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OPTIMAL CONTROL RESULTS FOR A STOCHASTIC ELASTIC SYSTEM WITH STRUCTURAL DAMPING AND HEMIVARIATIONAL INEQUALITIES

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Abstract. This paper investigates the optimal control results for a stochastic elastic system with structural damping and hemivariational inequalities. Initially, we utilize stochastic theory, fixed point theory, and some features of Clarke's subdifferential to demonstrate the existence of a mild solution for the considered problem. Further, we establish the existence of optimal control with a corresponding cost function by employing Balder's theorem for the proposed system. At last, an illustration is given to highlight the main results.

1. Introduction

In 1981, Panagiotopoulos initially established the hemivariational inequality (HVI) as a weak formulation for various mechanical issues involving non-smooth and non-convex energy functionals [32, 33]. Hemivariational inequalities (HVIs) represents a class of non-linear inclusions related with Clarke's subdifferential operator and has significance in both structural analysis and non-convex optimization. The control problems with HVIs have shown a great deal of interest from researchers recently. For instance, Migorski and Ochal [28] discussed how to solve the parabolic HVIs optimal control problem directly using the calculus of variations together with the Galerkin approach. Park and Park [35] employed the Faedo-Galerkin approach for optimal control pairs for a hyperbolic quasilinear HVIs. Using a fixed-point theory, Liu and Li [24] have examined approximate controllability for HVIs. Muthukumar et al. [30] employed the fixed point method for multi-valued maps together with Balder's theorem to handle the optimal control outcomes for second order stochastic HVIs. Li et al. [23] examined the existence and approximate controllability results for second order nonlinear evolution HVIs . In [42], Vijayakumar discussed the approximate controllability for a class of second-order stochastic evolution inclusions using fixed point theorems of multivalued maps. Recently, several authors have investigated existence, fixed point methods, and numerical aspects for nonlinear functional equations. For example, Pathak et al. [36] studied solvability results for nonlinear functional integral

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equations involving the Erdelyi-Kober fractional operator by employing generalized fixed point techniques and measures of noncompactness. Furthermore, Bhat et al. [4] developed existence and uniqueness results for weakly singular nonlinear functional Volterra integral equations using fixed point methods together with discretization techniques. In addition, authors [3] examined the augmented Urysohn-type nonlinear functional Volterra integral equations for solvability and convergence results via Picard iteration and discretization methods.

An optimal control problem describes an approach to control inputs centred on minimizing the cost functional or maximizing a payoff to the associated system over a collection of permissible control functions. Recently, there has been a significant advancement in optimal control theory, and its applications are beneficial in various domains, including science and engineering (see [13, 19, 22, 29]). Stochastic differential equations are helpful technique for characterising process and systems with random disturbances in numerous areas of engineering and science. For more information on stochastic differential equations, see the monograph [27]. It is also crucial for each artificial and natural systems to display stochastic problem or noise. The biological, physical, and medicinal sciences extensively utilize stochastic processes (see [20, 31, 41]). These are essential for modelling real-world problems where some kind of unpredictability is necessary. Stochastic evolution equations (SEEs) are based on the observation of random disturbances in the natural sciences, including control theory, physics, and chemistry. Several authors thoroughly investigated the existence of SEEs and their optimal control (see [9, 39, 43]). Applying the fixed point methodology, Shukla and Patel [40] investigated the optimal control results for second order semilinear systems.

The vibration equation of the structural damping beam has increasingly come into the focus of researchers as a significant mathematical model of spacecraft in recent years. In 1982, authors of [6] established the damped elastic system

$$\kappa''(t) + B\kappa'(t) + A\kappa(t) = 0,$$

where A is the elastic operator and B is the damping operator are both positive definite self-adjoint operators. An issue involving linear elastic system with structural damping was examined by Huang [18]. Luong and Tung examined the exponential decay for damped elastic system that involves infinite delay and nonlocal conditions in [25, 26]. Specifically, Fan et al. examined the existence, uniqueness, and asymptotic stability of elastic systems with structural damping in [11, 12]. Using sequence method, Gou and Li [15] investigated the approximate controllability of damped elastic systems. Peng et al. [38] studied the approximate controllability of a stochastic elastic systems.

The sine and cosine families indicates the existence and uniqueness of a mild solution for the second-order problems, as demonstrated by [1, 10, 16]. However, cosine family theory is not applicable to the damping elastic system; hence, we describe the result using semigroup theories. However, to the best of our knowledge, no study has yet addressed optimal control results for stochastic elastic systems involving HVIs.

Motivated by all the above-stated research, we investigate the optimal control results for a stochastic elastic system with structural damping and hemivariational inequalities:

$$
(1)\quad \begin{cases} \Big\langle -\kappa''(t) - \rho A\kappa'(t) - A^2\kappa(t) + B\mathfrak{u}(t) + \mathsf{g}(t)\frac{dW(t)}{dt}, v\Big\rangle_Z + \mathfrak{J}^0(t,\kappa(t);v) \geq 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } v \in Z,\ t \in I = [0,a], \\ \kappa(0) = \kappa_0, \quad \kappa'(0) = \kappa_1, \end{cases}
$$

where $\kappa(\cdot)$ values in separable Hilbert space Z with $\|\cdot\|_Z$ and $\langle\cdot,\cdot\rangle_Z$. $A : D(A) \subset Z \to Z$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$ on Z. the control function $\mathfrak{u}$ takes values in $\mathbb{U}$, $\mathbb{U}$ is a separable Hilbert space. $B : \mathbb{U} \to Z$ stands for a bounded linear operator, $\rho \geq 2$ represents a constant. $\mathsf{g}(t) \in \mathscr{G}(t,\kappa(t))$ is a multivalued function and a locally Lipschitz function $\mathfrak{J}(t,\cdot) : Z \to \mathbb{R}$ has a generalized Clarke's directional derivative, denoted as $\mathfrak{J}^0(t,\cdot;\cdot)$. Let $\mathbb{K}$ represents another separable Hilbert space and $\{W(t)\}_{t\geq 0}$ be a given $\mathbb{K}$-Wiener process with covariance operator $Q \geq 0$. Let E represent the expectation of a random variable, or the Lebesgue integral with regard to the probability measure $\mathbb{P}$ and $\mathcal{A}_{ad}$ become the set of all admissible control pairs $(\kappa,\mathfrak{u})$. On the set $\mathcal{A}_{ad}$, the cost functional is

$$
\mathcal{R}(\kappa,\mathfrak{u}) = E\Big\{\int_0^a L(t,\kappa^{\mathfrak{u}}(t),\mathfrak{u}(t))dt\Big\}.
$$

The primary contributions of this manuscript are outlined below:

- (i) To the best of our knowledge, this is the first work addressing the existence of mild solution and optimal control for stochastic elastic system with structural damping and hemivariational inequalities.
- (ii) By employing stochastic theory, the fixed-point approach, and certain aspects of the Clarke subdifferential, we demonstrate the existence of a mild solution for problem (1).
- (iii) The existence of optimal control pairs determined by stochastic elastic system with HVIs has also been presented.
- (iv) An example is given to illustrate the proposed theoretical outcomes.

The structure of this paper is as follows: Some fundamental concepts and preliminary outcomes are provided in Section 2. We illustrate that the system (1) exhibits a mild solution for suitable conditions in Section 3. We also address the optimal control problem for the system (1) in Section 4. At last, we provide an example in Section 5 to highlight our key findings.

2. Preliminaries

We provide definitions, lemmas, and initial findings that are used for understanding this article.

Let a complete probability space $(\Omega, F, \{F_t\}_{t\geq 0}, \mathbb{P})$ endowed with a right continuous increasing family, and F_0 includes all $\mathbb{P}$-null sets with normal filtration $\{F_t\}_{t\geq 0}$. Consider a $\mathbb{K}$-valued Wiener process $\{W(t), t \geq 0\}$ that is specified on $(\Omega, F, \{F_t\}_{t\geq 0}, \mathbb{P})$ with covariance operator $Q \geq 0$ such that $Tr(Q) < \infty$. Suppose that there is a complete orthonormal system $\{e_m\}_{m\geq 1}$ of $\mathbb{K}$ and a bounded sequence of $\{\lambda_m\}_{m=1}^{\infty}$ with respect to $Qe_m = \lambda_m e_m$, and β_m of independent Brownian motion such that

$$\langle W(t), \iota\rangle = \sum_{m=1}^{\infty} \sqrt{\lambda_m}\langle e_m, \iota\rangle \beta_m, \; t \geq 0, \quad \iota \in \mathbb{K},$$

and $F_t = F_t^W$, F_t^W is the σ-algebra produced by $\{W(\vartheta) : 0 \leq \vartheta \leq t\}$.

Let $L(\mathbb{K}, Z)$ represents the space of all bounded linear operators from $\mathbb{K}$ to Z. Let $L_0^2 = L_2(Q^{\frac{1}{2}}\mathbb{K}, Z)$ be a space of all Hilbert-Schmidt $Q^{\frac{1}{2}}\mathbb{K}$ into Z endowed with a norm

$$\|\sigma\|_{L_0^2}^2 = Tr((\sigma Q^{\frac{1}{2}})(\sigma Q^{\frac{1}{2}})^*), \text{ for all } \sigma \in L_0^2.$$

Obviously, for any bounded operators $\sigma \in L(\mathbb{K}, Z)$, this norm reduces to $\|\sigma\|_{L_0^2}^2 = Tr(\sigma Q \sigma^*)$. Let $L^2(F, Z)$ be a Banach space of all strongly F-measurable and Z-valued random parameters fulfilling $E\|\kappa\|_Z^2 < \infty$. Let $C(I, L^2(F, Z))$ stands for Banach space of all continuous map from I into $L^2(F, Z)$ with

$$\|\kappa\| = \sup_{t\in I} E\|\kappa(t)\|_Z^2.$$

In what follows, for every $B \in L_\infty(I, L(\mathbb{U}, Z))$, let $\|B\|_\infty$ represents the norm of B on $L_\infty(I, L(\mathbb{U}, Z))$, where $L_\infty(I, L(\mathbb{U}, Z))$ stands for the space of operator valued functions that are uniformly bounded on I and measurable in the strong operator topology. Let $L_F^2(I, \mathbb{U})$ be a closed subspace of $L_F^2(I \times \Omega, \mathbb{U})$, which comprises all measurable and F_t-adapted, $\mathbb{U}$-valued stochastic processes fulfill the criteria

$$E\int_0^a \|\mathfrak{u}(t)\|_{\mathbb{U}}^2 dt < \infty,$$

and equipped with

$$\|\mathfrak{u}(t)\|_{L_F^2(I,\mathbb{U})} = \left(E\int_0^a \|\mathfrak{u}(t)\|_{\mathbb{U}}^2 dt\right)^{\frac{1}{2}}.$$

For a Banach space X, we employ the notations shown below

$$P_{bd,cl,cv,cp}(X) = \{D \subseteq X : D \text{ is non-empty, bounded, closed, convex, compact}\}.$$

Now, we will provide some basic definitions of multivalued maps. For further details, please refer [8, 17] for more details.

Definition 2.1. [8] Let X be a Banach space. A multi-valued map $G : X \to P(X)$. Then

- (i) G is convex valued if $G(\kappa)$ is convex for every $\kappa \in X$.
- (ii) G is said to be upper semicontinuous on X if for each $\kappa_0 \in X$, the set $G(\kappa_0)$ is nonempty, closed subset of X and if for each open set U of X containing $G(\kappa_0)$, there exists an open neighborhood V of κ_0 such that $G(V) \subseteq U$.
- (iii) G is bounded on the bounded sets if $G(B) = \bigcup_{\kappa\in B} G(\kappa)$ is bounded in X for all $B \in P_{bd}(X)(i.e., \sup_{\kappa\in B}\{\sup\{\|y\| : y \in G(\kappa)\}\} < \infty)$.
- (iv) G is completely continuous if $G(B)$ is relatively compact for every bounded subset $B \in P(X)$.
- (v) G has a fixed point if there is an element $\kappa \in X$ such that $\kappa \in G(\kappa)$.

Definition 2.2. [7] Consider the Banach space X and X^* be its dual space and $\mathfrak{J} : X \to \mathbb{R}$ be a locally Lipschitz function on X. Clarke's generalised directional derivative of $\mathfrak{J}$ at $\kappa \in X$ in the direction $v \in X$ is denoted by

$$\mathfrak{J}^0(\kappa; v) = \lim_{\hbar \to 0^+} \sup_{y \to \kappa} \frac{\mathfrak{J}(y + \hbar v) - \mathfrak{J}(y)}{\hbar}.$$

The Clarke's generalised gradient of $\mathfrak{J}$ at $\kappa \in X$, indicated by $\partial\mathfrak{J}(\kappa) \subseteq X^*$ produced by

$$\partial\mathfrak{J}(\kappa) = \{\kappa^* \in X^* : \mathfrak{J}^0(\kappa; v) \geq \langle \kappa^*, v \rangle, \ \forall\ v \in X\}.$$

Lemma 2.3. [14] Let X and $\mathcal{Y}$ be two Hausdorff topological spaces. Let $\mathcal{Q} : X \to P(\mathcal{Y})$ is a multivalued map which has the property that $\mathcal{Q}(X) \subset \mathbb{K}$ and $\mathcal{Q}$ have a closed graph, $\mathbb{K}$ being a compact set. Then $\mathcal{Q}$ is upper semicontinuous (u.s.c.).

Lemma 2.4. [5] Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{D} \in P_{bd,cl,cv}(X)$. Suppose that $\mathcal{Q} : \mathcal{D} \to P_{bd,cl,cv}(\mathcal{D})$ is a u.s.c. multivalued map and $\mathcal{Q}(\mathcal{D})$ is relatively compact in X. Then $\mathcal{Q}$ admit a fixed point in $\mathcal{D}$.

3. Existence

We investigate the existence of a mild solution for the system (1). Now, we examine the existence of a mild solution for the following second-order stochastic inclusion:

$$\begin{cases} \kappa''(t) + \rho A \kappa'(t) + A^2 \kappa(t) \in B\mathfrak{u}(t) + \mathsf{g}(t)\frac{dW(t)}{dt} + \partial\mathfrak{J}(t, \kappa(t)), \text{ a.e. } t \in I, \\ \kappa(0) = \kappa_0, \quad \kappa'(0) = \kappa_1, \end{cases} \tag{2}$$

where $\partial\mathfrak{J}$ indicates generalised Clarke's subdifferential of $\mathfrak{J}(t, \cdot) : Z \to \mathbb{R}$. $\mathsf{g}(t) \in \mathscr{G}(t, \kappa(t))$ ($\mathscr{G}(t, \kappa(t))$ is a multivalued function to be specified later), and κ_0 is a F_0-measurable, Z-valued random variable independent of W.

We demonstrate that every solution of problem (2) is also a solution of problem(1) as well. For each $\kappa \in Z$, let us define the collection of $\mathscr{G}$ by

$$\mathsf{g} \in S_{\mathscr{G},\kappa} = \{\mathsf{g} \in L^2(I, L_0^2) : \mathsf{g}(t) \in \mathscr{G}(t, \kappa(t)) \text{ for a.e. } t \in I\}.$$

If $\kappa \in C(I, L^2(F, Z))$ is a solution of problem (2), then there exists $\mathsf{g} \in S_{\mathscr{G},\kappa}$ and a function $\eta(t) \in \partial\mathfrak{J}(t, \kappa(t))$ such that $\eta \in L_F^2(I, Z)$ and

$$\begin{cases} \kappa''(t) + \rho A \kappa'(t) + A^2 \kappa(t) = B\mathfrak{u}(t) + \mathsf{g}(t)\frac{dW(t)}{dt} + \eta(t), \ t \in I, \\ \kappa(0) = \kappa_0, \quad \kappa'(0) = \kappa_1. \end{cases}$$

which implies that

$$\begin{cases} \Big\langle -\kappa''(t) - \rho A \kappa'(t) - A^2 \kappa(t) + B\mathfrak{u}(t) + \mathsf{g}(t)\frac{dW(t)}{dt}, v \Big\rangle_Z + \langle \eta(t), v \rangle_Z = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \in I, \ \text{for all } v \in Z, \\ \kappa(0) = \kappa_0, \quad \kappa'(0) = \kappa_1, \end{cases}$$

Since $\eta(t) \in \partial\mathfrak{J}(t,\kappa(t))$ and $\langle \eta(t), v\rangle_Z \leq \mathfrak{J}^0(t,\kappa(t);v)$, we get

$$\begin{cases} \Big\langle -\kappa''(t) - \rho A\kappa'(t) - A^2\kappa(t) + B\mathfrak{u}(t) + \mathsf{g}(t)\frac{dW(t)}{dt}, v\Big\rangle_Z + \mathfrak{J}^0(t,\kappa(t);v) \geq 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \in I, \text{ for all } v \in Z, \\ \kappa(0) = \kappa_0, \quad \kappa'(0) = \kappa_1. \end{cases}$$

Therefore, in order to consider system (1), it is sufficient to discuss system (2). From [37], $A : D(A) \subset Z \to Z$ be the infinitesimal generator of $T(t)(t \geq 0)$ on Z, there exist a constant $\omega \in \mathbb{R}$ and $\mathbb{M}_\omega \geq 1$ such that

$$\|T(t)\| \leq \mathbb{M}_\omega e^{\omega t}, \quad \text{for all } t \geq 0.$$

Definition 3.1. For every $\mathfrak{u} \in L^2(I,\mathbb{U})$, an F_t-adapted stochastic process $\kappa(t) \in C(I, L^2(F,Z))$ is called a mild solution of (1), if $\kappa(0) = \kappa_0$, $\kappa'(0) = \kappa_1$ and there exist $\mathsf{g} \in S_{\mathscr{G},\kappa}$ and $\eta \in L^2_F(I,Z)$ such that $\eta(t) \in \partial\mathfrak{J}(t,\kappa(t))$ for a.e. $t \in I$ and

$$\begin{aligned}\kappa(t) =& \mathcal{V}_2(t)\kappa_0 + \int_0^t \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta)b_0 d\vartheta + \int_0^t\int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)[B\mathfrak{u}(\xi)+\eta(\xi)]d\xi d\vartheta \\ &+ \int_0^t\int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathsf{g}(\xi)dW(\xi)d\vartheta,\end{aligned}$$

where $b_0 = \kappa_1 + \sigma_2 A\kappa_0$ and $\mathcal{V}_i(t)(t \geq 0)(i = 1,2)$ are C_0-semigroups on Z and fulfill

$$\mathcal{V}_i(t) = T(\sigma_i t)(i = 1,2), \ t \geq 0, \tag{3}$$

where $\sigma_1 + \sigma_2 = \rho$, $\sigma_1\sigma_2 = 1$ and $\rho \geq 2$ is the damping coefficient.

According to properties of C_0-semigroups, let

$$\mathbb{M}_1 = \sup_{t\in I}\|\mathcal{V}_1(t)\|, \quad \mathbb{M}_2 = \sup_{t\in I}\|\mathcal{V}_2(t)\|. \tag{4}$$

To support our primary findings, we shall assume the following:

(H1) $T(t)$ is compact for $t > 0$.

(H2) $\mathscr{G}$ is an L^2-Caratheodory multivalued map satisfying the following criteria:

(i) For each $t \in I$, $\mathscr{G}(t,\cdot) : Z \to P_{bd,cl,cv}(L(\mathbb{K},Z))$ is u.s.c. such that, for each $\kappa \in Z$, $\mathscr{G}(\cdot,\kappa)$ is measurable, and each fixed $\kappa \in C(I, L^2(F,Z))$, the set

$$S_{\mathscr{G},\kappa} = \{\mathsf{g} \in L^2(I, L^2_0) : \mathsf{g}(t) \in \mathscr{G}(t,\kappa(t)), \text{ for a.e. } t \in I\}$$

is non-empty.

(ii) For each $k > 0$, there exists a positive function $\mathcal{M}_{\mathsf{g}}(k)$ independent on k such that

$$\sup_{E\|\kappa\|^2\leq k}\|\mathscr{G}(t,\kappa)\|^2 \leq \mathcal{M}_{\mathsf{g}}(k), \text{ for a.e. } t \in I,$$

where $\|\mathscr{G}(t,\kappa)\|^2 = \sup_{\mathsf{g}\in\mathscr{G}(t,\kappa)} E\|\mathsf{g}\|^2$.

(H3) The control function $\mathfrak{u} \in \mathbb{U}$, for all $\|B\|_\infty \in L_\infty(I, L(\mathbb{U},Z))$ and $\|B\|_\infty$ indicates norm of B in $L_\infty(I, L(\mathbb{U},Z))$.

(H4) The functional $\mathfrak{J} : I \times Z \to \mathbb{R}$ fulfills these requirements:

(i) $\mathfrak{J}(\cdot,\kappa) : I \to \mathbb{R}$ is measurable for each $\kappa \in Z$;

(ii) $\mathfrak{J}(t,\cdot) : Z \to \mathbb{R}$ is locally Lipschitz continuous for a.e. $t \in I$;

(iii) there exist $e \in L^2(I, \mathbb{R}^+)$ and $c > 0$ fulfilling

$$E\|\partial\mathfrak{J}(t,\kappa)\|^2 = \sup\{E\|\mathcal{Y}(t)\|^2 : \mathcal{Y}(t) \in \partial\mathfrak{J}(t,\kappa)\} \le e(t) + cE\|\kappa\|^2,$$

for all $\kappa \in Z$ and for almost every $t \in I$.

(H5) A multivalued map $\mathscr{U}(\cdot) : I \to P(\mathbb{U})$ has closed, bounded, and convex values so that $\mathscr{U}(\cdot)$ is graph measurable and $\mathscr{U}(\cdot) \subseteq \Omega$, where $\Omega \subset \mathscr{U}$ is bounded.

Describe the admissible set as

$$\mathcal{A}_{ad} = \Big\{ \mathfrak{u}(\cdot) \in L_F^2(I, \mathbb{U}),\ \mathfrak{u}(t) \in \mathscr{U}(t) \text{ a.e. } t \in I \Big\}.$$

By referring [17], we have $\mathcal{A}_{ad} \neq \emptyset$ and $\mathcal{A}_{ad}$ is a closed, bounded, and convex subset of $L^{\mathtt{p}}(I, \mathbb{U})$, $1 < \mathtt{p} < \infty$. It is evident that, $B\mathfrak{u} \in L^{\mathtt{p}}(I, Z)$ for every $\mathfrak{u} \in \mathcal{A}_{ad}$. Consider $\Psi : L_F^2(I, Z) \to 2^{L_F^2(I,Z)}$ as follows:

$$\Psi(\kappa) = \{\eta \in L_F^2(I, Z) : \eta(t) \in \partial\mathfrak{J}(t,\kappa) \text{ a.e. } t \in I\}, \quad \text{for all } \kappa \in L_F^2(I, Z).$$

Lemma 3.2. [30] If (H4) holds, then the set $\Psi(\kappa)$ is non-empty weakly compact and convex for every $\kappa \in L_F^2(I, Z)$.

Lemma 3.3. [30] Suppose that (H4) holds, Ψ fulfills that, if $\kappa_n \to \kappa$ in $L_F^2(I, Z)$, $z_n \xrightarrow{weakly} z \in L_F^2(I, Z)$ and $z_n \in \Psi(\kappa_n)$, then $z \in \Psi(\kappa)$.

Lemma 3.4. [21] Let I be a compact real interval and $\mathscr{G}$ be a multivalued map fulfilling (H2). Let Υ is a linear continuous operator from $L^2(I, L^2(F, Z))$ to $C(I, L^2(F, Z))$, then

$$\Upsilon \circ S_{\mathscr{G}} : C(I, L^2(F, Z)) \to P_{bd,cl,cv}(C(I, L^2(F, Z))), \quad \kappa \to \Upsilon \circ S_{\mathscr{G}}(\kappa) = \Upsilon S_{\mathscr{G},\kappa},$$

is a closed graph operator in $C(I, L^2(F, Z)) \times C(I, L^2(F, Z))$.

Lemma 3.5. [9] Let $\phi : I \times \Omega \to L_0^2$ be a strongly measurable maps such that $\int_0^a E\|\phi(\vartheta)\|_{L_0^2}^{\mathtt{p}} d\vartheta < \infty$. Then, for each $t \in I$ and $\mathtt{p} \ge 2$, we obtain

$$E\left\| \int_0^a \mathtt{g}(\vartheta) dW(\vartheta) \right\|^{\mathtt{p}} \le \mathtt{C}_{\mathscr{G}} \int_0^a E\|\mathtt{g}(\vartheta)\|_{L_0^2}^{\mathtt{p}} d\vartheta,$$

where $\mathtt{C}_{\mathscr{G}}$ is a constant employing $\mathtt{p}$ and a.

Theorem 3.6. *Suppose that (H1)-(H5) holds, then the problem* (1) *has a mild solution on I.*

Proof. For any $\kappa \in C(I, L^2(F, Z)) \subset L^2(I, L^2(F, Z))$, from Lemma 3.2, then the multi-valued map $\mathcal{Q} : C(I, L^2(F, Z)) \to P(C(I, L^2(F, Z)))$ defined by

$$\begin{aligned}\mathcal{Q}(\kappa) = \Big\{ \mathcal{E} \in C(I, L^2(F, Z)) : \mathcal{E}(t) &= \mathcal{V}_2(t)\kappa_0 + \int_0^t \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta) b_0 d\vartheta \\ &+ \int_0^t \int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)[B\mathfrak{u}(\xi) + \eta(\xi)] d\xi d\vartheta \\ &+ \int_0^t \int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathtt{g}(\xi) dW(\xi) d\vartheta, \mathtt{g} \in S_{\mathscr{G},\kappa},\ \eta \in \Psi(\kappa) \Big\}.\end{aligned}$$

In order to show that problem (1) has a mild solution, it is necessary to demonstrate that $\mathcal{Q}$ attains a fixed point in $C(I, L^2(F, Z))$. Now, we characterize the proof into several steps.

Step 1 : $\mathcal{Q}(\kappa)$ has nonempty, convex and weakly compact values for each $\kappa \in C(I, L^2(F, Z))$.

In view of Lemma 3.2, it is easy to check that $\mathcal{Q}(\kappa)$ has nonempty and weakly compact values. Moreover, as $\Psi(\kappa)$ has convex values; so that if $\eta_1, \eta_2 \in \Psi(\kappa)$ then $\gamma\eta_1 + (1-\gamma)\eta_2 \in \Psi(\kappa)$ for all $\gamma \in [0,1]$, which indicates that the operator $\mathcal{Q}(\kappa)$ is convex.

Step 2 : We show that $\mathcal{Q}(\kappa)$ is a bounded subset of $C(I, L^2(F, Z))$.

For any $\mathtt{r} > 0$, define $B_{\mathtt{r}} = \{\kappa \in C(I, L^2(F, Z)) : \|\kappa\|^2 \leq \mathtt{r}\}$. We claim that there exists a constant $R > 0$ such that, for each $\mathcal{E} \in \mathcal{Q}(\kappa)$ with $\kappa \in B_{\mathtt{r}}$, $\|\mathcal{E}\|^2 \leq R$. In fact, for any $\mathcal{E} \in \mathcal{Q}(\kappa)$, there exist a selection $\mathtt{g} \in S_{\mathscr{G},\kappa}$ and a function $\eta(t) \in \partial\mathfrak{J}(t, \kappa(t))$ such that

$$\begin{aligned}\mathcal{E}(t) =& \mathcal{V}_2(t)\kappa_0 + \int_0^t \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta)b_0 d\vartheta + \int_0^t \int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)[B\mathtt{u}(\xi) + \eta(\xi)]d\xi d\vartheta \\ &+ \int_0^t \int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathtt{g}(\xi)dW(\xi)d\vartheta,\end{aligned}$$

for $t \in I$. By conditions (H2), (H3), (H4)(iii), Lemma 3.5 and Hölder inequality, we derive

$$\begin{aligned}E\|\mathcal{E}(t)\|^2 \leq& 5E\left\|\mathcal{V}_2(t)\kappa_0\right\|^2 + 5E\left\|\int_0^t \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta)b_0 d\vartheta\right\|^2 \\ &+ 5E\left\|\int_0^t \int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)B\mathtt{u}(\xi)d\xi d\vartheta\right\|^2 \\ &+ 5E\left\|\int_0^t \int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\eta(\xi)d\xi d\vartheta\right\|^2 \\ &+ 5E\left\|\int_0^t \int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathtt{g}(\xi)dW(\xi)d\vartheta\right\|^2 \\ \leq& 5\mathbb{M}_2^2 E\|\kappa_0\|^2 + 5\mathbb{M}_1^2\mathbb{M}_2^2 a^2 E\|b_0\|^2 + \frac{5}{4}a^4\mathbb{M}_1^2\mathbb{M}_2^2\|B\|_\infty^2\|\mathtt{u}\|^2_{L^2(I;\mathbb{U})} \\ &+ \frac{5}{4}a^3\mathbb{M}_1^2\mathbb{M}_2^2(\|e\|_{L^2(I,\mathbb{R}^+)} + acr) + \frac{5}{3}\mathbb{M}_1^2\mathbb{M}_2^2 a^3 \mathtt{C}_{\mathscr{G}}\mathcal{M}_{\mathtt{g}}(r) \\ \leq& R. \end{aligned} \tag{5}$$

Therefore, $\mathcal{Q}(\kappa)$ is bounded in $C(I, L^2(F, Z))$.

Step 3 : $\{\mathcal{Q}(\kappa) : \kappa \in B_{\mathtt{r}}\}$ is equicontinuous. For any $\kappa \in B_{\mathtt{r}}$, we obtain

$$\begin{aligned}\mathcal{E}(t) =& \mathcal{V}_2(t)\kappa_0 + \int_0^t \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta)b_0 d\vartheta + \int_0^t \int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)[B\mathtt{u}(\xi) + \eta(\xi)]d\xi d\vartheta \\ &+ \int_0^t \int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathtt{g}(\xi)dW(\xi)d\vartheta.\end{aligned}$$

Let $0 \leq t_1 < t_2 \leq a$.

$$\begin{aligned}E\|\mathcal{E}(t_2) - \mathcal{E}(t_1)\|^2 \leq& 9E\|\mathcal{V}_2(t_2)\kappa_0 - \mathcal{V}_2(t_1)\kappa_0\|^2 + 9E\left\|\int_0^{t_1}[\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)]\mathcal{V}_1(\vartheta)b_0 d\vartheta\right\|^2 \\ &+ 9E\left\|\int_{t_1}^{t_2} \mathcal{V}_2(t_2-\vartheta)\mathcal{V}_1(\vartheta)b_0 d\vartheta\right\|^2\end{aligned}$$

$$
\begin{aligned}
&+ 9E\left\| \int_0^{t_1} \int_0^{\vartheta} [\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)]\mathcal{V}_1(\vartheta-\xi)B\mathfrak{u}(\xi)d\xi d\vartheta \right\|^2 \\
&+ 9E\left\| \int_{t_1}^{t_2} \int_0^{\vartheta} \mathcal{V}_2(t_2-\vartheta)\mathcal{V}_1(\vartheta-\xi)B\mathfrak{u}(\xi)d\xi d\vartheta \right\|^2 \\
&+ 9E\left\| \int_0^{t_1} \int_0^{\vartheta} [\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)]\mathcal{V}_1(\vartheta-\xi)\eta(\xi)d\xi d\vartheta \right\|^2 \\
&+ 9E\left\| \int_{t_1}^{t_2} \int_0^{\vartheta} \mathcal{V}_2(t_2-\vartheta)\mathcal{V}_1(\vartheta-\xi)\eta(\xi)d\xi d\vartheta \right\|^2 \\
&+ 9E\left\| \int_0^{t_1} \int_0^{\vartheta} [\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)]\mathcal{V}_1(\vartheta-\xi)\mathtt{g}(\xi)dW(\xi)d\vartheta \right\|^2 \\
&+ 9E\left\| \int_{t_1}^{t_2} \int_0^{\vartheta} \mathcal{V}_2(t_2-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathtt{g}(\xi)dW(\xi)d\vartheta \right\|^2 \\
=\ & 9\sum_{i=1}^{9} \mathtt{J}_i.
\end{aligned}
$$

By the strong continuity of $\mathcal{V}_2(t)\kappa_0$ for $t \geq 0$, we get $\mathtt{J}_1 \to 0$ as $t_2 \to t_1$. By Hölder inequality and Eq. 4, we get

$$
\begin{aligned}
\mathtt{J}_2 &= E\left\| \int_0^{t_1} [\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)]\mathcal{V}_1(\vartheta)b_0 d\vartheta \right\|^2 \\
&\leq a\int_0^{t_1} \|\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)\|^2 E\|\mathcal{V}_1(\vartheta)b_0\|^2 d\vartheta \\
&\leq a\mathbb{M}_1^2 E\|b_0\|^2 \int_0^{t_1} \|\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)\|^2 d\vartheta.
\end{aligned}
$$

Similarly, from (H4), Eq. 4 and Lemma 3.5, we can get

$$
\begin{aligned}
\mathtt{J}_4 &= E\left\| \int_0^{t_1} \int_0^{\vartheta} [\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)]\mathcal{V}_1(\vartheta-\xi)B\mathfrak{u}(\xi)d\xi d\vartheta \right\|^2 \\
&\leq \mathbb{M}_1^2 a^3 \|B\|_\infty^2 \|\mathfrak{u}\|_{L^2(I;\mathbb{U})}^2 \int_0^a \|\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)\|^2 d\vartheta, \\
\mathtt{J}_6 &= E\left\| \int_0^{t_1} \int_0^{\vartheta} [\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)]\mathcal{V}_1(\vartheta-\xi)\eta(\xi)d\xi d\vartheta \right\|^2 \\
&\leq \mathbb{M}_1^2 a^3 \int_0^a \|\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)\|^2 E\|\eta(\vartheta)\|^2 d\vartheta, \\
\mathtt{J}_8 &= E\left\| \int_0^{t_1} \int_0^{\vartheta} [\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)]\mathcal{V}_1(\vartheta-\xi)\mathtt{g}(\xi)dW(\xi)d\vartheta \right\|^2 \\
&\leq \mathbb{M}_1^2 a^2 \mathtt{C}_{\mathscr{G}} \int_0^a \|\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)\|^2 E\|\mathtt{g}(\vartheta)\|^2 d\vartheta \\
&\leq \mathbb{M}_1^2 a^2 \mathtt{C}_{\mathscr{G}} \mathcal{M}_{\mathtt{g}}(r) \int_0^a \|\mathcal{V}_2(t_2-\vartheta) - \mathcal{V}_2(t_1-\vartheta)\|^2 d\vartheta.
\end{aligned}
$$

As a result, given $t \in (0, a]$, we get that $\mathcal{V}_2(t)$ is continuous in uniform operator topology. Thus, $E\|\mathcal{V}_2(t_2 - \vartheta) - \mathcal{V}_2(t_1 - \vartheta)\|^2 \to 0$ as $t_2 \to t_1$. By employing the Lebesgue's dominated convergence theorem, we derive that $\mathtt{J}_2 = \mathtt{J}_4 = \mathtt{J}_6 = \mathtt{J}_8 \to 0$ as $t_2 \to t_1$. By Hölder inequality, Lemma 3.5 and Eq. 4, make it clear that

$$\mathtt{J}_3 = E\left\| \int_{t_1}^{t_2} \mathcal{V}_2(t_2 - \vartheta)\mathcal{V}_1(\vartheta) b_0 d\vartheta \right\|^2 \leq \mathbb{M}_1^2 \mathbb{M}_2^2 E\|b_0\|^2 |t_2 - t_1|^2,$$

$$\mathtt{J}_5 = E\left\| \int_{t_1}^{t_2} \int_0^{\vartheta} \mathcal{V}_2(t_2 - \vartheta)\mathcal{V}_1(\vartheta - \xi) B\mathfrak{u}(\xi) d\xi d\vartheta \right\|^2 \leq \frac{1}{4}\mathbb{M}_1^2 \mathbb{M}_2^2 \|B\|_\infty^2 \|\mathfrak{u}\|_{L^2(I;\mathbb{U})}^2 |t_2 - t_1|^4,$$

$$\mathtt{J}_7 = E\left\| \int_{t_1}^{t_2} \int_0^{\vartheta} \mathcal{V}_2(t_2 - \vartheta)\mathcal{V}_1(\vartheta - \xi) \eta(\xi) d\xi d\vartheta \right\|^2 \leq \frac{1}{4}\mathbb{M}_1^2 \mathbb{M}_2^2 |t_2^3 - t_1^3| (\|e\|_{L^2(I,\mathbb{R}^+)} + (t_2 - t_1)cr),$$

$$\mathtt{J}_9 = E\left\| \int_{t_1}^{t_2} \int_0^{\vartheta} \mathcal{V}_2(t_2 - \vartheta)\mathcal{V}_1(\vartheta - \xi) \mathtt{g}(\xi) dW(\xi) d\vartheta \right\|^2 \leq \frac{1}{3}\mathbb{M}_1^2 \mathbb{M}_2^2 \mathtt{C}_{\mathscr{G}} \mathcal{M}_{\mathtt{g}}(r) |t_2^3 - t_1^3|.$$

Therefore, $\mathtt{J}_3 = \mathtt{J}_5 = \mathtt{J}_7 = \mathtt{J}_9 \to 0$ as $t_2 \to t_1$, $E\|\mathcal{E}(t_2) - \mathcal{E}(t_1)\|^2 \to 0$ independently as $t_2 \to t_1$. Hence $\{\mathcal{Q}(\kappa) : \kappa \in B_{\mathtt{r}}\}$ is equicontinuous.

Step 4 : The multi-valued map $\mathcal{Q}$ is compact. We shall prove that, for any $t \in [0, a]$, $\Pi(t) = \{\mathcal{E}(t) : \mathcal{E} \in \mathcal{Q}(B_{\mathtt{r}})\}$ is relatively compact subset of Z. It is easy to see that $\Pi(0)$ is relatively compact when $t = 0$. Let $0 < t \leq a$ be fixed. Then, for every $\kappa \in B_{\mathtt{r}}$, there exist a selection $\mathtt{g} \in S_{\mathscr{G},\kappa}$ of $\mathscr{G}(t, \kappa)$ and a function $\eta(t) \in \Psi(\kappa)$ satisfying

$$\begin{aligned}\mathcal{E}(t) =& \mathcal{V}_2(t)\kappa_0 + \int_0^t \mathcal{V}_2(t - \vartheta)\mathcal{V}_1(\vartheta) b_0 d\vartheta + \int_0^t \int_0^{\vartheta} \mathcal{V}_2(t - \vartheta)\mathcal{V}_1(\vartheta - \xi)[B\mathfrak{u}(\xi) + \eta(\xi)] d\xi d\vartheta \\ &+ \int_0^t \int_0^{\vartheta} \mathcal{V}_2(t - \vartheta)\mathcal{V}_1(\vartheta - \xi) \mathtt{g}(\xi) dW(\xi) d\vartheta,\end{aligned}$$

for $t \in I$. For any $\epsilon \in (0, t)$ with $0 < t \leq a$ and $\kappa \in B_{\mathtt{r}}$, define $\Pi_\epsilon(t) = \{\mathcal{E}_\epsilon(t) : \mathcal{E}_\epsilon \in \mathcal{Q}(B_{\mathtt{r}})\}$, where

$$\begin{aligned}\mathcal{E}_\epsilon(t) =& \mathcal{V}_2(t)\kappa_0 + \int_0^{t-\epsilon} \mathcal{V}_2(t - \vartheta)\mathcal{V}_1(\vartheta) b_0 d\vartheta + \int_0^{t-\epsilon} \int_0^{\vartheta} \mathcal{V}_2(t - \vartheta)\mathcal{V}_1(\vartheta - \xi)[B\mathfrak{u}(\xi) + \eta(\xi)] d\xi d\vartheta \\ &+ \int_0^{t-\epsilon} \int_0^{\vartheta} \mathcal{V}_2(t - \vartheta)\mathcal{V}_1(\vartheta - \xi) \mathtt{g}(\xi) dW(\xi) d\vartheta \\ =& \mathcal{V}_2(t)\kappa_0 + \mathcal{V}_2(\epsilon) \int_0^{t-\epsilon} \mathcal{V}_2(t - \epsilon - \vartheta)\mathcal{V}_1(\vartheta) b_0 d\vartheta \\ &+ \mathcal{V}_2(\epsilon) \int_0^{t-\epsilon} \int_0^{\vartheta} \mathcal{V}_2(t - \epsilon - \vartheta)\mathcal{V}_1(\vartheta - \xi)[B\mathfrak{u}(\xi) + \eta(\xi)] d\xi d\vartheta \\ &+ \mathcal{V}_2(\epsilon) \int_0^{t-\epsilon} \int_0^{\vartheta} \mathcal{V}_2(t - \epsilon - \vartheta)\mathcal{V}_1(\vartheta - \xi) \mathtt{g}(\xi) dW(\xi) d\vartheta.\end{aligned}$$

It is simple to determine that $\mathcal{V}_2(t)(t > 0)$ is compact by employing **(H1)** and Eq. 3. Hence, $\Pi_\epsilon(t)$ is relatively compact subset of Z for any $\epsilon \in (0, t)$. Moreover, for each $0 < \epsilon < t$, we have

$$E\|\mathcal{E}(t) - \mathcal{E}_\epsilon(t)\|^2 \leq 4E\left\| \int_{t-\epsilon}^{t} \mathcal{V}_2(t - \vartheta)\mathcal{V}_1(\vartheta) b_0 d\vartheta \right\|^2$$

$$+ 4E\left\| \int_{t-\epsilon}^{t} \int_{0}^{\vartheta} \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\eta(\xi)d\xi d\vartheta \right\|^2$$
$$+ 4E\left\| \int_{t-\epsilon}^{t} \int_{0}^{\vartheta} \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)B\mathfrak{u}(\xi)d\xi d\vartheta \right\|^2$$
$$+ 4E\left\| \int_{t-\epsilon}^{t} \int_{0}^{\vartheta} \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathsf{g}(\xi)dW(\xi)d\vartheta \right\|^2 \to 0, \text{ as } \epsilon \to 0.$$

The above discussion demonstrates that the relatively compact sets that are arbitrarily close to $\Pi(t)$. As a result, for any $t \in I$, we also know that $\Pi(t),\ t > 0$ provides a relatively compact subset of Z. By Step 1-3, the relative compactness of $\Pi(t),\ t > 0$ and the Ascoli-Arzela theorem, we deduce that $\{\mathcal{Q}(\kappa) : \kappa \in B_{\mathtt{r}}\}$ is relatively compact subset of Z. This shows that multivalued map $\mathcal{Q}$ is compact.

Step 5 : $\mathcal{Q}$ has a closed graph.

Let $\kappa_n \to \kappa_*$ in $C(I, L^2(F, Z))$ and $\mathcal{E}_n \in \mathcal{Q}(\kappa_n)$ and $\mathcal{E}_n \to \mathcal{E}_*$ in $C(I, L^2(F, Z))$. Next, we must demonstrate that $\mathcal{E}_* \in \mathcal{Q}(\kappa_*)$. Let $\mathcal{E}_n \in \mathcal{Q}(\kappa_n)$ implies there exist $\mathsf{g}_n \in S_{\mathscr{G},\kappa_n}$ and $\eta_n \in \Psi(\kappa_n)$ such that for every $t \in I$,

$$\begin{aligned}\mathcal{E}_n(t) =& \mathcal{V}_2(t)\kappa_0 + \int_0^t \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta)b_0 d\vartheta + \int_0^t \int_0^{\vartheta} \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)[B\mathfrak{u}(\xi) + \eta_n(\xi)]d\xi d\vartheta \\ &+ \int_0^t \int_0^{\vartheta} \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathsf{g}_n(\xi)dW(\xi)d\vartheta.\end{aligned} \tag{6}$$

According to (H2) and (H4)(iii), we get the boundedness of $\{\mathsf{g}_n\}$ and $\{\eta_n\}$. Considering the reflexivity of $L^2_F(I, Z)$ and L^2_0, without loss of generality, we are assuming that

$$(\eta_n, \mathsf{g}_n) \xrightarrow{weakly} (\eta_*, \mathsf{g}_*) \in L^2_F(I, Z) \times L^2_0. \tag{7}$$

Define an operator $\Upsilon : L^2_0 \to C(I, L^2(F, Z))$ by

$$\Upsilon(\mathsf{g})(t) = \int_0^t \int_0^{\vartheta} \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathsf{g}(\xi)dW(\xi)d\vartheta, \quad \text{for all } \mathsf{g} \in L^2_0.$$

Obviously, Υ is linear and continuous. Moreover, similar to the proof of Step 2, condition (H2) infers

$$E\|\Upsilon(\mathsf{g})(t)\|^2 \le \frac{1}{3}\mathbb{M}_1^2\mathbb{M}_2^2 a^3 \mathtt{c}_{\mathscr{G}} \mathcal{M}_{\mathsf{g}}(r). \tag{8}$$

By (H1) and (3), it is simple to get $\mathcal{V}_i(t)(t \ge 0),\ i = 1, 2$ is compact, it follows from (H2),(H4)(iii),(6) $-$ (8), we get

$$\begin{aligned}\mathcal{E}_n(t) \to& \mathcal{V}_2(t)\kappa_0 + \int_0^t \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta)b_0 d\vartheta + \int_0^t \int_0^{\vartheta} \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)[B\mathfrak{u}(\xi) + \eta_*(\xi)]d\xi d\vartheta \\ &+ \int_0^t \int_0^{\vartheta} \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathsf{g}_*(\xi)dW(\xi)d\vartheta.\end{aligned}$$

We observe that $\mathcal{E}_n \to \mathcal{E}_*$ in $C(I, L^2(F, Z)),\ \mathsf{g}_n \in S_{\mathscr{G},\kappa_n}$ and $\eta_n \in \Psi(\kappa_n)$ from Lemma 3.3, Eq. 4 and 7, we have $\eta_* \in \Psi(\kappa_*)$ and $\mathsf{g}_* \in S_{\mathscr{G},\kappa_*}$. Hence $\mathcal{E}_* \in \mathcal{Q}(\kappa_*)$, which shows that $\mathcal{Q}$ has a closed graph.

According to Step 4 and 5, it yields from Lemma 2.3 that $\mathcal{Q}$ is u.s.c. Consequently, we demonstrate that $\mathcal{Q}$ is compact, upper semicontinuous, and has convex closed values as a result of Steps 1-5. Therefore, summarizing the above, we conclude that the multivalued map $\mathcal{Q}$ fulfills Lemma 2.4. Thus, from Lemma 2.4, $\mathcal{Q}$ has a fixed point on $B_{\mathtt{r}}$. $\square$

4. Optimal control result

We consider the Lagrange problem (P):

Examine a pair $(\kappa^0, \mathfrak{u}^0) \in C(I, L^2(F, Z)) \times \mathcal{A}_{ad}$ such that

$$\mathcal{R}(\kappa^0, \mathfrak{u}^0) \leq \mathcal{R}(\kappa, \mathfrak{u}), \ \forall \ (\kappa, \mathfrak{u}) \in C(I, L^2(F, Z)) \times \mathcal{A}_{ad},$$

where

$$\mathcal{R}(\kappa, \mathfrak{u}) = E \int_0^a L(t, \kappa^{\mathfrak{u}}(t), \mathfrak{u}(t))dt.$$

Here $\kappa^{\mathfrak{u}}$ indicates the mild solution of (1) that corresponds to $\mathfrak{u} \in \mathcal{A}_{ad}$. We utilize the subsequent hypothesis (H6) to examine the (P):

$(a1)$ $L : I \times Z \times \mathbb{U} \to \mathbb{R} \cup \{\infty\}$ is Borel measurable;

$(a2)$ For all $t \in I$, L is sequentially lower semicontinuous on $Z \times \mathbb{U}$;

$(a3)$ For each $\kappa \in Z$ and $t \in I$, $L(t, \kappa, \cdot)$ is convex on $\mathbb{U}$;

$(a4)$ there exists $\mathtt{d}_1 \geq 0$, $\mathtt{d}_2 > 0$, α is positive and $\alpha \in L^1(I, \mathbb{R})$ such that

$$L(t, \kappa, \mathfrak{u}) \geq \alpha(t) + \mathtt{d}_1 E\|\kappa\|_Z^2 + \mathtt{d}_2 E\|\mathfrak{u}\|_{\mathbb{U}}^{\mathtt{p}}.$$

THEOREM 4.1. *If the hypothesis (H1)-(H6) holds, then the Lagrange problem admits at least one optimal pair.*

Proof. If $\inf\{\mathcal{R}(\kappa, \mathfrak{u}) \mid \mathfrak{u} \in \mathcal{A}_{ad}\} = +\infty$, then it is simple to conclude that there is only one optimal pair for the Lagrange problem (P). Without loss of generality, assuming $\inf\{\mathcal{R}(\kappa, \mathfrak{u}) : \mathfrak{u} \in \mathcal{A}_{ad}\} = \mu < +\infty$. From (H6), we get $\mu > -\infty$. According to infimum definition, there exists a minimizing sequence feasible pair $\{(\kappa^n, \mathfrak{u}^n)\} \subset \mathcal{A}_{ad}$ such that $\mathcal{R}(\kappa^n, \mathfrak{u}^n) \to \mu$ as $n \to +\infty$. Hence $\{\mathfrak{u}^n\} \subseteq \mathcal{A}_{ad}$, $n = 1, 2, \cdots$, $\{\mathfrak{u}^n\}$ is bounded in $L_F^{\mathtt{p}}(I, \mathbb{U})$, by the reflexivity of $L_F^{\mathtt{p}}(I, \mathbb{U})$, there exist a subsequence of $\{\mathfrak{u}^n\}$, again indicated by $\{\mathfrak{u}^n\}$ and $\mathfrak{u}^0 \in L_F^{\mathtt{p}}(I, \mathbb{U})$ satisfying

$$\mathfrak{u}^n \xrightarrow{weakly} \mathfrak{u}^0 \in L_F^{\mathtt{p}}(I, \mathbb{U}).$$

Since $\mathcal{A}_{ad}$ is closed and convex, it follows from Mazur's lemma, $\mathfrak{u}^0 \in \mathcal{A}_{ad}$. Let $\{\kappa^n\}$ represent the corresponding sequence of solutions of the following integral equation:

$$\kappa^n(t) = \mathcal{V}_2(t)\kappa_0 + \int_0^t \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta)b_0 d\vartheta + \int_0^t \int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)[B\mathfrak{u}^n(\xi) + \eta^n(\xi)]d\xi d\vartheta$$
$$+ \int_0^t \int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathtt{g}^n(\xi)dW(\xi)d\vartheta, \quad t \in I,$$

where $\mathtt{g}^n \in S_{\mathscr{G},\kappa^n}$ and $\eta^n \in \Psi(\kappa^n)$.

Afterwards, we demonstrate that $\{\kappa^n\}$ is relatively compact subset of $C(I, L^2(F, Z))$. Initially, comparable to Eq. 5's proof, we get

$$E\|\kappa^n(t)\|^2 \leq 5\mathbb{M}_2^2 E\|\kappa_0\|^2 + 5\mathbb{M}_1^2\mathbb{M}_2^2 a^2 E\|b_0\|^2 + \frac{5}{4}a^4\mathbb{M}_1^2\mathbb{M}_2^2\|B\|_\infty^2\|\mathfrak{u}^n\|^2_{L_F^2(I;\mathbb{U})}$$

$$+\frac{5}{4}a^3\mathbb{M}_1^2\mathbb{M}_2^2(\|e\|_{L^2(I,\mathbb{R}^+)}+acr)+\frac{5}{3}\mathbb{M}_1^2\mathbb{M}_2^2a^3\mathtt{C}_{\mathscr{G}}\mathcal{M}_{\mathtt{g}}(r). \tag{9}$$

Considering that $\{\mathfrak{u}^n\}$ is bounded, Eq. 9 and the Gronwall's inequality, we deduce that there exists $\lambda > 0$ such that $\|\kappa^n\| \le \lambda$, implying that $\{\kappa^n\}$ is uniformly bounded. Next, in the same way of Theorem 3.6's Step 3 and 4, we might prove that $\{\kappa^n(t)\}$ is equicontinuous on I and $\{\kappa^n(t)\}$ is relatively compact for every $t \in I$. According to the Ascoli-Arzela theorem, $\{\kappa^n\} \subset C(I, L^2(F,Z))$ is relatively compact, and so there exists $\kappa^0 \in C(I, L^2(F,Z))$ with respect to

$$\kappa^n \to \kappa^0 \text{ in } C(I, L^2(F,Z)) \subset L^2(I, L^2(F,Z)). \tag{10}$$

The boundedness of $\{\mathfrak{u}^n\}$ and the compactness of $\mathcal{V}_i(t)(t \ge 0),\ i = 1,2$ implies that

$$\int_0^t\int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)B\mathfrak{u}^n(\xi)d\xi d\vartheta \to \int_0^t\int_0^\vartheta \mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)B\mathfrak{u}^0(\xi)d\xi d\vartheta. \tag{11}$$

Similar to the proof of Step 5 in Theorem 3.6, (H2), (H4)(iii), Eq. 10, Lemma 3.3 and Eq. 4, we obtain

$$\begin{aligned}
&\mathcal{V}_2(t)\kappa_0+\int_0^t\mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta)b_0d\vartheta+\int_0^t\int_0^\vartheta\mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\eta^n(\xi)d\xi d\vartheta\\
&\quad+\int_0^t\int_0^\vartheta\mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathtt{g}^n(\xi)dW(\xi)d\vartheta\\
&\to\mathcal{V}_2(t)\kappa_0+\int_0^t\mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta)b_0d\vartheta+\int_0^t\int_0^\vartheta\mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\eta^0(\xi)d\xi d\vartheta\\
&\quad+\int_0^t\int_0^\vartheta\mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathtt{g}^0(\xi)dW(\xi)d\vartheta.
\end{aligned} \tag{12}$$

Hence, it follows from Eq. 11 and 12 that

$$\begin{aligned}
\kappa^0(t)&=\mathcal{V}_2(t)\kappa_0+\int_0^t\mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta)b_0d\vartheta+\int_0^t\int_0^\vartheta\mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)[B\mathfrak{u}^0(\xi)+\eta^0(\xi)]d\xi d\vartheta\\
&\quad+\int_0^t\int_0^\vartheta\mathcal{V}_2(t-\vartheta)\mathcal{V}_1(\vartheta-\xi)\mathtt{g}^0(\xi)dW(\xi)d\vartheta.
\end{aligned}$$

This indicates that κ^0 has a mild solution of problem (1) corresponding to $\mathfrak{u}^0 \in \mathcal{A}_{ad}$.

We notice that (H6) fulfills the requirements of Balder theorem [2]. Consequently, Balder's theorem demonstrates that

$$(\kappa, \mathfrak{u}) \to E\int_0^a L(t, \kappa^{\mathfrak{u}}(t), \mathfrak{u}(t))dt$$

is sequentially lower semicontinuous in the strong topology of $L^1_F(I,Z)$ and weak topology of $L^{\mathtt{p}}_F(I,Z) \subset L^1_F(I,Z)$. We infer that, $\mathcal{R}$ is weakly lower semicontinuous on $L^{\mathtt{p}}_F(I,Z)$. By (H6), we are aware of that $\mathcal{R} > -\infty$. As a result, we conclude that $\mathcal{R}$ reaches its inf at $\mathfrak{u}^0 \in \mathcal{A}_{ad}$ i.e.,

$$\mu = \lim_{n\to\infty} E\int_0^a L(t,\kappa^n(t),\mathfrak{u}^n(t))dt \ge E\int_0^a L(t,\kappa^0(t),\mathfrak{u}^0(t))dt \ge \mu.$$

This completes the proof. □

5. Example

We consider the IVP for a damped elastic stochastic system as follows to clarify our major findings:

(13)
$$\begin{cases} \frac{\partial^2\kappa(t,\mathcal{E})}{\partial t^2} - 4\frac{\partial^3\kappa(t,\mathcal{E})}{\partial t\partial\mathcal{E}^2} + \frac{\partial^4\kappa(t,\mathcal{E})}{\partial\mathcal{E}^4} = \int_0^1 q(\mathcal{E},\xi)\mathfrak{u}(\xi,t)d\xi + \partial\mathfrak{J}(t,\kappa(t,\mathcal{E})) + \mathsf{g}(t)\frac{\partial\breve{W}(t)}{\partial t}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t\in[0,1],\ \mathcal{E}\in[0,\pi], \\ \kappa(t,0)=\kappa(t,\pi)=0, \quad t\in[0,1], \\ \kappa(\epsilon,\mathcal{E})=\chi(\epsilon,\mathcal{E}), \quad \epsilon\le 0,\ \mathcal{E}\in[0,\pi], \\ \frac{\partial}{\partial t}\kappa(0,\mathcal{E})=\kappa_1, \end{cases}$$

where $\chi(\cdot,\cdot)$ is F_0-measurable and $\breve{W}$ represents a one-dimensional standard Brownian motion specified on $(\Omega, F, \{F_t\}_{t\ge 0}, \mathbb{P})$, $\mathsf{g}(t) \in \mathscr{G}(t,\kappa(t,\mathcal{E}))$ and $\mathsf{g} : I\times Z \to P(Z)$ is a non-empty, closed, bounded and convex multi-valued map which fulfills (H2), $q:[0,1]\times[0,1]\to\mathbb{R}$ is continuous.

Let $Z = \mathbb{U} = L^2([0,\pi])$ with $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, we clarified $\kappa(t)(\cdot) = \kappa(t,\cdot)$ and $\chi(t)(\cdot) = \chi(t,\cdot)$. $A : D(A)\to Z$ represents the linear operator by

$$A\kappa = -\frac{\partial^2\kappa}{\partial\mathcal{E}^2},$$

with $D(A) = \{\kappa\in Z : \kappa, \kappa_t$ are absolutely continuous, $\kappa_{tt}\in Z,\ \kappa(t,0)=\kappa(t,\pi)=0\}$. Following that, $-A$ produces a compact, analytic, and self-adjoint C_0-semigroup $T(t)(t\ge 0)$. Let $-A$ have discrete spectrum made up of the eigenvalues $-m^2,\ m\in\mathbb{N}$, and the associated normalized eigenvectors are $e_m(\kappa) = \sqrt{\frac{2}{\pi}}\sin(m\kappa)$, and they exhibit the characteristics:

(i) For every $\kappa\in D(A)$,

$$A\kappa = -\sum_{m=1}^{\infty} m^2\langle\kappa, e_m\rangle e_m.$$

(ii) For any $\kappa\in Z$, the compact analytic C_0-semigroup $T(t)(t\ge 0)$ formed by A can be expressed as

$$T(t)\kappa = \sum_{m=1}^{\infty} e^{-m^2 t}\langle\kappa, e_m\rangle e_m. \tag{14}$$

By (3) and (14) generated by $-\sigma_i A$ are described below:

$$\mathcal{V}_i(t)\kappa = \sum_{m=1}^{\infty} e^{-m^2\sigma_i t}\langle\kappa, e_m\rangle e_m,$$

where $\sigma_1 = 2-\sqrt{3}$ and $\sigma_2 = 2+\sqrt{3}$.

The set $\mathcal{A}_{ad} = \{\mathfrak{u}\in\mathbb{U} \mid \|\mathfrak{u}\|_{(L^2[0,1],\mathbb{U})}\le 1\}$. Let us examine the control $\mathfrak{u}(t,\mathcal{E})$ to minimize the cost functional

$$\mathcal{R}(\kappa,\mathfrak{u}) = \int_0^1\int_0^\pi |\kappa(t,\mathcal{E})|^2 d\mathcal{E}dt + \int_0^1\int_0^\pi |\mathfrak{u}(t,\mathcal{E})|^2 d\mathcal{E}dt,$$

subject to system (13). Define

$$\kappa(t,\mathcal{E}) = \kappa(\mathcal{E})(t),\ B(t)\mathfrak{u}(t)(\mathcal{E}) = \int_0^1 q(\mathcal{E},\xi)\mathfrak{u}(\xi,t)d\xi.$$

Let us define a functional $\mathfrak{J} : (0,1)\times Z \to \mathbb{R}$ as follows:

$$\mathfrak{J}(t,\kappa) = \int_0^1 j(t,\mathcal{E},\kappa(\mathcal{E}))d\mathcal{E}, \quad t\in(0,1),\ \kappa\in Z,$$

where

$$j(t,\mathcal{E},\mathfrak{p}) = \int_0^{\mathfrak{p}} \beta(t,\mathcal{E},\theta)d\theta, \quad (t,\mathcal{E})\in(0,1)\times(0,\pi),\ \mathfrak{p}\in\mathbb{R}.$$

Let $\beta : (0,\pi)\times(0,1)\times\mathbb{R}\to\mathbb{R}$ follows:

(a) for each $\mathcal{E}\in(0,\pi),\ \mathfrak{p}\in\mathbb{R},\ \beta(\cdot,\mathcal{E},\mathfrak{p}) : (0,1)\to\mathbb{R}$ is measurable;
(b) for every $t\in(0,1),\ \mathfrak{p}\in\mathbb{R},\ \beta(t,\cdot,\mathfrak{p}) : (0,\pi)\to\mathbb{R}$ is continuous;
(c) for all $\mathfrak{p}\in\mathbb{R}$, there exists $\mathfrak{f}_1>0$ fulfilling $\beta(\cdot,\cdot,\mathfrak{p})\le\mathfrak{f}_1(1+|\mathfrak{p}|)$;
(d) for all $\mathfrak{p}\in\mathbb{R}$, $\beta(\cdot,\cdot,\mathfrak{p}\pm 0)$ exists.

If β fulfills (c), then we have $\partial j(\mathfrak{p})\subset[\underline{\beta}(\mathfrak{p}),\overline{\beta}(\mathfrak{p})]$ for $\mathfrak{p}\in\mathbb{R}$ (we omit $(t,\mathcal{E})$ here), where $\underline{\beta}(\mathfrak{p})$ and $\overline{\beta}(\mathfrak{p})$ represents for the essential inf and essential sup of η at a certain $\mathfrak{p}$ (see [7]). If β fulfills (a) $-$ (d), $j(\cdot,\cdot,\cdot)$ defined previously as follows [34]:

(e) for every $\mathcal{E}\in(0,\pi),\ \mathfrak{p}\in\mathbb{R},\ j(\cdot,\mathcal{E},\mathfrak{p}) : (0,1)\to\mathbb{R}$ is measurable and $j(\cdot,\cdot,0)\in L^2((0,1)\times(0,\pi))$;
(f) for every $t\in(0,1),\ \mathfrak{p}\in\mathbb{R},\ j(t,\cdot,\mathfrak{p}) : (0,\pi)\to\mathbb{R}$ is continuous;
(g) for all $(t,\mathcal{E})\in(0,1)\times(0,\pi)$, $j(t,\mathcal{E},\cdot) : \mathbb{R}\to\mathbb{R}$ is locally Lipschitz;
(h) there exist $\mathfrak{f}_2>0$ fulfilling
$$|e|\le\mathfrak{f}_2(1+|\mathfrak{p}|),\ \text{for all } e\in\partial j(t,\mathcal{E},\mathfrak{p}),\ (t,\mathcal{E})\in(0,1)\times(0,\pi);$$
(i) there exist $\mathfrak{f}_3>0$ fulfilling
$$j^0(t,\mathcal{E},\mathfrak{p},-\mathfrak{p})\le\mathfrak{f}_3(1+|\mathfrak{p}|),\ \text{for all } (t,\mathcal{E})\in(0,1)\times(0,\pi).$$

Furthermore, condition (H4) is satisfied by the functional $\mathfrak{J}(\cdot,\cdot)$ as described before. As a result, (13) can be expressed as the abstract form of (1) with the cost function

$$\mathcal{R}(\kappa,\mathfrak{u}) = E\int_0^1[\|\kappa\|^2+\|\mathfrak{u}\|^2]dt.$$

Considering that all assumptions of Theorem 3.6 and 4.1 have been fulfilled, we may infer that there is at least one optimal pair and a mild solution for problem (13).

6. Conclusion

In this study, optimal control results for a stochastic elastic system with structural damping and HVIs have been explored. A fixed point approach for multivalued maps, stochastic analysis methods, features of generalized Clarke subdifferential operators, and the semigroup theory have all been used to show that the proposed system has a mild solution. Moreover, an investigation has been conducted on the optimal control for the given system. Finally, an example is included to demonstrate our theoretical results. In future work, we plan to study the approximate controllability of damped elastic systems with non-instantaneous impulsive by using a fixed point approach.

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