Korean J. Math. **34** (2026), No. 1, pp. 51–64
https://dx.doi.org/10.11568/kjm.2026.34.1.51

GENERALIZED STABILITY OF A GENERAL SEXTIC FUNCTIONAL EQUATION

Sun-Sook Jin†,∗ and Yang-Hi Lee

Abstract. The general sextic functional equation is a generalization of many functional equations such as Jensen, general quadratic, general cubic, general quartic, and general quintic functional equations. In this paper, we investigate the generalized stability of the general sextic functional equation.

1. Introduction

In this paper, let V, X, and Y be a real vector space, a real normed space, and a real Banach space, respectively.

Ulam's question about the stability of group isomorphism motivated Hyers' study of the stability of the additive functional equation, and since then many mathematicians have investigated the stability of functional equations (see [3,11]). For more historical information, see [2,5,7–9].

Consider the general sextic functional equation

$$\sum_{i=0}^{7}(-1)^{7-i}\binom{7}{i}f(x+iy)=0 \tag{1}$$

for all $x, y \in V$. If $f : V \to Y$ is a solution mapping of the functional equation (1), then we call the mapping f a general sextic mapping.

A result obtained by Y.-H. Lee for the Hyers-Ulam-Rassas stability of the functional equation (1) is presented in the following theorem.

Theorem 1.1. ([6, Theorem 3]) *Let $p \notin \{1,2,3,4,5,6\}$ be a nonnegative real number. Suppose $f : X \to Y$ is a mapping such that*

$$\left\|\sum_{i=0}^{7}(-1)^{7-i}\binom{7}{i}f(x+iy)\right\| \le \theta(\|x\|^p+\|y\|^p)$$

Received January 24, 2025. Revised December 6, 2025. Accepted December 6, 2025.
2010 Mathematics Subject Classification: 39B82, 39B52.
Key words and phrases: stability of a functional equation, general sextic functional equation, general sextic mapping.
† This work was supported by Gongju National University of Education.
∗ Corresponding author.

for all $x, y \in X$. Then there exists a general sextic mapping $F : X \to Y$ with $F(0) = 0$ and a constant $K(p)$ such that

$$\|f(x) - f(0) - F(x)\| \le K(p)\theta\|x\|^p$$

for all $x \in X$.

J. Roh et al. [10] studied the generalized stability of functional equation (1) using the fixed point method, and I.-S. Chang et al. [1] also obtained results on the generalized stability of (1) by utilizing the idea of P. Găvruta.

In this paper, we improve previous results on the stability of general sextic functional equation by providing a more explicit proof based on the idea of Găvruta. In particular, we extend the special conditions imposed on the control function related to the generalized stability of the functional equation (1) obtained by Chang et al. [1] and Roh et al. [10] to much more general conditions.

2. Stability of a general sextic functional equation

Throughout this paper, for a given mapping $f : V \to Y$, we use the following abbreviations:

$$\begin{aligned}
\tilde{f}(x) :=& f(x) - f(0),\\
f_1(x) :=& -\frac{1}{312480}\big(\tilde{f}(32x) - 124\tilde{f}(16x) + 4960\tilde{f}(8x) - 79360\tilde{f}(4x)\\
&+ 507904\tilde{f}(2x) - 1048576\tilde{f}(x)\big),\\
f_2(x) :=& \frac{1}{161280}\big(\tilde{f}(32x) - 122\tilde{f}(16x) + 4720\tilde{f}(8x) - 70400\tilde{f}(4x)\\
&+ 385024\tilde{f}(2x) - 524288\tilde{f}(x)\big),\\
f_3(x) :=& -\frac{1}{258048}\big(\tilde{f}(32x) - 118\tilde{f}(16x) + 4264\tilde{f}(8x) - 55168\tilde{f}(4x)\\
&+ 225280\tilde{f}(2x) - 262144\tilde{f}(x)\big),\\
f_4(x) :=& \frac{1}{1032192}\big(\tilde{f}(32x) - 110\tilde{f}(16x) + 3448\tilde{f}(8x) - 34112\tilde{f}(4x)\\
&+ 120832\tilde{f}(2x) - 131072\tilde{f}(x)\big),\\
f_5(x) :=& -\frac{1}{10321920}\big(\tilde{f}(32x) - 94\tilde{f}(16x) + 2200\tilde{f}(8x) - 18880\tilde{f}(4x)\\
&+ 62464\tilde{f}(2x) - 65536\tilde{f}(x)\big),\\
f_6(x) :=& \frac{1}{319979520}\big(\tilde{f}(32x) - 62\tilde{f}(16x) + 1240\tilde{f}(8x) - 9920\tilde{f}(4x)\\
&+ 31744\tilde{f}(2x) - 32768\tilde{f}(x)\big),\\
\underset{y}{\overset{7}{\Delta}} f(x) :=& \sum_{i=0}^{7}(-1)^{7-i}\binom{7}{i} f(x + iy),\\
\Gamma f(x) :=& f(64x) - 126f(32x) + 5208f(16x) - 89280f(8x)\\
&+ 666624f(4x) - 2064384f(2x) + 2097152f(x)
\end{aligned}$$

for all $x, y \in V$.

By some careful calculations, we can obtain some useful equations given in the following lemma. Therefore, we can omit their proofs.

Lemma 2.1. *For a given mapping $f : V \to Y$, the equalities*

$$\mathop{\Delta}\limits_{y}^{7} \tilde{f}(x) = \mathop{\Delta}\limits_{y}^{7} f(x),$$

$$\begin{aligned}\Gamma \tilde{f}(x) = {} & \mathop{\Delta}\limits_{8x}^{7} f(8x) - 7 \mathop{\Delta}\limits_{-8x}^{7} f(56x) + 28 \mathop{\Delta}\limits_{4x}^{7} f(20x) + 196 \mathop{\Delta}\limits_{4x}^{7} f(16x) + 672 \mathop{\Delta}\limits_{4x}^{7} f(12x) \\ & + 1568 \mathop{\Delta}\limits_{4x}^{7} f(8x) + 2828 \mathop{\Delta}\limits_{4x}^{7} f(4x) - 4116 \mathop{\Delta}\limits_{-4x}^{7} f(28x) + 4480 \mathop{\Delta}\limits_{2x}^{7} f(10x) \\ & + 31360 \mathop{\Delta}\limits_{2x}^{7} f(8x) + 95872 \mathop{\Delta}\limits_{2x}^{7} f(6x) + 169344 \mathop{\Delta}\limits_{2x}^{7} f(4x) + 191744 \mathop{\Delta}\limits_{2x}^{7} f(2x) \\ & - 137984 \mathop{\Delta}\limits_{-2x}^{7} f(14x) + 43008 \mathop{\Delta}\limits_{x}^{7} f(5x) + 301056 \mathop{\Delta}\limits_{x}^{7} f(4x) + 888832 \mathop{\Delta}\limits_{x}^{7} f(3x) \\ & + 1404928 \mathop{\Delta}\limits_{x}^{7} f(2x) + 1214464 \mathop{\Delta}\limits_{x}^{7} f(x) - 473088 \mathop{\Delta}\limits_{-x}^{7} f(7x),\end{aligned} \tag{2}$$

$$\tilde{f}_1(x) - \frac{1}{2}\tilde{f}_1(2x) = \frac{1}{624960}\Gamma \tilde{f}(x), \quad \tilde{f}_1(x) - 2\tilde{f}_1\left(\frac{1}{2}x\right) = -\frac{1}{312480}\Gamma \tilde{f}\left(\frac{1}{2}x\right), \tag{3}$$

$$\tilde{f}_2(x) - \frac{1}{4}\tilde{f}_2(2x) = -\frac{1}{645120}\Gamma \tilde{f}(x), \quad \tilde{f}_2(x) - 4\tilde{f}\left(\frac{1}{2}x\right) = \frac{1}{161280}\Gamma \tilde{f}\left(\frac{1}{2}x\right), \tag{4}$$

$$\tilde{f}_3(x) - \frac{1}{8}\tilde{f}_3(2x) = \frac{1}{2064384}\Gamma \tilde{f}(x), \quad \tilde{f}_3(x) - 8\tilde{f}_3\left(\frac{1}{2}x\right) = -\frac{1}{258048}\Gamma \tilde{f}\left(\frac{1}{2}x\right), \tag{5}$$

$$\tilde{f}_4(x) - \frac{1}{16}\tilde{f}_4(2x) = -\frac{1}{16515072}\Gamma \tilde{f}(x), \quad \tilde{f}_4(x) - 16\tilde{f}_4\left(\frac{1}{2}x\right) = \frac{1}{1032192}\Gamma \tilde{f}\left(\frac{1}{2}x\right), \tag{6}$$

$$\tilde{f}_5(x) - \frac{1}{32}\tilde{f}_5(2x) = \frac{\Gamma \tilde{f}(x)}{330301440}, \quad \tilde{f}_5(x) - 32\tilde{f}_5\left(\frac{1}{2}x\right) = -\frac{1}{10321920}\Gamma \tilde{f}\left(\frac{1}{2}x\right), \tag{7}$$

$$\tilde{f}_6(x) - \frac{1}{64}\tilde{f}_6(2x) = -\frac{\Gamma \tilde{f}(x)}{20478689280}, \quad \tilde{f}_6(x) - 64\tilde{f}_6\left(\frac{1}{2}x\right) = \frac{1}{319979520}\Gamma \tilde{f}\left(\frac{1}{2}x\right), \tag{8}$$

$$f(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x) + f_6(x) \tag{9}$$

hold for all $x, y \in V$.

Lemma 2.2. *If $f : V \to Y$ satisfies the functional equation $\mathop{\Delta}\limits_{y}^{7} f(x) = 0$ for all $x, y \in V$, then the mappings $\tilde{f}_1, \dots, \tilde{f}_6 : V \to Y$ satisfy*

$$\tilde{f}_k(2x) = 2^k \tilde{f}_k(x) \tag{10}$$

for all $x \in V$ *and each* $k \in \{1, 2, 3, 4, 5, 6\}$*, where* $\tilde{f}, \tilde{f}_1, \dots, \tilde{f}_6$ *are defined at the beginning of this section.*

Proof. If $f : V \to Y$ satisfies the functional equation $\overset{7}{\underset{y}{\Delta}} f(x) = 0$ for all $x, y \in V$, then f satisfies the functional equation $\Gamma \tilde{f}(x) = 0$ due to (2). Therefore, the equality (10) follows from the equalities (3), (4), (5), (6), (7), and (8). □

According to [4, Corollary 6], we obtain the following lemma.

Lemma 2.3. *For any given mapping* $f : V \to Y$*, if there exist a mapping* $F : V \to Y$ *and a function* $\phi : V \setminus \{0\} \to [0, \infty)$ *that satisfy either*

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{1}{2^i} \phi(2^i x) < \infty \text{ or}$$

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{1}{2^{(\ell+1)i}} \phi(2^i x) + \sum_{i=0}^{\infty} 2^{\ell i} \phi\left(\frac{1}{2^i} x\right) < \infty \text{ or}$$

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} 2^{6i} \phi\left(\frac{1}{2^i} x\right) < \infty$$

for all $x \in V \setminus \{0\}$ *and for some* $\ell \in \{1, 2, 3, 4, 5\}$*, where* $F(x) = \sum_{k=1}^{6} F_k(x)$ *and every* F_k *has the property* (10)*, then the mapping* F *is uniquely determined.*

Lemma 2.4. *If a mapping* $f : V \to Y$ *satisfies the functional equation* $\overset{7}{\underset{y}{\Delta}} f(x) = 0$ *for all* $x, y \in V \setminus \{0\}$*, then it is a general sextic mapping.*

Proof. It is clear that $\overset{7}{\underset{0}{\Delta}} f(x) = 0$ for all $x \in V$ and

$$\overset{7}{\underset{y}{\Delta}} f(0) = -\overset{7}{\underset{-y}{\Delta}} f(7y) = 0$$

for all $y \in V \setminus \{0\}$. Thus, $\overset{7}{\underset{y}{\Delta}} f(x) = 0$ for all $x, y \in V$, as desired. □

Theorem 2.5. *Let* $\varphi : (V \setminus \{0\})^2 \to [0, \infty)$ *be a function that satisfies one of the following conditions*

$$\sum_{i=0}^{\infty} 2^{-i} \varphi(2^i x, 2^i y) < \infty, \tag{11}$$

$$\sum_{i=0}^{\infty} 4^{-i} \varphi(2^i x, 2^i y) < \infty \text{ and } \sum_{i=0}^{\infty} 2^i \varphi\left(\frac{1}{2^i} x, \frac{1}{2^i} y\right) < \infty, \tag{12}$$

$$\sum_{i=0}^{\infty} 8^{-i} \varphi(2^i x, 2^i y) < \infty \text{ and } \sum_{i=0}^{\infty} 4^i \varphi\left(\frac{1}{2^i} x, \frac{1}{2^i} y\right) < \infty, \tag{13}$$

$$\sum_{i=0}^{\infty} 16^{-i} \varphi(2^i x, 2^i y) < \infty \text{ and } \sum_{i=0}^{\infty} 8^i \varphi\left(\frac{1}{2^i} x, \frac{1}{2^i} y\right) < \infty, \tag{14}$$

$$\sum_{i=0}^{\infty} 32^{-i}\varphi(2^i x, 2^i y) < \infty \text{ and } \sum_{i=0}^{\infty} 16^i \varphi\left(\frac{1}{2^i}x, \frac{1}{2^i}y\right) < \infty, \tag{15}$$

$$\sum_{i=0}^{\infty} 64^{-i}\varphi(2^i x, 2^i y) < \infty \text{ and } \sum_{i=0}^{\infty} 32^i \varphi\left(\frac{1}{2^i}x, \frac{1}{2^i}y\right) < \infty, \tag{16}$$

$$\sum_{i=0}^{\infty} 64^i \varphi\left(\frac{1}{2^i}x, \frac{1}{2^i}y\right) < \infty \tag{17}$$

for all $x, y \in V \setminus \{0\}$. *Suppose* $f : V \to Y$ *is a mapping such that*

$$\left\| \overset{7}{\underset{y}{\Delta}} f(x) \right\| \le \varphi(x, y) \tag{18}$$

for all $x, y \in V \setminus \{0\}$. *Then there exists a unique mapping* $F : V \to Y$ *such that* $\overset{7}{\underset{y}{\Delta}} F(x) = 0$ *and*

$$\|\tilde{f}(x) - F(x)\| \le \frac{1}{624960} \sum_{i=0}^{\infty} \frac{1}{2^i} \Phi(2^i x), \tag{19}$$

$$\|\tilde{f}(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{2^i}{312480} \Phi\left(\frac{1}{2^{i+1}}x\right) + \sum_{i=0}^{\infty} \frac{1}{645120 \cdot 4^i} \Phi(2^i x), \tag{20}$$

$$\|\tilde{f}(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{4^i}{161280} \Phi\left(\frac{1}{2^{i+1}}x\right) + \sum_{i=0}^{\infty} \frac{1}{2064384 \cdot 8^i} \Phi(2^i x), \tag{21}$$

$$\|\tilde{f}(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{8^i}{258048} \Phi\left(\frac{1}{2^{i+1}}x\right) + \sum_{i=0}^{\infty} \frac{1}{16515072 \cdot 16^i} \Phi(2^i x), \tag{22}$$

$$\|\tilde{f}(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{16^i}{1032192} \Phi\left(\frac{1}{2^{i+1}}x\right) + \sum_{i=0}^{\infty} \frac{1}{330301440 \cdot 32^i} \Phi(2^i x), \tag{23}$$

$$\|\tilde{f}(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{32^i}{10321920} \Phi\left(\frac{1}{2^{i+1}}x\right) + \sum_{i=0}^{\infty} \frac{1}{20478689280 \cdot 64^i} \Phi(2^i x), \tag{24}$$

$$\|\tilde{f}(x) - F(x)\| \le \sum_{i=5}^{\infty} \frac{64^i}{319979520} \Phi\left(\frac{1}{2^{i+1}}x\right) \tag{25}$$

for all $x \in V \setminus \{0\}$ *if* φ *satisfies* (11), (12), (13), (14), (15), (16), *or* (17), *respectively, where* $\Phi : V \setminus \{0\} \to [0, \infty)$ *is the function defined by*

$$\begin{aligned}\Phi(x) :=& \varphi(8x, 8x) + 7\varphi(56x, -8x) + 28\varphi(20x, 4x) + 196\varphi(16x, 4x) + 672\varphi(12x, 4x) \\ &+ 1568\varphi(8x, 4x) + 2828\varphi(4x, 4x) + 4116\varphi(28x, -4x) + 4480\varphi(10x, 2x) \\ &+ 31360\varphi(8x, 2x) + 95872\varphi(6x, 2x) + 169344\varphi(4x, 2x) + 191744\varphi(2x, 2x) \\ &+ 137984\varphi(14x, -2x) + 43008\varphi(5x, x) + 301056\varphi(4x, x) + 888832\varphi(3x, x) \\ &+ 1404928\varphi(2x, x) + 1214464\varphi(x, x) + 473088\varphi(7x, -x).\end{aligned}$$

Proof. Note that, by (2) and (18), we have

$$\begin{aligned}\|\Gamma\tilde{f}(x)\| =&\Big\| \underset{8x}{\overset{7}{\Delta}} f(8x) - 7 \underset{-8x}{\overset{7}{\Delta}} f(56x) + 28 \underset{4x}{\overset{7}{\Delta}} f(20x) + 196 \underset{4x}{\overset{7}{\Delta}} f(16x) + 672 \underset{4x}{\overset{7}{\Delta}} f(12x)\\ &+ 1568 \underset{4x}{\overset{7}{\Delta}} f(8x) + 2828 \underset{4x}{\overset{7}{\Delta}} f(4x) - 4116 \underset{-4x}{\overset{7}{\Delta}} f(28x) + 4480 \underset{2x}{\overset{7}{\Delta}} f(10x)\\ &+ 31360 \underset{2x}{\overset{7}{\Delta}} f(8x) + 95872 \underset{2x}{\overset{7}{\Delta}} f(6x) + 169344 \underset{2x}{\overset{7}{\Delta}} f(4x) + 191744 \underset{2x}{\overset{7}{\Delta}} f(2x)\\ &- 137984 \underset{-2x}{\overset{7}{\Delta}} f(14x) + 43008 \underset{x}{\overset{7}{\Delta}} f(5x) + 301056 \underset{x}{\overset{7}{\Delta}} f(4x) + 888832 \underset{x}{\overset{7}{\Delta}} f(3x)\\ &+ 1404928 \underset{x}{\overset{7}{\Delta}} f(2x) + 1214464 \underset{x}{\overset{7}{\Delta}} f(x) - 473088 \underset{-x}{\overset{7}{\Delta}} f(7x)\Big\|\\ &\le \Phi(x) \qquad (26)\end{aligned}$$

for all $x \in V$. We prove the theorem in two steps.

Step 1. Let $k \in \{1,2,3,4,5,6\}$ and $\delta \in \{-1,1\}$, and let φ satisfy

$$\sum_{n=0}^{\infty} \frac{1}{2^{\delta kn}} \varphi(2^{\delta n}x, 2^{\delta n}y) < \infty \qquad (27)$$

for all $x, y \in V \setminus \{0\}$. Together with

$$\frac{1}{2^{\delta kn}} \tilde{f}_k(2^{\delta n}x) - \frac{1}{2^{\delta k(n+m)}} \tilde{f}_k(2^{\delta(n+m)}x) = \sum_{i=n}^{n+m-1} \left(\frac{1}{2^{\delta ki}} \tilde{f}_k(2^{\delta i}x) - \frac{1}{2^{\delta k(i+1)}} \tilde{f}_k(2^{\delta(i+1)}x) \right)$$

and using (3), (4), (5), (6), (7), (8), (26), we have the inequalities

$$\begin{aligned}\left\| \frac{1}{2^n} \tilde{f}_1(2^n x) - \frac{1}{2^{n+m}} \tilde{f}_1(2^{n+m}x) \right\| &\le \frac{1}{624960} \sum_{i=n}^{n+m-1} \left\| \frac{1}{2^i} \Gamma\tilde{f}(2^i x) \right\|\\ &\le \frac{1}{624960} \sum_{i=n}^{n+m-1} \frac{1}{2^i} \Phi(2^i x),\end{aligned}$$

$$\begin{aligned}\left\| 2^n \tilde{f}_1\left(\frac{1}{2^n}x\right) - 2^{n+m} \tilde{f}_1\left(\frac{1}{2^{n+m}}x\right) \right\| &\le \frac{1}{312480} \sum_{i=n}^{n+m-1} \left\| 2^i \Gamma\tilde{f}\left(\frac{1}{2^{i+1}}x\right) \right\|\\ &\le \frac{1}{312480} \sum_{i=n}^{n+m-1} 2^i \Phi\left(\frac{1}{2^{i+1}}x\right),\end{aligned}$$

$$\left\| \frac{1}{4^n} \tilde{f}_2(2^n x) - \frac{1}{4^{n+m}} \tilde{f}_2(2^{n+m}x) \right\| \le \frac{1}{645120} \sum_{i=n}^{n+m-1} \frac{1}{4^i} \Phi(2^i x),$$

$$\left\| 4^n \tilde{f}_2\left(\frac{1}{2^n}x\right) - 4^{n+m} \tilde{f}_2\left(\frac{1}{2^{n+m}}x\right) \right\| \le \frac{1}{161280} \sum_{i=n}^{n+m-1} 4^i \Phi\left(\frac{1}{2^{i+1}}x\right),$$

$$\left\| \frac{1}{8^n} \tilde{f}_3(2^n x) - \frac{1}{8^{n+m}} \tilde{f}_3(2^{n+m}x) \right\| \le \frac{1}{2064384} \sum_{i=n}^{n+m-1} \frac{1}{8^i} \Phi(2^i x),$$

$$\left\|8^n\tilde{f}_3\left(\frac{1}{2^n}x\right)-8^{n+m}\tilde{f}_3\left(\frac{1}{2^{n+m}}x\right)\right\|\leq\frac{1}{258048}\sum_{i=n}^{n+m-1}8^i\Phi\left(\frac{1}{2^{i+1}}x\right),$$

$$\left\|\frac{1}{16^n}\tilde{f}_4(2^nx)-\frac{1}{16^{n+m}}\tilde{f}_4(2^{n+m}x)\right\|\leq\frac{1}{16515072}\sum_{i=n}^{n+m-1}\frac{1}{16^i}\Phi(2^ix),$$

$$\left\|16^n\tilde{f}_4\left(\frac{1}{2^n}x\right)-16^{n+m}\tilde{f}_4\left(\frac{1}{2^{n+m}}x\right)\right\|\leq\frac{1}{1032192}\sum_{i=n}^{n+m-1}16^i\Phi\left(\frac{1}{2^{i+1}}x\right),$$

$$\left\|\frac{1}{32^n}\tilde{f}_5(2^nx)-\frac{1}{32^{n+m}}\tilde{f}_5(2^{n+m}x)\right\|\leq\frac{1}{330301440}\sum_{i=n}^{n+m-1}\frac{1}{32^i}\Phi(2^ix),$$

$$\left\|32^n\tilde{f}_5\left(\frac{1}{2^n}x\right)-32^{n+m}\tilde{f}_5\left(\frac{1}{2^{n+m}}x\right)\right\|\leq\frac{1}{10321920}\sum_{i=n}^{n+m-1}32^i\Phi\left(\frac{1}{2^{i+1}}x\right),$$

$$\left\|\frac{1}{64^n}\tilde{f}_6(2^nx)-\frac{1}{64^{n+m}}\tilde{f}_6(2^{n+m}x)\right\|\leq\frac{1}{20478689280}\sum_{i=n}^{n+m-1}\frac{1}{64^i}\Phi(2^ix),$$

and

$$\left\|64^n\tilde{f}_6\left(\frac{1}{2^n}x\right)-64^{n+m}\tilde{f}_6\left(\frac{1}{2^{n+m}}x\right)\right\|\leq\frac{1}{319979520}\sum_{i=n}^{n+m-1}64^i\Phi\left(\frac{1}{2^{i+1}}x\right)$$

for all $x\in V\setminus\{0\}$ and $n,m\in\mathbb{N}\cup\{0\}$. It leads us to prove that $\{\frac{1}{2^{\delta kn}}\tilde{f}_k(2^{\delta n}x)\}$ is a Cauchy sequence for all $x\in V\setminus\{0\}$ if φ satisfies (27). Moreover, since Y is complete and $\tilde{f}_k(0)=0$, the sequence converges for all $x\in V$. It follows that we can define a mapping $F_{\delta k}:V\to Y$ by

$$F_{\delta k}(x):=\lim_{n\to\infty}\frac{1}{2^{\delta kn}}\tilde{f}_k(2^{\delta n}x) \tag{28}$$

for all $x\in V$ if φ satisfies (27). Now we observe that the equality

$$\begin{aligned}\underset{y}{\overset{7}{\Delta}}F_{\delta k}(x)&=\sum_{i=0}^{7}(-1)^{7-i}\binom{7}{i}F_{\delta k}(x+iy)\\&=\lim_{n\to\infty}\sum_{i=0}^{7}(-1)^{7-i}\binom{7}{i}\frac{1}{2^{\delta kn}}\tilde{f}_k(2^{\delta n}(x+iy))\end{aligned}$$

holds for all $x,y\in V\setminus\{0\}$. Together with the definition of $\tilde{f}_1$, if φ satisfies (27) for $k=1$, then we have

$$\begin{aligned}\left\|\underset{y}{\overset{7}{\Delta}}F_{\delta 1}(x)\right\|=\lim_{n\to\infty}\Bigg\|&\frac{-1}{312480}\sum_{i=0}^{7}(-1)^{7-i}\binom{7}{i}\frac{\tilde{f}(2^{\delta n+5}(x+iy))}{2^{\delta n}}\\&+\frac{124}{312480}\sum_{i=0}^{7}(-1)^{7-i}\binom{7}{i}\frac{\tilde{f}(2^{\delta n+4}(x+iy))}{2^{\delta n}}-\frac{4960}{312480}\sum_{i=0}^{7}(-1)^{7-i}\binom{7}{i}\frac{\tilde{f}(2^{\delta n+3}(x+iy))}{2^{\delta n}}\end{aligned}$$

$$+\frac{79360}{312480}\sum_{i=0}^{7}(-1)^{7-i}\binom{7}{i}\frac{\tilde{f}(2^{\delta n+2}(x+iy))}{2^{\delta n}}-\frac{507904}{312480}\sum_{i=0}^{7}(-1)^{7-i}\binom{7}{i}\frac{\tilde{f}(2^{\delta n+1}(x+iy))}{2^{\delta n}}$$

$$+\frac{1048576}{312480}\sum_{i=0}^{7}(-1)^{7-i}\binom{7}{i}\frac{\tilde{f}(2^{\delta n}(x+iy))}{2^{\delta n}}\Bigg\|$$

$$=\lim_{n\to\infty}\Bigg\|\frac{1048576\overset{7}{\underset{2^{\delta n}y}{\Delta}}f(2^{\delta n}x)}{312480\cdot 2^{\delta n}}-\frac{507904\overset{7}{\underset{2^{\delta n+1}y}{\Delta}}f(2^{\delta n+1}x)}{312480\cdot 2^{\delta n}}+\frac{79360\overset{7}{\underset{2^{\delta n+2}y}{\Delta}}f(2^{\delta n+2}x)}{312480\cdot 2^{\delta n}}$$

$$-\frac{4960\overset{7}{\underset{2^{\delta n+3}y}{\Delta}}f(2^{\delta n+3}x)}{312480\cdot 2^{\delta n}}+\frac{124\overset{7}{\underset{2^{\delta n+4}y}{\Delta}}f(2^{\delta n+4}x)}{312480\cdot 2^{\delta n}}-\frac{\overset{7}{\underset{2^{\delta n+5}y}{\Delta}}f(2^{\delta n+5}x)}{312480\cdot 2^{\delta n}}\Bigg\|$$

$$\leq\lim_{n\to\infty}\Bigg(\frac{1048576\varphi(2^{\delta n}x,2^{\delta n}y)}{312480\cdot 2^{\delta n}}+\frac{507904\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{312480\cdot 2^{\delta n}}+\frac{79360\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{312480\cdot 2^{\delta n}}$$

$$+\frac{4960\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{312480\cdot 2^{\delta n}}+\frac{124\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{312480\cdot 2^{\delta n}}+\frac{\varphi(2^{\delta n+5}x,2^{\delta n+5}y)}{312480\cdot 2^{\delta n}}\Bigg)$$

$$=0$$

for all $x, y \in V \setminus \{0\}$. In a similar way, by the definition of $\tilde{f}_k$, if φ satisfies (27) for $k \in \{2, 3, 4, 5, 6\}$, respectively, then we get

$$\Big\|\overset{7}{\underset{y}{\Delta}}F_{\delta 2}(x)\Big\|$$

$$\leq\lim_{n\to\infty}\Bigg(\frac{524288\varphi(2^{\delta n}x,2^{\delta n}y)}{161280\cdot 4^{\delta n}}+\frac{385024\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{161280\cdot 4^{\delta n}}+\frac{70400\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{161280\cdot 4^{\delta n}}$$

$$+\frac{4720\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{161280\cdot 4^{\delta n}}+\frac{122\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{161280\cdot 4^{\delta n}}+\frac{\varphi(2^{\delta n+5}x,2^{\delta n+5}y)}{161280\cdot 4^{\delta n}}\Bigg)$$

$$=0,$$

$$\Big\|\overset{7}{\underset{y}{\Delta}}F_{\delta 3}(x)\Big\|$$

$$\leq\lim_{n\to\infty}\Bigg(\frac{262144\varphi(2^{\delta n}x,2^{\delta n}y)}{258048\cdot 8^{\delta n}}+\frac{225280\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{258048\cdot 8^{\delta n}}+\frac{55168\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{258048\cdot 8^{\delta n}}$$

$$+\frac{4264\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{258048\cdot 8^{\delta n}}+\frac{118\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{258048\cdot 8^{\delta n}}+\frac{\varphi(2^{\delta n+5}x,2^{\delta n+5}y)}{258048\cdot 8^{\delta n}}\Bigg)$$

$$=0,$$

$$\Big\|\overset{7}{\underset{y}{\Delta}}F_{\delta 4}(x)\Big\|$$

$$\leq\lim_{n\to\infty}\Bigg(\frac{131072\varphi(2^{\delta n}x,2^{\delta n}y)}{1032192\cdot 16^{\delta n}}+\frac{120832\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{1032192\cdot 16^{\delta n}}+\frac{34112\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{1032192\cdot 16^{\delta n}}$$

$$+\frac{3448\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{1032192\cdot 16^{\delta n}}+\frac{110\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{1032192\cdot 16^{\delta n}}+\frac{\varphi(2^{\delta n+5}x,2^{\delta n+5}y)}{1032192\cdot 16^{\delta n}}\Bigg)$$
$$=0,$$

$$\left\|\underset{y}{\overset{7}{\Delta}} F_{\delta 5}(x)\right\|$$
$$\leq \lim_{n\to\infty}\Bigg(\frac{65536\varphi(2^{\delta n}x,2^{\delta n}y)}{10321920\cdot 32^{\delta n}}+\frac{62464\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{10321920\cdot 32^{\delta n}}+\frac{18880\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{10321920\cdot 32^{\delta n}}$$
$$+\frac{2200\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{10321920\cdot 32^{\delta n}}+\frac{94\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{10321920\cdot 32^{\delta n}}+\frac{\varphi(2^{\delta n+5}x,2^{\delta n+5}y)}{10321920\cdot 32^{\delta n}}\Bigg)$$
$$=0,$$

$$\left\|\underset{y}{\overset{7}{\Delta}} F_{\delta 6}(x)\right\|$$
$$\leq \lim_{n\to\infty}\Bigg(\frac{32768\varphi(2^{\delta n}x,2^{\delta n}y)}{319979520\cdot 64^{\delta n}}+\frac{31744\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{319979520\cdot 64^{\delta n}}+\frac{9920\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{319979520\cdot 64^{\delta n}}$$
$$+\frac{1240\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{319979520\cdot 64^{\delta n}}+\frac{62\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{319979520\cdot 64^{\delta n}}+\frac{\varphi(2^{\delta n+5}x,2^{\delta n+5}y)}{319979520\cdot 64^{\delta n}}\Bigg)$$
$$=0$$

for all $x,y\in V\setminus\{0\}$. And then, since $\underset{y}{\overset{7}{\Delta}} F_{\delta k}(x)=0$ for all $x,y\in V\setminus\{0\}$, the mapping $F_{\delta k}$ is a general sextic mapping for all $k\in\{1,2,3,4,5,6\}$ and $\delta\in\{+1,-1\}$ by Lemma 2.4.

Step 2. Now we define the desired general sextic mapping F for all cases.

(1) Let φ satisfy the condition (11), then F_1, F_2, F_3, F_4, F_5, and F_6 are defined by (28). We put a general sextic mapping $F:V\to Y$ by

$$F(x):=F_1(x)+F_2(x)+F_3(x)+F_4(x)+F_5(x)+F_6(x)$$

for all $x\in V$. Observe that by (3), (4), (5), (6), (7), and (8), we have

$$\left\|\tilde f(x)-\sum_{k=1}^{6}\frac{1}{2^{kn}}\tilde f_k(2^n x)\right\|\leq\sum_{i=0}^{n-1}\left\|\sum_{k=1}^{6}\left(\frac{1}{2^{ki}}\tilde f_k(2^i x)-\frac{1}{2^{k(i+1)}}\tilde f_k(2^{i+1}x)\right)\right\|$$
$$=\sum_{i=0}^{n-1}\Bigg(\frac{1}{624960\cdot 2^i}-\frac{1}{645120\cdot 4^i}+\frac{1}{2064384\cdot 8^i}-\frac{1}{16515072\cdot 16^i}$$
$$+\frac{1}{330301440\cdot 32^i}-\frac{1}{20478689280\cdot 64^i}\Bigg)\|\Gamma\tilde f(2^i x)\|$$
$$\leq\sum_{i=0}^{n-1}\left\|\frac{1}{624960\cdot 2^i}\Gamma\tilde f(2^i x)\right\|$$

$$\leq \frac{1}{624960} \sum_{i=0}^{n-1} \frac{1}{2^i} \Phi(2^i x)$$

for all $x \in V \setminus \{0\}$, where, if $n \to \infty$, we obtain the inequality (19).

(2) Let φ satisfy the condition (12), then F_{-1}, F_2, F_3, F_4, F_5, and F_6 are defined by (28). Putting a general sextic mapping $F : V \to Y$ by

$$F(x) := F_{-1}(x) + F_2(x) + F_3(x) + F_4(x) + F_5(x) + F_6(x)$$

for all $x \in V$. Then we have

$$\begin{aligned}
&\left\| \tilde{f}(x) - 2^n \tilde{f}_1\left(\frac{1}{2^n}x\right) - \sum_{k=2}^{6} \frac{1}{2^{kn}} \tilde{f}_k(2^n x) \right\| \\
&\quad \leq \sum_{i=0}^{n-1} \left\| 2^i \tilde{f}_1\left(\frac{1}{2^i}x\right) - 2^{i+1} \tilde{f}_1\left(\frac{1}{2^{i+1}}x\right) \right\| \\
&\qquad + \sum_{i=0}^{n-1} \left\| \sum_{k=2}^{6} \left(\frac{1}{2^{ki}} \tilde{f}_k(2^i x) - \frac{1}{2^{k(i+1)}} \tilde{f}_k(2^{i+1} x) \right) \right\| \\
&\quad \leq \sum_{i=0}^{n-1} \frac{2^i}{312480} \left\| \Gamma \tilde{f}\left(\frac{1}{2^{i+1}}x\right) \right\| + \sum_{i=0}^{n-1} \left(\frac{1}{645120 \cdot 4^i} - \frac{1}{2064384 \cdot 8^i} \right. \\
&\qquad \left. + \frac{1}{16515072 \cdot 16^i} - \frac{1}{330301440 \cdot 32^i} + \frac{1}{20478689280 \cdot 64^i} \right) \| \Gamma \tilde{f}(2^i x) \| \\
&\quad \leq \frac{2^i}{312480} \sum_{i=0}^{n-1} 2^i \Phi\left(\frac{1}{2^{i+1}}x\right) + \frac{1}{645120} \sum_{i=0}^{n-1} \frac{1}{4^i} \Phi(2^i x)
\end{aligned}$$

for all $x \in V \setminus \{0\}$ by (3), (4), (5), (6), (7), and (8). Now, if we let $n \to \infty$, we obtain the inequality (20).

(3) Let φ satisfy the condition (13), then F_{-1}, F_{-2}, F_3, F_4, F_5, and F_6 are defined by (28). Putting a general sextic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_3(x) + F_4(x) + F_5(x) + F_6(x)$$

for all $x \in V$. We have the inequality

$$\begin{aligned}
&\left\| \tilde{f}(x) - \sum_{k=1}^{2} 2^{kn} \tilde{f}_k\left(\frac{1}{2^n}x\right) - \sum_{k=3}^{6} \frac{1}{2^{kn}} \tilde{f}_k(2^n x) \right\| \\
&\quad \leq \sum_{i=0}^{n-1} \left\| \sum_{k=1}^{2} \left(2^{ki} \tilde{f}_k\left(\frac{1}{2^i}x\right) - 2^{k(i+1)} \tilde{f}_k\left(\frac{1}{2^{i+1}}x\right) \right) \right\| \\
&\qquad + \sum_{i=0}^{n-1} \left\| \sum_{k=3}^{6} \left(\frac{1}{2^{ki}} \tilde{f}_k(2^i x) - \frac{1}{2^{k(i+1)}} \tilde{f}_k(2^{i+1} x) \right) \right\| \\
&\quad \leq \sum_{i=0}^{n-1} \left(\frac{4^i}{161280} - \frac{2^i}{312480} \right) \left\| \Gamma \tilde{f}\left(\frac{1}{2^{i+1}}x\right) \right\| + \sum_{i=0}^{n-1} \left(\frac{1}{2064384 \cdot 8^i} \right.
\end{aligned}$$

$$- \frac{1}{16515072 \cdot 16^i} + \frac{1}{330301440 \cdot 32^i} - \frac{1}{20478689280 \cdot 64^i}\Big) \|\Gamma \tilde{f}(2^i x)\|$$

$$\le \frac{1}{161280} \sum_{i=0}^{n-1} 4^i \Phi\left(\frac{1}{2^{i+1}} x\right) + \frac{1}{2064384} \sum_{i=0}^{n-1} \frac{1}{8^i} \Phi(2^i x)$$

for all $x \in V \setminus \{0\}$ by (3), (4), (5), (6), (7), and (8). Let $n \to \infty$, then we obtain inequality (21).

(4) Let φ satisfy the condition (14), then F_{-1}, F_{-2}, F_{-3}, F_4, F_5, and F_6 are defined by (28). We define a general sextic mapping by

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_{-3}(x) + F_4(x) + F_5(x) + F_6(x)$$

for all $x \in V$. We have the inequality

$$\begin{aligned}
&\left\| \tilde{f}(x) - \sum_{k=1}^{3} 2^{kn} \tilde{f}_k\left(\frac{1}{2^n} x\right) - \sum_{k=4}^{6} \frac{1}{2^{kn}} \tilde{f}_k(2^n x) \right\| \\
&\quad \le \sum_{i=0}^{n-1} \left\| \sum_{k=1}^{3} \left(2^{ki} \tilde{f}_k\left(\frac{1}{2^i} x\right) - 2^{k(i+1)} \tilde{f}_k\left(\frac{1}{2^{i+1}} x\right)\right) \right\| \\
&\qquad + \sum_{i=0}^{n-1} \left\| \sum_{k=4}^{6} \left(\frac{1}{2^{ki}} \tilde{f}_k(2^i x) - \frac{1}{2^{k(i+1)}} \tilde{f}_k(2^{i+1} x)\right) \right\| \\
&\quad \le \sum_{i=0}^{n-1} \left(\frac{8^i}{258048} - \frac{4^i}{161280} + \frac{2^i}{312480}\right) \left\| \Gamma \tilde{f}\left(\frac{1}{2^{i+1}} x\right) \right\| \\
&\qquad + \sum_{i=0}^{n-1} \left(\frac{1}{16515072 \cdot 16^i} - \frac{1}{330301440 \cdot 32^i} + \frac{1}{20478689280 \cdot 64^i}\right) \|\Gamma \tilde{f}(2^i x)\| \\
&\quad \le \frac{1}{258048} \sum_{i=0}^{n-1} 8^i \Phi\left(\frac{1}{2^{i+1}} x\right) + \frac{1}{16515072} \sum_{i=0}^{n-1} \frac{1}{16^i} \Phi(2^i x)
\end{aligned}$$

for all $x \in V \setminus \{0\}$ by (3), (4), (5), (6), (7), and (8), which gives the inequality (22) by letting $n \to \infty$.

(5) Let φ satisfy the condition (15), then F_{-1}, F_{-2}, F_{-3}, F_{-4}, F_5, and F_6 are defined by (28). We define a general sextic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_{-3}(x) + F_{-4}(x) + F_5(x) + F_6(x)$$

for all $x \in V$. We have the inequality

$$\begin{aligned}
&\left\| \tilde{f}(x) - \sum_{k=1}^{4} 2^{kn} \tilde{f}_k\left(\frac{1}{2^n} x\right) - \sum_{k=5}^{6} \frac{1}{2^{kn}} \tilde{f}_k(2^n x) \right\| \\
&\quad \le \sum_{i=0}^{n-1} \left\| \sum_{k=1}^{4} \left(2^{ki} \tilde{f}_k\left(\frac{1}{2^i} x\right) - 2^{k(i+1)} \tilde{f}_k\left(\frac{1}{2^{i+1}} x\right)\right) \right\| \\
&\qquad + \sum_{i=0}^{n-1} \left\| \sum_{k=5}^{6} \left(\frac{1}{2^{ki}} \tilde{f}_k(2^i x) - \frac{1}{2^{k(i+1)}} \tilde{f}_k(2^{i+1} x)\right) \right\|
\end{aligned}$$

$$\leq \sum_{i=0}^{n-1}\left(\frac{16^i}{1032192}-\frac{8^i}{258048}+\frac{4^i}{161280}-\frac{2^i}{312480}\right)\left\|\Gamma\tilde{f}\left(\frac{1}{2^{i+1}}x\right)\right\|$$
$$+\sum_{i=0}^{n-1}\left(\frac{1}{330301440\cdot 32^i}-\frac{1}{20478689280\cdot 64^i}\right)\|\Gamma\tilde{f}(2^ix)\|$$
$$\leq \frac{1}{1032192}\sum_{i=0}^{n-1}16^i\Phi\left(\frac{1}{2^{i+1}}x\right)+\frac{1}{330301440}\sum_{i=0}^{n-1}\frac{1}{32^i}\Phi(2^ix)$$

for all $x \in V \setminus \{0\}$ by (3), (4), (5), (6), (7), and (8). If $n \to \infty$, the inequality (23) follows.

(6) Let φ satisfy the condition (16), then F_{-1}, F_{-2}, F_{-3}, F_{-4}, F_{-5}, and F_6 are defined by (28). Put a general sextic mapping

$$F(x) := F_{-1}(x)+F_{-2}(x)+F_{-3}(x)+F_{-4}(x)+F_{-5}(x)+F_6(x)$$

for all $x \in V$. We have the inequality

$$\left\|\tilde{f}(x)-\sum_{k=1}^{5}2^{kn}\tilde{f}_k\left(\frac{1}{2^n}x\right)-\frac{1}{2^{6n}}\tilde{f}_6(2^nx)\right\|$$
$$\leq \sum_{i=0}^{n-1}\left\|\sum_{k=1}^{5}\left(2^{ki}\tilde{f}_k\left(\frac{1}{2^i}x\right)-2^{k(i+1)}\tilde{f}_k\left(\frac{1}{2^{i+1}}x\right)\right)\right\|$$
$$+\sum_{i=0}^{n-1}\left\|\frac{1}{2^{6i}}\tilde{f}_6(2^ix)-\frac{1}{2^{6(i+1)}}\tilde{f}_6(2^{i+1}x)\right\|$$
$$\leq \sum_{i=0}^{n-1}\left(\frac{32^i}{10321920}-\frac{16^i}{1032192}+\frac{8^i}{258048}-\frac{4^i}{161280}+\frac{2^i}{312480}\right)\left\|\Gamma\tilde{f}\left(\frac{1}{2^{i+1}}x\right)\right\|$$
$$+\frac{1}{20478689280}\sum_{i=0}^{n-1}\frac{1}{64^i}\|\Gamma\tilde{f}(2^ix)\|$$
$$\leq \frac{1}{10321920}\sum_{i=0}^{n-1}32^i\Phi\left(\frac{1}{2^{i+1}}x\right)+\frac{1}{20478689280}\sum_{i=0}^{n-1}\frac{1}{64^i}\Phi(2^ix)$$

for all $x \in V \setminus \{0\}$ by (3), (4), (5), (6), (7), and (8), from which the inequality (24) follows if we set $n \to \infty$.

(7) Let φ satisfy the condition (17), then F_{-1}, F_{-2}, F_{-3}, F_{-4}, F_{-5}, and F_{-6} are defined by (28). Put a general sextic mapping

$$F(x) := F_{-1}(x)+F_{-2}(x)+F_{-3}(x)+F_{-4}(x)+F_{-5}(x)+F_{-6}(x)$$

for all $x \in V$. We have the inequality

$$\left\|\tilde{f}(x)-\sum_{k=1}^{6}2^{kn}\tilde{f}_k\left(\frac{1}{2^n}x\right)\right\| \leq \sum_{i=0}^{n-1}\left\|\sum_{k=1}^{6}\left(2^{ki}\tilde{f}_k\left(\frac{1}{2^i}x\right)-2^{k(i+1)}\tilde{f}_k\left(\frac{1}{2^{i+1}}x\right)\right)\right\|$$
$$\leq \sum_{i=0}^{n-1}\left\|\left(\frac{2^i}{312480}-\frac{4^i}{161280}+\frac{8^i}{258048}-\frac{16^i}{1032192}+\frac{32^i}{10321920}-\frac{64^i}{319979520}\right)\Gamma\tilde{f}\left(\frac{1}{2^{i+1}}x\right)\right\|$$

$$\leq \sum_{i=5}^{n-1}\left(\frac{64^i}{319979520}-\frac{32^i}{10321920}+\frac{16^i}{1032192}-\frac{8^i}{258048}+\frac{4^i}{161280}-\frac{2^i}{312480}\right)\left\|\Gamma\tilde{f}\left(\frac{1}{2^{i+1}}x\right)\right\|$$
$$\leq \frac{1}{319979520}\sum_{i=5}^{n-1}64^i\Phi\left(\frac{1}{2^{i+1}}x\right)$$

for all $x \in V \setminus \{0\}$ by (3), (4), (5), (6), (7), and (8), since $\frac{2^i}{312480} - \frac{4^i}{161280} + \frac{8^i}{258048} - \frac{16^i}{1032192} + \frac{32^i}{10321920} - \frac{64^i}{319979520} = 0$ when $i \in \{0, 1, 2, 3, 4\}$. If $n \to \infty$, inequality (25) is derived.

Moreover, by the definition, we easily get

$$F_{\delta k}(2x) = 2^k F_{\delta k}(x)$$

and $\overset{7}{\underset{y}{\Delta}} F_{\delta k}(x) = 0$ for all $x, y \in V$. According to Lemma 2.4, F is the unique general sextic mapping. □

The stability results for the functional equation (1) proved by Chang et al. [1] and Roh et al. [10] only deal with the conditions (11) and (17) of Theorem 2.4.

Compare the following concise theorem obtained from Theorem 2.4 with Theorem 1.1 obtained by Y.-H. Lee [6].

Theorem 2.6. *Let θ be a positive real constant and p a real number such that $p \notin \{1, 2, 3, 4, 5, 6\}$. If $f : X \to Y$ satisfies the inequality*

$$\left\|\overset{7}{\underset{y}{\Delta}} f(x)\right\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X \setminus \{0\}$, then there exists a unique mapping F such that $\overset{7}{\underset{y}{\Delta}} F(x) = 0$ and

$$\|\tilde{f}(x) - F(x)\| \leq \begin{cases} \dfrac{M\theta\|x\|^p}{312480(2-2^p)} & (\text{for } p < 1), \\ \dfrac{M\theta\|x\|^p}{312480(2^p-2)} + \dfrac{M\theta\|x\|^p}{161280(4-2^p)} & (\text{for } 1 < p < 2), \\ \dfrac{M\theta\|x\|^p}{161280(2^p-4)} + \dfrac{M\theta\|x\|^p}{258048(8-2^p)} & (\text{for } 2 < p < 3), \\ \dfrac{M\theta\|x\|^p}{258048(2^p-8)} + \dfrac{M\theta\|x\|^p}{1032192(16-2^p)} & (\text{for } 3 < p < 4), \\ \dfrac{M\theta\|x\|^p}{1032192(2^p-16)} + \dfrac{M\theta\|x\|^p}{10321920(32-2^p)} & (\text{for } 4 < p < 5), \\ \dfrac{M\theta\|x\|^p}{10321920(2^p-32)} + \dfrac{M\theta\|x\|^p}{319979520(64-2^p)} & (\text{for } 5 < p < 6), \\ \dfrac{32768M\theta\|x\|^p}{9765 \cdot 32^p(2^p-64)} & (\text{for } 6 < p) \end{cases}$$

for all $x \in X \setminus \{0\}$, *where*

$$\begin{aligned} M :=& 7 \cdot 56^p + 4116 \cdot 28^p + 28 \cdot 20^p + 196 \cdot 16^p + 137984 \cdot 14^p \\ &+ 672 \cdot 12^p + 4480 \cdot 10^p + 32937 \cdot 8^p + 473088 \cdot 7^p + 95872 \cdot 6^p \\ &+ 43008 \cdot 5^p + 482636 \cdot 4^p + 888832 \cdot 3^p + 2227456 \cdot 2^p + 5539840. \end{aligned}$$

References

[1] I.-S. Chang, Y.-H. Lee, and J. Roh, *On the stability of the general sextic functional equation*, J. Chungcheong Math. Soc. **34** (2021), 295–306.
https://doi.org/10.14403/jcms.2021.34.3.295

[2] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
https://doi.org/10.1006/jmaa.1994.1211

[3] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.
https://doi.org/10.1073/pnas.27.4.222

[4] S.-M. Jung, Y.-H. Lee, and J. Roh, *A uniqueness theorem for stability problems of functional equations*, Symmetry **16** (10) (2024), 1298.
https://doi.org/10.3390/sym16101298

[5] K.-W. Jun and H.-M. Kim, *On the Hyers-Ulam-Rassias stability of a general cubic functional equation*, Math. Inequal. Appl. **6** (2003), 289–302.

[6] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of a general quintic functional equation and a general sextic functional equation*, Mathematics **7** (6) (2019), 510.
https://doi.org/10.3390/math7060510

[7] Y.-H. Lee and S.-M. Jung, *A fixed point approach to the stability of a general quartic functional equation*, J. Math. Comput. Sci. **20** (2020), 207–215.

[8] Y.-H. Lee and S.-M. Jung, *A fixed point approach to the stability of a general quintic functional equation*, Nonlinear Funct. Anal. Appl. **28** (2023), 671–684.

[9] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
https://doi.org/10.1090/S0002-9939-1978-0507327-1

[10] J. Roh, Y.-H. Lee, and S.-M. Jung, *The stability of a general sextic functional equation by fixed point theory*, J. Funct. Spaces **2020** (2020), Art. ID 6497408.
https://doi.org/10.1155/2020/6497408

[11] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York (1960).

Sun-Sook Jin
Department of Mathematics Education, Gongju National University of Education, Gongju 32553, Korea
E-mail: ssjin@gjue.ac.kr

Yang-Hi Lee
Department of Mathematics Education, Gongju National University of Education, Gongju 32553, Korea
E-mail: yanghi22@naver.com