pi}{2}, \frac{\pi}{2}); \aleph \geq 0; z, w \in \Delta_U$ and the function g is given by(13).

Example 1.3. If we set $h(z) = \frac{1+Az}{1+Bz}$,$-1 \leq B < A \leq 1$, then $\mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\aleph, h) \equiv \mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\varpi, \aleph, A, B)$ which is define as $f \in \Sigma$,

$$e^{i\varpi}\left(\frac{z^{1-\aleph}(\mathfrak{M}^{\sigma}_{\wp,b}f(z))'}{[\mathfrak{M}^{\sigma}_{\wp,b}f(z)]^{1-\aleph}}\right) \prec \frac{1+Az}{1+Bz}cos\varpi + isin\varpi,$$

and

$$e^{i\varpi}\left(\frac{w^{1-\aleph}(\mathfrak{M}^{\sigma}_{\wp,b}g(w))'}{[\mathfrak{M}^{\sigma}_{\wp,b}g(w)]^{1-\aleph}}\right) \prec \frac{1+Aw}{1+Bw}cos\varpi + isin\varpi.$$

Example 1.4. If we set $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leq \alpha < 1$ then the class $\mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\aleph, h) \equiv \mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\varpi, \aleph, \alpha)$ which is define as $f \in \Sigma$,

$$\Re\left[e^{i\varpi}\left(\frac{z^{1-\aleph}(\mathfrak{M}^{\sigma}_{\wp,b}f(z))'}{[\mathfrak{M}^{\sigma}_{\wp,b}f(z)]^{1-\aleph}}\right)\right] > \alpha cos\varpi$$

and

$$\Re\left[e^{i\varpi}\left(\frac{w^{1-\aleph}(\mathfrak{M}^{\sigma}_{\wp,b}g(w))'}{[\mathfrak{M}^{\sigma}_{\wp,b}g(w)]^{1-\aleph}}\right)\right] > \alpha cos\varpi$$

On specializing the parameters $\aleph$ one can specify the different new subclasses of Σ as illustrated in the following examples.

Example 1.5. For $\aleph = 0$ and a function $f \in \Sigma$, given by (1) is said to be in the class $\mathcal{S}^{\sigma,b}_{\wp,\Sigma}(\varpi, h)$ if the following conditions are satisfied:

$$e^{i\varpi}\left(\frac{z(\mathfrak{M}^{\sigma}_{\wp,b}f(z))'}{\mathfrak{M}^{\sigma}_{\wp,b}f(z)}\right) \prec h(z)cos\varpi + isin\varpi \tag{17}$$

and

$$e^{i\varpi}\left(\frac{w(\mathfrak{M}^{\sigma}_{\wp,b}g(w))'}{\mathfrak{M}^{\sigma}_{\wp,b}g(w)}\right) \prec h(w)cos\varpi + isin\varpi \tag{18}$$

where $\varpi \in (\frac{-\pi}{2}, \frac{\pi}{2})$; $z, w \in \Delta_U$ and the function g is given by(13).

Example 1.6. For $\aleph = 1$, $f \in \Sigma$, given by (1) and is said to be in the class $\mathcal{H}^{\sigma,b}_{\wp,\Sigma}(\varpi, h)$ if the following conditions are satisfied:

$$e^{i\varpi}\left(\mathfrak{M}^{\sigma}_{\wp,b}f(z))'\right) \prec h(w)cos\varpi + isin\varpi \tag{19}$$

and

$$e^{i\varpi}\left((\mathfrak{M}^{\sigma}_{\wp,b}g(w))'\right) \prec h(w)cos\varpi + isin\varpi \tag{20}$$

where $\varpi \in (\frac{-\pi}{2}, \frac{\pi}{2})$; $z, w \in \Delta_U$ and the function g is given by(13).

Example 1.7. For $\varpi = 0$, and a function $f \in \Sigma$, given by (1)

(1) is said to be in the class $\mathcal{B}^{\sigma,b}_{\wp,\Sigma}(\aleph, h)$ if the following conditions are satisfied:

$$\left(\frac{z^{1-\aleph}(\mathfrak{M}^{\sigma}_{\wp,b}f(z))'}{[\mathfrak{M}^{\sigma}_{\wp,b}f(z)]^{1-\aleph}}\right) \prec h(z) \text{ and } \left(\frac{w^{1-\aleph}(\mathfrak{M}^{\sigma}_{\wp,b}g(w))'}{[\mathfrak{M}^{\sigma}_{\wp,b}g(w)]^{1-\aleph}}\right) \prec h(w)$$

(2) is said to be in the class $\mathcal{B}^{\sigma,b}_{\wp,\Sigma},(0,h) \equiv \mathcal{S}^{\sigma,b}_{\wp,\Sigma}(h)$ if the following conditions are satisfied:

$$\left(\frac{z(\mathfrak{M}^{\sigma}_{\wp,b}f(z))'}{\mathfrak{M}^{\sigma}_{\wp,b}f(z)}\right) \prec h(z) \text{ and } \left(\frac{w(\mathfrak{M}^{\sigma}_{\wp,b}g(w))'}{\mathfrak{M}^{\sigma}_{\wp,b}g(w)}\right) \prec h(w)$$

(3) is said to be in the class $\mathcal{B}^{\sigma,b}_{\wp,\Sigma}(1,h) \equiv \mathcal{H}^{\sigma,b}_{\wp,\Sigma}(h)$ if the following conditions are satisfied:

$$\left(\mathfrak{M}^{\sigma}_{\wp,b}f(z))'\right) \prec h(w) \text{ and } e^{i\varpi}\left((\mathfrak{M}^{\sigma}_{\wp,b}g(w))'\right) \prec h(w)$$

where the function g is given by(13).

In the following section we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the bi-bazilevic function class $\mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\varpi, \aleph, h)$ of the function class Σ.

In order to derive our main results, we shall need the following lemmas:

Lemma 1.8. *(see [36]). If a function $p \in \mathcal{P}$ is given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \Delta_U),$$

then

$$|p_k| \leqq 2 \qquad (k \in \mathbb{N}),$$

where $\mathcal{P}$ is the family of all functions p, analytic in Δ_U, for which

$$p(0) = 1 \qquad \text{and} \qquad \Re\big(p(z)\big) > 0 \qquad (z \in \Delta_U).$$

LEMMA 1.9. *(see [37]; see also [4]). Let the function $\psi(z)$ given by*

$$\psi(z) = \sum_{n=1}^{\infty} \mathcal{B}_n z^n \qquad (z \in \Delta_U)$$

be convex in Δ_U. Suppose also that the function $\mathfrak{h}(z)$ given by

$$\mathfrak{h}(z) = \sum_{n=1}^{\infty} \mathfrak{h}_n z^n$$

is holomorphic in Δ_U. If

$$\mathfrak{h}(z) \prec \psi(z) \qquad (z \in \Delta_U),$$

then

$$|\mathfrak{h}_n| \leq |\mathcal{B}_1| \qquad (n \in \mathbb{N}).$$

2. Coefficient Bounds for the Function Class $\mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\varpi, \aleph, h)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the bi-bazelvic class $\mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\varpi, \aleph, h)$.

THEOREM 2.1. *Let the function $f(z)$ given by (1) be in the class $\mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\varpi, \aleph, h)$. Then*

$$|a_2| \leqq \sqrt{\frac{2|\mathcal{B}_1|cos\varpi}{[(\aleph-1)(\aleph+2)\mathfrak{y}_2(\sigma,\wp)^2 + 2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)]}} \tag{21}$$

and

$$|a_3| \leqq \frac{|\mathcal{B}_1|cos\varpi}{(\aleph+2)\mathfrak{y}_3(\sigma,\wp)} + \frac{|\mathcal{B}_1|^2 cos^2\varpi}{(1+\aleph)^2\mathfrak{y}_2(\sigma,\wp)^2}. \tag{22}$$

where $\mathfrak{y}_n(\sigma,\wp)$ is given by (12).

Proof. It follows from (15) and (16) that

$$e^{i\varpi}\left(\frac{z^{1-\aleph}(\mathfrak{M}^{\sigma}_{\wp,b}f(z))'}{[\mathfrak{M}^{\sigma}_{\wp,b}f(z)]^{1-\aleph}}\right) = p(z)cos\varpi + isin\varpi \tag{23}$$

and

$$e^{i\varpi}\left(\frac{w^{1-\aleph}(\mathfrak{M}^{\sigma}_{\wp,b}g(w))'}{[\mathfrak{M}^{\sigma}_{\wp,b}g(w)]^{1-\aleph}}\right) = q(w)cos\varpi + isin\varpi, \tag{24}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the following forms:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{25}$$

and

$$q(w) = 1 + q_1 z + q_2 z^2 + \cdots \tag{26}$$

respectively. Since by definition, $p(z), q(w) \in h(\Delta_U)$, by applying Lemma1.9 in conjunction with the Taylor-Maclaurin expansions (14), (25) and (26), we find that

$$|p_n| := \left|\frac{p^{(n)}(0)}{n!}\right| \leqq |\mathcal{B}_1| \qquad (n \in \mathbb{N}) \tag{27}$$

and

$$|q_n| := \left|\frac{q^{(n)}(0)}{n!}\right| \leqq |\mathcal{B}_1| \qquad (n \in \mathbb{N}). \tag{28}$$

Now, equating the coefficients in (23) and (24), we get

$$e^{i\varpi}(1+\aleph)\mathfrak{y}_2(\sigma,\wp)a_2 = p_1 cos\varpi, \tag{29}$$

$$e^{i\varpi}\left[\frac{(\aleph-1)(\aleph+2)}{2}\mathfrak{y}_2(\sigma,\wp)^2a_2^2 + (\aleph+2)\mathfrak{y}_3(\sigma,\wp)a_3\right] = p_2 cos\varpi \tag{30}$$

$$-e^{i\varpi}(\aleph+1)\mathfrak{y}_2(\sigma,\wp)a_2 = q_1 cos\varpi \tag{31}$$

and

(32)

$$e^{i\varpi}\left[\left(2(\aleph+2)\mathfrak{y}_3(\sigma,\wp) + \frac{(\aleph-1)(\aleph+2)}{2}\mathfrak{y}_2(\sigma,\wp)^2\right)a_2^2 - (\aleph+2)\mathfrak{y}_3(\sigma,\wp)a_3\right] = q_2 cos\varpi$$

From (29) and (31), we find that

$$a_2 = \frac{p_1 cos\varpi e^{-i\varpi}}{(\aleph+1)\mathfrak{y}_2(\sigma,\wp)} = -\frac{q_1 cos\varpi e^{-i\varpi}}{(\aleph+1)\mathfrak{y}_2(\sigma,\wp)}, \tag{33}$$

which implies

$$p_1 = -q_1. \tag{34}$$

and

$$2(\aleph+1)^2\mathfrak{y}_2(\sigma,\wp)^2a_2^2 = (p_1^2+q_1^2)cos^2\varpi\ e^{-2i\varpi}. \tag{35}$$

Adding (30)and (32), by using(33) and (34) ,we obtain

$$e^{i\varpi}\left[(\aleph-1)(\aleph+2)\mathfrak{y}_2(\sigma,\wp)^2 + 2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)\right]a_2^2 = (p_2+q_2)cos\varpi. \tag{36}$$

Thus,

$$a_2^2 = \frac{(p_2+q_2)cos\varpi}{[(\aleph-1)(\aleph+2)\mathfrak{y}_2(\sigma,\wp)^2 + 2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)]}e^{-i\varpi}. \tag{37}$$

Applying Lemma 1.8 for the coefficients p_2 and q_2, we immediately have

$$|a_2|^2 = \frac{2|\mathcal{B}_1|cos\varpi}{[(\aleph-1)(\aleph+2)\mathfrak{y}_2(\sigma,\wp)^2 + 2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)]}. \tag{38}$$

This gives the bound on $|a_2|$ as asserted in (21).

Next, in order to find the bound on $|a_3|$, by subtracting (32) from (30), we get

$$\begin{aligned} e^{i\varpi}\left[2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)a_3 - 2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)a_2^2\right] &= (p_2-q_2)cos\varpi \\ a_3 &= a_2^2 + \frac{(p_2-q_2)cos\varpi e^{-i\varpi}}{2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)}. \end{aligned} \tag{39}$$

It follows from (33), (34) and (39) that

$$a_3 = \frac{(p_2-q_2)cos\varpi\ e^{-i\varpi}}{2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)} + \frac{(p_1^2+q_1^2)cos^2\varpi\ e^{-i2\varpi}}{2(1+\aleph)^2\mathfrak{y}_2(\sigma,\wp)^2}. \tag{40}$$

Applying Lemma 1.8 once again for the coefficients p_2 and q_2, we readily get

$$|a_3| \leqq \frac{|\mathcal{B}_1|cos\varpi}{(\aleph+2)\mathfrak{y}_3(\sigma,\wp)} + \frac{|\mathcal{B}_1|^2cos^2\varpi}{(1+\aleph)^2\mathfrak{y}_2(\sigma,\wp)^2}.$$

This completes the proof of Theorem 2.1. □

Putting $\aleph = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *If* $f \in \mathcal{S}^{\sigma,b}_{\wp,\Sigma}(\varpi, h)$ *and as given in*(1), *then*

$$|a_2| \leqq \sqrt{\frac{|\mathcal{B}_1| cos\varpi}{2\mathfrak{y}_3(\sigma,\wp) - \mathfrak{y}_2(\sigma,\wp)^2}} \tag{41}$$

and

$$|a_3| \leqq \frac{|\mathcal{B}_1| cos\varpi}{2\mathfrak{y}_3(\sigma,\wp)} + \frac{|\mathcal{B}_1|^2 cos^2\varpi}{\mathfrak{y}_2(\sigma,\wp)^2}. \tag{42}$$

Fixing $\aleph = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.3. *If* $f \in \mathcal{H}^{\sigma,b}_{\wp,\Sigma}(\varpi, h)$ *and is as given in* (1), *then*

$$|a_2| \leqq \sqrt{\frac{|\mathcal{B}_1| cos\varpi}{3\mathfrak{y}_3(\sigma,\wp)}} \tag{43}$$

and

$$|a_3| \leqq \frac{|\mathcal{B}_1| cos\varpi}{3\mathfrak{y}_3(\sigma,\wp)} + \frac{|\mathcal{B}_1|^2 cos^2\varpi}{4\mathfrak{y}_2(\sigma,\wp)^2}. \tag{44}$$

By setting $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, in Theorem 2.1, we sate the following theorem .

Theorem 2.4. *If* $f \in \mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\varpi, \aleph, A, B)$ *and as given in* (1) *then*

$$|a_2| \leqq \sqrt{\frac{2(A-B)cos\varpi}{[(\aleph-1)(\aleph+2)\mathfrak{y}_2(\sigma,\wp)^2 + 2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)]}} \tag{45}$$

and

$$|a_3| \leqq \frac{(A-B)cos\varpi}{(\aleph+2)\mathfrak{y}_3(\sigma,\wp)} + \frac{(A-B)^2 cos^2\varpi}{(1+\aleph)^2\mathfrak{y}_2(\sigma,\wp)^2}. \tag{46}$$

Putting $\aleph = 0$ in Theorem 2.4, we have the following corollary.

Corollary 2.5. *If* $f \in \mathcal{S}^{\sigma,b}_{\wp,\Sigma}(\varpi, A, B)$ *is given by* (1) , *then*

$$|a_2| \leqq \sqrt{\frac{(A-B)cos\varpi}{2\mathfrak{y}_3(\sigma,\wp) - \mathfrak{y}_2(\sigma,\wp)^2}} \tag{47}$$

and

$$|a_3| \leqq \frac{(A-B)cos\varpi}{2\mathfrak{y}_3(\sigma,\wp)} + \frac{(A-B)^2 cos^2\varpi}{\mathfrak{y}_2(\sigma,\wp)^2}. \tag{48}$$

Putting $\aleph = 1$ in Theorem 2.4, we have the following corollary.

Corollary 2.6. *If* $f \in \mathcal{H}^{\sigma,b}_{\wp,\Sigma}(\varpi, A, B)$ *is given by* (1) *then*

$$|a_2| \leqq \sqrt{\frac{(A-B)cos\varpi}{3\mathfrak{y}_3(\sigma,\wp)}} \tag{49}$$

and

$$|a_3| \leqq \frac{(A-B)cos\varpi}{3\mathfrak{y}_3(\sigma,\wp)} + \frac{(A-B)^2cos^2\varpi}{4\mathfrak{y}_2(\sigma,\wp)^2}. \tag{50}$$

Further, by setting $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leq \alpha < 1$ in Theorem 2.1 we get the following result.

THEOREM 2.7. *If $f \in \mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\varpi, \aleph, \alpha)$ and is given by (1), then*

$$|a_2| \leqq \sqrt{\frac{4(1-\alpha)cos\varpi}{[(\aleph-1)(\aleph+2)\mathfrak{y}_2(\sigma,\wp)^2 + 2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)]}} \tag{51}$$

and

$$|a_3| \leqq \frac{2(1-\alpha)cos\varpi}{(\aleph+2)\mathfrak{y}_3(\sigma,\wp)} + \frac{4(1-\alpha)^2cos^2\varpi}{(1+\aleph)^2\mathfrak{y}_2(\sigma,\wp)^2}. \tag{52}$$

Putting $\aleph = 0$ in Theorem 2.7, we have the following corollary.

COROLLARY 2.8. *If $f \in \mathcal{S}^{\sigma,b}_{\wp,\Sigma}(\varpi, \alpha)$ and as assued as in (1) , then*

$$|a_2| \leqq \sqrt{\frac{2(1-\alpha)cos\varpi}{2\mathfrak{y}_3(\sigma,\wp) - \mathfrak{y}_2(\sigma,\wp)^2}} \tag{53}$$

and

$$|a_3| \leqq \frac{(1-\alpha)cos\varpi}{\mathfrak{y}_3(\sigma,\wp)} + \frac{4(1-\alpha)^2cos^2\varpi}{\mathfrak{y}_2(\sigma,\wp)^2}. \tag{54}$$

Putting $\aleph = 1$ in Theorem 2.7, we have the following corollary.

COROLLARY 2.9. *If $f \in \mathcal{H}^{\sigma,b}_{\wp,\Sigma}(\varpi, \alpha)$ is given by (1) , then*

$$|a_2| \leqq \sqrt{\frac{2(1-\alpha)cos\varpi}{3\mathfrak{y}_3(\sigma,\wp)}} \tag{55}$$

and

$$|a_3| \leqq \frac{2(1-\alpha)cos\varpi}{3\mathfrak{y}_3(\sigma,\wp)} + \frac{(1-\alpha)^2cos^2\varpi}{\mathfrak{y}_2(\sigma,\wp)^2}. \tag{56}$$

By fixing $\varpi = 0$ we deduce the following results

THEOREM 2.10. *Let the function $f(z)$ given by (1) be in the class $\mathfrak{B}^{\sigma,b}_{\wp,\Sigma}(\aleph, h)$. Then*

$$|a_2| \leqq \sqrt{\frac{2|\mathcal{B}_1|}{[(\aleph-1)(\aleph+2)\mathfrak{y}_2(\sigma,\wp)^2 + 2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)]}} \tag{57}$$

and

$$|a_3| \leqq \frac{|\mathcal{B}_1|}{(\aleph+2)\mathfrak{y}_3(\sigma,\wp)} + \frac{|\mathcal{B}_1|^2}{(1+\aleph)^2\mathfrak{y}_2(\sigma,\wp)^2}. \tag{58}$$

where $\mathfrak{y}_n(\sigma,\wp)$ is given by (12).

REMARK 2.11. Suitably fixing the parameters as in Example 1.7 we can deduce the coefficient inequalities for the new subclasses stated in Example 1.7 of Section 1 .

3. Fekete-Szegö inequality Results

In 1933, Fekete and Szegö [38] established the maximum value of the expression $|a_3 - \lambda a_2^2|$ for a univalent function f, where the real parameter satisfies $0 \leq \lambda \leq 1$. This significant finding gave rise to the Fekete-Szegö problem, which focuses on maximizing the modulus of the functional $\Psi_\lambda(f) = a_3 - \lambda a_2^2$ for $f \in \mathcal{A}$, with λ being any complex number. A multitude of researchers have explored the Fekete-Szegö functional and related coefficient estimation issues. In this section, we prove Fekete-Szegö inequalities for functions in the class $\mathfrak{M}_{\wp,\Sigma}^{\sigma,b}(\varpi, \aleph, h)$. We shall use the following lemmas, which were introduced by Zaprawa in [32, 39] .

Lemma 3.1. *Let $k, l \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$, then*

$$|(k+l)z_1 + (k-l)z_2| \leq \begin{cases} 2\,|k|\,R, & |k| \geq |l|\,, \\ 2\,|l|\,R, & |k| \leq |l|\,. \end{cases}$$

Theorem 3.2. *For $\hbar \in \mathbb{R}$, if $f \in \mathfrak{M}_{\wp,\Sigma}^{\sigma,b}(\varpi, \aleph, h)$ is given by (1) then*

$$| \, a_3 - \hbar a_2^2 \, | \leq \begin{cases} \frac{\mathcal{B}_1|cos\varpi|}{(\aleph+2)\mathfrak{y}_3(\sigma,\wp)}, & 0 \leq | \, \Lambda(\hbar, \wp, \aleph) \, | \leq \frac{\mathcal{B}_1|cos\varpi|}{2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)} \\ 2\mathcal{B}_1|\Lambda(\hbar, \wp, \aleph)|, & |\Lambda(\hbar, \wp, \aleph)| \geq \frac{\mathcal{B}_1|cos\varpi|}{2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)}, \end{cases} \tag{59}$$

where

$$\Lambda(\hbar, \wp, \aleph) = \frac{1-\hbar}{[(\aleph-1)(\aleph+2)\mathfrak{y}_2(\sigma,\wp)^2 + 2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)]}.$$

Proof. From (37) and (39) it follows that

$$\begin{aligned} a_3 - \hbar a_2^2 \quad &= \quad \frac{(1-\hbar)(p_2+q_2)cos\varpi e^{-i\varpi}}{[(\aleph-1)(\aleph+2)\mathfrak{y}_2(\sigma,\wp)^2 + 2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)]} \\ &+ \quad \frac{(p_2-q_2)cos\varpi e^{-i\varpi}}{2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)} \\ &= \quad cos\varpi e^{-i\varpi}\left\{\left[\Lambda(\hbar,\wp,\aleph) + \frac{\mathcal{B}_1}{2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)}\right]p_2\right. \\ &+ \quad \left.\left[\Lambda(\hbar,\wp,\aleph) - \frac{\mathcal{B}_1}{2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)}\right]q_2\right\}, \end{aligned}$$

where

$$\Lambda(\hbar, \wp, \aleph) = \frac{1-\hbar}{[(\aleph-1)(\aleph+2)\mathfrak{y}_2(\sigma,\wp)^2 + 2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)]}.$$

Thus by applying Lemma 3.1 and we get the desired result as given below

$$| \, a_3 - \hbar a_2^2 \, | \leq \begin{cases} \frac{\mathcal{B}_1|cos\varpi|}{(\aleph+2)\mathfrak{y}_3(\sigma,\wp)}, & 0 \leq | \, \Lambda(\hbar, \wp, \aleph) \, | \leq \frac{|\mathcal{B}_1 cos\varpi|}{2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)} \\ 2\mathcal{B}_1|\Lambda(\hbar, \wp, \aleph)|, & |\Lambda(\hbar, \wp, \aleph)| \geq \frac{\mathcal{B}_1|cos\varpi|}{2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)}. \end{cases}$$

In particular by taking $\hbar = 1$, we get

$$| \, a_3 - a_2^2 \, | \leq \frac{\mathcal{B}_1|cos\varpi|}{(1+\aleph)\mathfrak{y}_3(\wp,\ell)}.$$

□

By fixing $\varpi = 0$ we state the following Fekete-Szegö inequality result:

THEOREM 3.3. *For* $\hbar \in \mathbb{R}$, *if* $f \in \mathfrak{B}^{\sigma,b}_{\wp,\Sigma}(\aleph, h)$ *is given by* (1) *then*

$$| a_3 - \hbar a_2^2 | \leq \begin{cases} \frac{\mathcal{B}_1}{(\aleph+2)\mathfrak{y}_3(\sigma,\wp)}, & 0 \leq | \Lambda(\hbar, \wp, \aleph) | \leq \frac{\mathcal{B}_1}{2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)} \\ 2\mathcal{B}_1|\Lambda(\hbar, \wp, \aleph)|, & |\Lambda(\hbar, \wp, \aleph)| \geq \frac{\mathcal{B}_1}{2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)}, \end{cases} \tag{60}$$

where

$$\Lambda(\hbar, \wp, \aleph) = \frac{1-\hbar}{[(\aleph-1)(\aleph+2)\mathfrak{y}_2(\sigma,\wp)^2 + 2(\aleph+2)\mathfrak{y}_3(\sigma,\wp)]}.$$

REMARK 3.4. Suitably fixing the parameters as in Corollaries 2.2–2.9 we can deduce the Fekete-Szegö inequalities for the new subclasses stated in Examples1.3- 1.7 of Section 1 .

4. Concluding remarks

We introduced new operator based on HLZ in association with EPD and defined new subfamilies of bi-Bazilevic functions of the function class Σ, denoted by $\mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\varpi, \aleph, h)$. The non-sharp estimates on the coefficients $|a_2|$ and $|a_3|$ for $f \in \mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\varpi, \aleph, h)$. More over few special cases are sated as corollaries which are also not yet been discussed relating with HLZ in association with EPD. Further, Fekete-Szegö inequalities and approximated coefficient constraints are obtained for $f \in \mathfrak{M}^{\sigma,b}_{\wp,\Sigma}(\varpi, \aleph, h)$. In fact, by specializing the values of $\varpi = 0$ and setting either by $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, $\left(\text{or } h(z) = \frac{1+(1-2\alpha)z}{1-z},\ 0 \leq \alpha < 1\right)$ the results presented in this paper would find further applications for the class of bi-univalent functions in Examples 1.3- 1.7 of Section 1 we left this as an exercise to interested readers. If we consider $h(z)$ as functions related to Van Der Pol numbers (VPN) [40] one can establish better values for $|a_2|$ and $|a_3|$. The results of the research could be expanded to develop *q-fractional calculus*, extending the results for bi-univalent functions. In image processing and computer vision, scale-space representations of images can be created and modified using the estimates. Further it can be used for edge identification, texture analysis, and multi-scale image analysis. This initial coefficients can be used to analyze the statistical properties of textures, including the distribution of gray levels and the spatial organization of textures.

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