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FUZZY HILBERT $C^\star$-MODULES

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Abstract. In the present article, we introduce and study the notion of fuzzy inner product A-module, where A is an arbitrary unital C^*-algebra. Moreover, we construct some examples of particular classes of C^*-algebras. As an application, we obtain some $M_n(A)$-valued fuzzy inner product, where $M_n(A)$ denotes the $n \times n$ matrix C^*-algebra of a unital C^*-algebra A. Moreover, we obtain some relations with the notion of C^*-valued fuzzy normed spaces.

1. Introduction and preliminaries

The field of fuzzy theory plays an important role in mathematics and applied sciences, with fuzzy sets of Zadeh [28] having wide applications in various fields of mathematics. See [14,27] for more information on fuzzy theory and applications. Our paper builds upon the existing literature and presents new results that could have significant implications for various fields. George and Viramani [6] presented new results by making changes in the definitions, and Bag and Samantha [3] presented a new concept of a fuzzy norm. New fuzzy norm concepts were later developed by Saadati and Vaezpour [25] and Ameri [1]. Recently, Azzam *et al.* [2] studied spherical fuzzy topology and soft topology, Malik *et al.* [17] investigated T-rough bipolar fuzzy sets, Khan *et al.* [11] introduced double sequences in neutrosophic fuzzy G-metric spaces and Qumami *et al.* [23] studied impulsive fuzzy integro-differential equations.

An operation $\star : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if $([0,1],\star)$ is a commutative monoid with unit 1 such that $\varsigma_1 \star \varsigma_3 \leqslant \varsigma_2 \star \varsigma_4$ whenever $\varsigma_1 \leqslant \varsigma_3$ and $\varsigma_2 \leqslant \varsigma_4 \quad (\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \in [0,1])$. Examples of continuous t-norms include $\varsigma_1 \star \varsigma_2 = \min(\varsigma_1, \varsigma_2)$ and $\varsigma_1 \star \varsigma_2 = \varsigma_1 \cdot \varsigma_2$.

Definition 1.1. [4] Let Ξ be a linear space and $\star$ be a continuous t-norm. A fuzzy set μ is considered a fuzzy norm on $\Xi \times (0, +\infty)$ such that the following conditions hold: For all $\xi, \zeta \in \Xi$ and $\tau, \varsigma > 0$,

1) $\mu(\zeta, \varsigma) > 0$;
2) $\mu(\zeta, \varsigma) = 1,\ iff\ \zeta = 0$;

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3) $\mu(a\zeta, \varsigma) = \mu\left(\zeta, \frac{\varsigma}{|a|}\right), \ \forall a \neq 0;$

4) $\mu(\xi + \zeta, \varsigma + \varsigma) \geq \mu(\xi, \varsigma) \star \mu(\zeta, \varsigma);$

5) $\mu(\zeta, \cdot)$ *is continuous for each* $\zeta \in \Xi$;

6) $\lim_{\varsigma \to +\infty} \mu(\zeta, \varsigma) = 1.$

The 3-tuple $(\Xi, \mu, \star)$ is called a fuzzy normed space (FNS).

Definition 1.2. [22] Let Ξ be a complex linear space and δ represent a fuzzy set from $\Xi \times \Xi \times \mathbb{C}$ to $[0, 1]$. Then the pair (Ξ, δ) is called a fuzzy inner product (FIP) space if

1) $\delta(\xi, \xi, \iota) = 0, \quad \forall \xi \in \Xi, \quad \forall \iota \in \mathbb{C} \setminus (0, +\infty);$

2) $\delta(\xi, \xi, \iota) = 1, \quad \forall \iota \in (0, +\infty) \quad iff \quad \xi = 0;$

3) $\delta(\alpha\xi, \zeta, \iota) = \delta(\xi, \zeta, \frac{\iota}{|\alpha|}), \quad \forall \xi, \zeta \in \Xi, \quad \forall \iota \in \mathbb{C}, \ \forall \alpha \in \mathbb{C} \setminus \{0\};$

4) $\delta(\xi, \zeta, \iota) = \delta(\zeta, \xi, \bar{\iota}), \quad \forall \xi, \zeta \in \Xi, \quad \forall \iota \in \mathbb{C};$

5) $\delta(\xi + \zeta, \varsigma, |\iota| + |\kappa|) \geq \min\{\delta(\xi, \varsigma, |\iota|), \delta(\zeta, \varsigma, |\kappa|)\}, \ \forall \xi, \zeta, \varsigma \in \Xi, \ \forall \iota, \kappa \in \mathbb{C};$

6) $\delta(\xi, \xi, \cdot) : \mathbb{R}_+ \to [0, 1], \ \xi \in \Xi,$ *is left continuous and* $\lim_{\iota \to \infty} \delta(\xi, \xi, \iota) = 1;$

7) $\delta(\xi, \zeta, |\iota\kappa|) \geq \min\{\delta(\xi, \xi, |\iota|^2), \delta(\zeta, \zeta, |\kappa|^2)\}, \quad \forall \xi, \zeta \in \Xi, \quad \forall \iota, \kappa \in \mathbb{C}.$

Our paper contributes to the field of FIP spaces and their applications. We introduce the concept of FIP A-module, which provides a more general framework for studying fuzzy inner product spaces. Our results can inspire further research in this area and lead to new insights and applications in mathematics and physics [5,13,18,19].

Hilbert spaces are important in functional analysis and have many applications in mathematics and physics, such as differential equations, quantum mechanics, and Fourier analysis. However, there are few papers dedicated to studying fuzzy norms and inner product spaces. Therefore, every breakthrough in this area is significant, and discovering a correct definition of FIP space could have countless applications in various fields.

In our paper, we highlight some notable works in this area, including Majumdar and Samanta's [16] definition of inner product space and our introduction of new properties of the FIP function. Additionally, Goudarzi, Vaezpour and Saadati [8] introduced the concept of intuitionistic FIP space and established several theorems, while Goudarzi and Vaezpour [7] altered the definition of FIP space and proved interesting results related to fuzzy Hilbert (FH) space and a fuzzy version of the Riesz representation theorem [7, 9, 19, 20, 24, 26].

Our contribution to this area is the introduction of the notion of FIP A-modules, which is a fuzzy version of Hilbert C^*-modules, where A is a unital C^*-algebra. This concept generalizes other known notions such as FIP and FH spaces [22]. We also construct some examples on special classes of C^*-algebras and obtain some $M_n(A)$-valued FIP, where $M_n(A)$ denotes the matrix algebra of a C^*-algebra. Moreover, we establish some relations with the notion of C^*-valued fuzzy metric spaces, which were introduced recently by Khaofong and Khammahawong [12].

In conclusion, our paper provides new insights into FIP spaces and their applications. We hope that our results will inspire further research in this area and lead to new discoveries and applications in mathematics and physics.

2. FIP C^*-modules

In the field of mathematics, the concept of a Hilbert C^*-module is a generalization of the concept of a Hilbert space. It was first introduced by Kaplansky in [10]. Before discussing Hilbert C^*-modules, it is necessary to recall some basic definitions related to C^*-algebras. These definitions can be found in [15, 21].

Let A be a C^*-algebra. An element $\iota \in A$ is considered positive if it is self-adjoint and has a non-negative spectrum. The set of all positive elements of A is denoted by A^+. We also introduce the partial order $\geq$ on A as follows: For any $\iota, \kappa \in A$, we say that $\iota \geq \kappa$ if $\iota - \kappa \in A^+$.

We note that $\iota \in A$ is positive if and only if there is a unique element $\kappa \in A^+$ such that $\iota = \kappa^2$ [21]. In this case, we write $\kappa = \iota^{\frac{1}{2}}$. Moreover, it is known that

$$A^+ = \{\iota^*\iota : \iota \in A\}$$

and according this for each $\iota \in A$ the absolute value of ι can be introduced by

$$|\iota| = (\iota^*\iota)^{\frac{1}{2}}.$$

Furthermore, for each $\iota, \kappa \in A$, we set

$$\min\{\iota, \kappa\} = \frac{(\iota + \kappa) - |\iota - \kappa|}{2}.$$

Definition 2.1. Let Ξ be a complex linear space equiped with a compatible right A-module action (i.e., $\lambda(\xi\iota) = (\lambda\xi)\iota = \xi(\lambda\iota)$ for $\xi \in \Xi$, $\iota \in A$, $\lambda \in \mathbb{C}$). We say a map $\langle\cdot,\cdot\rangle : \Xi \times \Xi \longrightarrow A$ is an inner product if for each $\xi, \zeta, \varsigma \in \Xi, \alpha, \beta \in \mathbb{C}, \iota \in A$,

(i) $\langle\xi, \alpha\zeta + \beta\varsigma\rangle = \alpha\langle\xi, \zeta\rangle + \beta\langle\xi, \varsigma\rangle$;
(ii) $\langle\xi, \zeta\iota\rangle = \langle\xi, \zeta\rangle\iota$;
(iii) $\langle\xi, \zeta\rangle = \langle\zeta, \xi\rangle^*$;
(iv) $\langle\xi, \xi\rangle \geq 0$ and $\langle\xi, \xi\rangle = 0$ if and only if $\xi = 0$.

Furthermore, Ξ is considered a Hilbert C^*-module over A if it is complete with the induced norm introduced by

$$\|\xi\| := \|\langle\xi, \xi\rangle\|^{\frac{1}{2}}, \qquad (\xi \in \Xi).$$

As an example, it is easy to see that every complex Hilbert space H is a Hilbert C^*-module over $\mathbb{C}$, with its inner product. Moreover, every C^*-algebra A can be regarded as a Hilbert C^*-module over A, where

$$\langle\iota, \kappa\rangle := \iota^*\kappa, \qquad (\iota, \kappa \in A).$$

Motivated by these facts, we aim to introduce the notions of FIP C^*-modules and FH C^*-modules.

Definition 2.2. Let A be an arbitrary C^*-algebra. A map $f : A^+ \to ball(A^+) = \{\iota \in A^+ : \|\iota\| \leq 1\}$ is considered vanish at infinity if for every $\varepsilon > 0$ the set of $\{\iota \in A^+ : \|f(\iota)\| \geq \varepsilon\}$ is compact. Also, $F_0(A^+)$ is the notation used to represent the set of all maps which are vanish at infinity.

Notice that if $A = \mathbb{C}$, then we obtain

$$F_0(A^+) = \{f : \mathbb{R}^+ \to [0,1] : \lim_{\xi \to +\infty} f(\xi) = 0\}.$$

The introduction of the notion of FIP A-module, where A is a unital C^*-algebra, is now at hand. This concept can be viewed as an extension of previously established notions such as FIP and FH spaces, which were initially proposed by Lorena and Sida [22].

Definition 2.3. Let Ξ be a complex linear space, A be a unital C^*-algebra and μ represent a fuzzy set from $\Xi \times \Xi \times A$ to ball(A^+). Then the pair (Ξ, μ) is called an FIP A-module if the following hold:

FIPA1) $\mu(\xi, \xi, \iota) = 0, \quad \forall \xi \in \Xi, \quad \forall \iota \in A \setminus A^+;$

FIPA2) $\mu(\xi, \xi, \iota) = 1_A, \quad \forall \iota \in A^+ \quad iff \quad \xi = 0;$

FIPA3) $\mu(\alpha\xi, \zeta, \iota) = \mu(\xi, \zeta, \frac{\iota}{|\alpha|}), \quad \forall \xi, \zeta \in \Xi, \quad \forall \iota \in A, \ \forall \alpha \in \mathbb{C} \setminus \{0\};$

FIPA4) $\mu(\xi, \zeta, \iota) = \mu(\zeta, \xi, \iota^*), \quad \forall \xi, \zeta \in \Xi, \quad \forall \iota \in A;$

FIPA5) $\mu(\xi + \zeta, \varsigma, |\iota| + |\kappa|) \geq \min\{\mu(\xi, \varsigma, |\iota|), \mu(\zeta, \varsigma, |\kappa|)\},$
$\forall \xi, \zeta, \varsigma \in \Xi, \ \forall \iota, \kappa \in A;$

FIPA6) $\mu(\xi, \xi, \cdot) : A^+ \to ball(A^+), \xi \in \Xi,$ *is left continuous*
and $\mu(\xi, \xi, \cdot) - 1_A \in F_0(A^+);$

FIPA7) $\mu(\xi, \zeta, |\iota\kappa|) \geq \min\{\mu(\xi, \xi, |\iota|^2), \mu(\zeta, \zeta, |\kappa|^2)\}, \ \forall \xi, \zeta \in \Xi, \ \forall \iota, \kappa \in A.$

Theorem 2.4. *Let (Ξ, μ) be an FIP A-module. Then for each $\xi, \zeta \in \Xi, \iota \in A, \alpha \in \mathbb{C} \setminus \{0\}$, $\mu(\xi, \alpha\zeta, \iota) = \mu\left(\xi, \zeta, \frac{\iota}{|\alpha|}\right)$.*

Proof. Using (FIPA3) and (FIPA4), we conclude that

$$\mu(\xi, \alpha\zeta, \iota) = \mu(\alpha\zeta, \xi, \iota^*) = \mu\left(\zeta, \xi, \frac{\iota^*}{|\alpha|}\right) = \mu\left(\xi, \zeta, \frac{(\iota^*)^*}{|\alpha|}\right) = \mu\left(\xi, \zeta, \frac{\iota}{|\alpha|}\right).$$

This completes the proof. □

Theorem 2.5. *Assume that (Ξ, μ) is an FIP A-module and $\xi \in \Xi, \iota \in A^+$. Then $\mu(\xi, 0, \iota) = \mu(0, \xi, \iota) = 1_A$.*

Proof. By (FIPA5) and (FIPA6), we conclude that

$$\begin{aligned} 1_A \geq \mu(\xi, 0, \iota) &= \mu(\xi, 0, 2n\iota) = \mu(\xi, \xi - \xi, n\iota + n\iota) \\ &\geq \min\{\mu(\xi, \xi, n\iota), \mu(\xi, \xi, n\iota)\} = \mu(\xi, \xi, n\iota). \end{aligned} \tag{1}$$

Moreover, by (FIPA6), we know that $\mu(\xi, \xi, \cdot) - 1_A \in F_0(A^+)$. So, for any $\epsilon > 0$, the set

$$K = \{\iota \in A^+ : \|\mu(\xi, \xi, \iota) - 1_A\| \geq \epsilon\}$$

is compact. It follows that there is $N \in \mathbb{N}$ such that for any $n \geq N$, the element of $n\iota$ does not belong to K and so

$$\|\mu(\xi, \xi, n\iota) - 1_A\| < \epsilon.$$

Now, by taking the limit $n \to \infty$ in (1), it follows that $\mu(\xi, 0, \iota) = 1_A$, as required. □

THEOREM 2.6. *Assume that* (Ξ, μ) *is an* FIP A*-module. Then for any* $\xi, \zeta \in \Xi$, *the map* $\mu(\xi, \zeta, \cdot) : A^+ \longrightarrow ball(A^+)$ *is monotonic nondecreasing.*

Proof. For any $\iota, \kappa \in A^+$ with $\iota \leq \kappa$, there is $c \in A^+$ such that $\kappa = \iota + c$. Now, it follows that

$$\begin{aligned}\mu(\xi, \zeta, \kappa) &= \mu(\xi + 0, \zeta, \iota + c) \geq \min\{\mu(\xi, \zeta, \iota), \mu(0, \zeta, c)\} \\ &= \min\{\mu(\xi, \zeta, \iota), 1_A\} = \mu(\xi, \zeta, \iota),\end{aligned}$$

as required. □

THEOREM 2.7. *Assume that* (Ξ, μ) *is an* FIP A*-module such that* $(A^+, \leq)$ *is totally ordered. Then for any* $\xi, \zeta \in \Xi, \iota, \kappa \in A^+$,

$$\mu(\xi, \zeta, \iota\kappa) \geq \min\{\mu(\xi, \zeta, \iota^2), \mu(\xi, \zeta, \kappa^2)\}.$$

Proof. Since $(A^+, \leq)$ is totally ordered, without loss of generality, we can consider $\iota \leq \kappa$. Then it is easy to see that $\iota^2 \leq \iota\kappa \leq \kappa^2$. So we conclude that

$$\mu(\xi, \zeta, \iota^2) \leq \mu(\xi, \zeta, \iota\kappa) \leq \mu(\xi, \zeta, \kappa^2).$$

Thus it follows that

$$\mu(\xi, \zeta, |\iota\kappa|) \geq \min\{\mu(\xi, \zeta, |\iota|^2), \mu(\xi, \zeta, |\kappa|^2)\}.$$

This completes the proof. □

THEOREM 2.8. *Assume that* (Ξ, μ) *is an* FIP A*-module. Then for any* $\xi, \zeta, \varsigma \in \Xi, \iota \in A^+$,

$$\mu(\xi, \zeta, \iota) \geq \min\{\mu(\xi, \zeta - \varsigma, \iota), \mu(\xi, \zeta + \varsigma, \iota)\}.$$

Proof. Using (FIPA3), (FIPA4) and (FIPA5), we deduce that

$$\begin{aligned}\mu(\xi, \zeta, \iota) &= \mu(\xi, 2\zeta, 2\iota) = \mu(\xi, \zeta + \varsigma + \zeta - \varsigma, \iota + \iota) \\ &\geq \min\{\mu(\xi, \zeta + \varsigma, \iota), \mu(\xi, \zeta - \varsigma, \iota)\}.\end{aligned}$$

This completes the proof. □

THEOREM 2.9. *Assume that* (Ξ, μ) *is an* FIP A*-module such that* $(A^+, \leq)$ *is totally ordered. Then the map* $\eta : \Xi \times A^+ \longrightarrow ballA^+$, *given by*

$$\eta(\xi, \iota) = \mu(\xi, \xi, |\iota|^2) \qquad (\xi \in \Xi, \iota \in A^+),$$

is a fuzzy norm on Ξ.

Proof. (N1) By (FIPA1), for each $\xi \in \Xi$, we obtain $\eta(\xi, 0_A) = \mu(\xi, \xi, 0_A) = 0$.

(N2) By (FIPA2), for each $\iota \in A^+$, $\eta(\xi, \iota) = 1$ if and only if $\mu(\xi, \xi, |\iota|^2) = 1_A$ if and only if $\xi = 0_A$.

(N3) We have

$$\begin{aligned}\eta(\alpha\xi, \iota) &= \mu(\alpha\xi, \alpha\xi, |\iota|^2) = \mu\left(\xi, \alpha\xi, \frac{|\iota|^2}{|\alpha|}\right) \\ &= \mu\left(\alpha\xi, \xi, \frac{|\iota|^{2*}}{|\alpha|}\right) = \mu\left(\xi, \xi, \frac{|\iota|^2}{|\alpha|^2}\right) \\ &= \eta\left(\xi, \frac{\iota}{|\alpha|}\right).\end{aligned}$$

(N4) We show that $\eta(\xi+\zeta,\iota+\kappa)\ge\min\{\eta(\xi,\iota),\eta(\zeta,\kappa)\}$. In the case that $\iota=0_A$ or $\kappa=0_A$, the inequality is obvious. Otherwise, we obtain

$$\begin{aligned}\eta(\xi+\zeta,\iota+\kappa)&=\mu(\xi+\zeta,\xi+\zeta,|\iota+\kappa|^2)=\mu(\xi+\zeta,\xi+\zeta,(\iota+\kappa)^2)\\&=\mu(\xi+\zeta,\xi+\zeta,\iota^2+\kappa^2+\iota\kappa+\kappa\iota)\\&\ge\min\{\mu(\xi,\xi+\zeta,\iota^2+\iota\kappa),\mu(\zeta,\xi+\zeta,\kappa^2+\kappa\iota)\}\\&\ge\min\{\mu(\xi,\xi,\iota^2),\mu(\xi,\zeta,\iota\kappa),\mu(\zeta,\xi,\kappa^2),\mu(\zeta,\zeta,\kappa\iota)\}\\&\ge\min\{\mu(\xi,\xi,\iota^2),\mu(\zeta,\zeta,\kappa^2)\}=\min\{\eta(\xi,\iota),\eta(\zeta,\kappa)\}.\end{aligned}$$

(N5) From (FIAP6), we conclude that the map $\eta(\xi,\cdot)$ is left continuous and also we obtain $\eta(\xi,\cdot)-1_A\in F_0(A^+)$. □

THEOREM 2.10. *Assume that A is a unital C^*-algebra and the map*

$$\langle\cdot,\cdot\rangle:\Xi\times\Xi\to A,$$

is an inner product. Then the following statements are equivalent:

(1) *The map $\mu:\Xi\times\Xi\times A\to ball(A^+)$, defined by*

$$\mu(\xi,\zeta,\iota)=\begin{cases}\left(\dfrac{\|\iota\|}{\|\iota\|+\|\langle\xi,\zeta\rangle\|}\right)1_A & \text{if}\quad \iota\in A^+,\\[2mm] 0 & \text{if}\quad \iota\in A\setminus A^+,\end{cases}\tag{2}$$

is an FIP on Ξ.

(2) *A is finite dimensional.*

Proof. (1) ⇒ (2): By the condition (FIPA6), it follows that the map $\mu(\xi,\xi,\cdot)-1_A$ is vanish at infinity. So, by choosing $\varepsilon=\frac12$, we conclude that the set

$$\left\{\iota\in A^+:\left\|\left(\frac{\|\iota\|}{\|\iota\|+\|\xi\|}-1\right)1_A\right\|\ge\frac12\right\}$$

is compact. Thus we obtain that the set

$$\left\{\iota\in A^+:\frac{\|\xi\|}{\|\iota\|+\|\xi\|}\ge\frac12\right\}=\{\iota\in A^+:\|\iota\|\le\|\xi\|\}$$

is compact. It follows that $ball(A)$ is compact and hence A is finite dimensional.

(2) ⇒ (1): It suffices to show that all the conditions (FIPA1)-(FIPA7) are satisfied.

FIPA1) It is obvious from the definition of μ that $\mu(\xi,\xi,\iota)=0$ for each $\xi\in\Xi,\iota\in A\setminus A^+$.

FIPA2) For each $\xi\in\Xi,\iota\in A^+$, we obtain

$$\begin{aligned}\mu(\xi,\xi,\iota)=1_A&\Leftrightarrow\ \|\iota\|+\|\langle\xi,\xi\rangle\|=\|\iota\|\\&\Leftrightarrow\ \|\langle\xi,\xi\rangle\|=0\\&\Leftrightarrow\ \xi=0.\end{aligned}$$

FIAP3) Assume that $\xi,\zeta\in\Xi,\iota\in A,\alpha\in\mathbb{C}\setminus\{0\}$. Then

$$\mu(\alpha\xi,\zeta,\iota)=\frac{\|\iota\|}{\|\iota\|+\|\langle\alpha\xi,\zeta\rangle\|}=\frac{\|\iota\|}{\|\iota\|+|\alpha|\|\langle\xi,\zeta\rangle\|}=\frac{\dfrac{\|\iota\|}{|\alpha|}}{\dfrac{\|\iota\|}{|\alpha|}+\|\langle\xi,\zeta\rangle\|}=\frac{\left\|\dfrac{\iota}{|\alpha|}\right\|}{\left\|\dfrac{\iota}{|\alpha|}\right\|+\|\langle\xi,\zeta\rangle\|}$$

$$=\mu\left(\xi,\zeta,\frac{\iota}{|\alpha|}\right).$$

FIAP4) For each $\xi, \zeta \in \Xi, \iota \in A$, we know that $\|\iota\| = \|\iota^*\|$ and so

$$\begin{aligned}\mu(\xi, \zeta, \iota) &= \left(\frac{\|\iota\|}{\|\iota\| + \|\langle \xi, \zeta \rangle\|}\right) 1_A \\ &= \left(\frac{\|\iota^*\|}{\|\iota^*\| + \|\langle \zeta, \xi \rangle\|}\right) 1_A \\ &= \mu(\zeta, \xi, \iota^*).\end{aligned}$$

FIAP5) We show that for any $\xi, \zeta, \varsigma \in \Xi$ and $\iota, \kappa \in A$,

$$\mu(\xi + \zeta, \varsigma, |\iota| + |\kappa|) \geq \min\{\mu(\xi, \varsigma, |\iota|), \mu(\zeta, \varsigma, |\kappa|)\}.$$

In the case that at least one of ι and κ is from $A \backslash A^+$, the result is obvious. Without loss of generality, let us choose ι and κ from the set A^+ such that $\mu(\xi, \varsigma, |\iota|) \leq \mu(\zeta, \varsigma, |\kappa|)$. On the other hand, we note that

$$\||\iota|\|^2 = \||\iota|^*|\iota|\| = \||\iota|^2\| = \|\iota^*\iota\| = \|\iota\|^2.$$

Then

$$\begin{aligned}\frac{\|\iota\|}{\|\iota\| + \|\langle \xi, \varsigma \rangle\|} \leq \frac{\|\kappa\|}{\|\kappa\| + \|\langle \zeta, \varsigma \rangle\|} &\Rightarrow \frac{\|\iota\| + \|\langle \xi, \varsigma \rangle\|}{\|\iota\|} \geq \frac{\|\kappa\| + \|\langle \zeta, \varsigma \rangle\|}{\|\kappa\|} \\ &\Rightarrow 1 + \frac{\|\langle \xi, \varsigma \rangle\|}{\|\iota\|} \geq 1 + \frac{\|\langle \zeta, \varsigma \rangle\|}{\|\kappa\|} \\ &\Rightarrow \frac{\|\kappa\|}{\|\iota\|}\|\langle \xi, \varsigma \rangle\| \geq \|\langle \zeta, \varsigma \rangle\| \\ &\Rightarrow \|\langle \xi, \varsigma \rangle\| + \frac{\|\kappa\|}{\|\iota\|}\|\langle \xi, \varsigma \rangle\| \geq \|\langle \xi, \varsigma \rangle\| + \|\langle \zeta, \varsigma \rangle\| \\ &\Rightarrow \frac{\|\iota\| + \|\kappa\|}{\|\iota\|}\|\langle \xi, \varsigma \rangle\| \geq \|\langle \xi, \varsigma \rangle\| + \|\langle \zeta, \varsigma \rangle\| \\ &\Rightarrow \frac{\|\iota\| + \|\kappa\|}{\|\iota\|}\|\langle \xi, \varsigma \rangle\| \geq \|\langle \xi + \zeta, \varsigma \rangle\| \\ &\Rightarrow \frac{\|\langle \xi, \varsigma \rangle\|}{\|\iota\|} \geq \frac{\|\langle \xi + \zeta, \varsigma \rangle\|}{\|\iota\| + \|\kappa\|} \\ &\Rightarrow \frac{\|\langle \xi, \varsigma \rangle\|}{\|\iota\|} + 1 \geq \frac{\|\langle \xi + \zeta, \varsigma \rangle\|}{\|\iota\| + \|\kappa\|} + 1 \\ &\Rightarrow \frac{\|\iota\| + \|\langle \xi, \varsigma \rangle\|}{\|\iota\|} \geq \frac{\|\langle \xi + \zeta, \varsigma \rangle\| + \|\iota\| + \|\kappa\|}{\|\iota\| + \|\kappa\|} \\ &\Rightarrow \frac{\|\iota\|}{\|\iota\| + \|\langle \xi, \varsigma \rangle\|} \leq \frac{\|\iota\| + \|\kappa\|}{(\|\iota\| + \|\kappa\|) + \|\langle \xi + \zeta, \varsigma \rangle\|} \\ &\Rightarrow \mu(\xi, \varsigma, |\iota|) \leq \mu(\xi + \zeta, \varsigma, |\iota| + |\kappa|).\end{aligned}$$

It follows that $\mu(\xi + \zeta, \varsigma, |\iota| + |\kappa|) \geq \min\{\mu(\xi, \varsigma, |\iota|), \mu(\zeta, \varsigma, |\kappa|)\}$, as required.

FIAP6) It is easy to show that the map $\mu(\xi,\xi,\cdot)$ is left continuous. On the other hand, according to the assumption, we know that A is finite dimensioanl and so $ball(A)$ is compact. It follows that the map $\mu(\xi,\xi,\cdot)-1_A$ is vanish at infinity.

FIAP7) If at least one of ι and κ is from $A\backslash A^+$, then the result is clear. Without loss of generality, let us choose ι and κ from the set A^+ such that $\mu(\xi,\xi,|\iota|^2)\le\mu(\zeta,\zeta,|\kappa|^2)$. Then we conclude that

$$\begin{aligned}\frac{\|\iota\|^2}{\|\iota\|^2+\|\langle\xi,\xi\rangle\|}\le\frac{\|\kappa\|^2}{\|\kappa\|^2+\|\langle\zeta,\zeta\rangle\|} &\Leftrightarrow \frac{\|\iota\|^2+\|\langle\xi,\xi\rangle\|}{\|\iota\|^2}\ge\frac{\|\kappa\|^2+\|\langle\zeta,\zeta\rangle\|}{\|\kappa\|^2}\\ &\Leftrightarrow 1+\frac{\|\langle\xi,\xi\rangle\|}{\|\iota\|^2}\ge\frac{\|\langle\zeta,\zeta\rangle\|}{\|\kappa\|^2}+1\\ &\Leftrightarrow \frac{\|\langle\xi,\xi\rangle\|}{\|\iota\|^2}\ge\frac{\|\langle\zeta,\zeta\rangle\|}{\|\kappa\|^2}\\ &\Leftrightarrow \|\kappa\|^2\|\langle\xi,\xi\rangle\|\ge\|\iota\|^2\|\langle\zeta,\zeta\rangle\|.\end{aligned}$$

Now, the Cauchy-Schwartz inequality says that

$$\begin{aligned}\|\iota\|\|\langle\xi,\zeta\rangle\| &\le \sqrt{\|\langle\xi,\xi\rangle\|}\cdot\|\iota\|\sqrt{\|\langle\zeta,\zeta\rangle\|}\\ &\le \sqrt{\|\langle\xi,\xi\rangle\|}\cdot\|\kappa\|\sqrt{\|\langle\xi,\xi\rangle\|}\\ &= \|\kappa\|\|\langle\xi,\xi\rangle\|.\end{aligned}$$

So we conclude that

$$\begin{aligned}\|\iota\|^2\|\langle\xi,\zeta\rangle\|\le\|\iota\|\|\kappa\|\|\langle\xi,\xi\rangle\| &\Rightarrow \|\iota\|^3\|\kappa\|+\|\iota\|^2\|\langle\xi,\zeta\rangle\|\le\|\iota\|^3\|\kappa\|+\|\iota\|\|\kappa\|\|\langle\xi,\xi\rangle\|\\ &\Rightarrow \|\iota\|^2(\|\iota\|\|\kappa\|+\|\langle\xi,\zeta\rangle\|)\le\|\iota\|\|\kappa\|(\|\iota\|^2+\|\langle\xi,\xi\rangle\|)\\ &\Rightarrow \frac{\|\iota\|^2}{\|\iota\|^2+\|\langle\xi,\xi\rangle\|}\le\frac{\|\iota\|\|\kappa\|}{\|\iota\|\|\kappa\|+\|\langle\xi,\zeta\rangle\|}\\ &\Rightarrow \frac{\|\iota\|^2}{\|\iota\|^2+\|\langle\xi,\xi\rangle\|}\le\frac{\|\iota\kappa\|}{\|\iota\kappa\|+\|\langle\xi,\zeta\rangle\|}.\end{aligned}$$

Thus we deduce that

$$\mu(\xi,\zeta,|\iota\kappa|)\ge\min\{\mu(\xi,\xi,|\iota|^2),\mu(\zeta,\zeta,|\kappa|^2)\}.$$

This completes the proof. □

3. Some applications

In this section, we aim to give some applications related to FIP C^*-modules. Indeed, we construct some examples by matrix-valued FIP. Also, we obtain some relations with the notion of C^*-valued FNS.

For an arbitrary C^*-algebra A, by Gelfand-Naimark-Segal representation theorem, there is a Hilbert space H and an injective $*$-homomorphism $\phi: A\longrightarrow B(H)$. So, for any $n\ge 1$, we can obtain an injective $*$-homomorphism

$$\overline{\phi}: M_n(A)\longrightarrow M_n(B(H))\cong B(H^{(n)}),$$

where $H^{(n)} = \oplus_{i=1}^{n} H$. Using this fact, it is easy to see that the matrix algebra $M_n(A)$ is a C^*-algebra with the norm defined by

$$\|(a_{ij})\| = \|\overline{\phi}((a_{ij}))\|, \quad ((a_{ij}) \in M_n(A)).$$

See [21] for more details.

Example 3.1. Let A be a unital C^*-algebra with finite dimension and $n \geq 1$. Then $M_n(A)$ is a unital C^*-algebra with finite dimension. As an application of Theorem 2.10, it follows that if Ξ is a Hilbert C^*-module over $M_n(A)$, then the map $\mu : \Xi \times \Xi \times A \to \mathrm{ball}(M_n(A)^+)$ introduced as in (2), is an $M_n(A)$-valued FIP.

Recently, Khaofong and Khammahawong [12] introduced and studied a new concept of C^*-algebra valued FNS.

Definition 3.2. Let A be a unital C^*-algebra, Ξ be an arbitrary nonempty set and $\star$ be a continuous t-norm. A fuzzy set $\mu_A : S \times (0, +\infty) \to \mathrm{ball}(A^+)$ is considered a C^*-algebra valued fuzzy norm if it satisfies the following: For any $\xi, \zeta \in S$ and $\tau, \varsigma > 0$,

(i) $\mu_A(\zeta, \tau) > 0_A$;
(ii) $\mu_A(\zeta, \tau) = 1_A$ if and only if $\zeta = 0$;
(iii) $\mu_A(a\zeta, \tau) = \mu\left(\zeta, \frac{\tau}{|a|}\right)$ for each $a \neq 0$;
(iv) $\mu_A(\xi + \zeta, \tau + \varsigma) \geq \mu_A(\xi, \tau) \star \mu_A(\zeta, \varsigma)$;
(v) $\forall \zeta \in S,\ \mu_A(\zeta, \cdot)$ is continuous;
(vi) $\lim_{\tau \to +\infty} \mu_A(\zeta, \tau) = 1_A$.

The 4-tuple $(S, A, \mu_A, \star)$ is called a C^*-algebra valued FNS.

Example 3.3. Let $(S, \|\cdot\|)$ be a normed space and A be a unital C^*-algebra. Then

$$\mu_A(s, \varsigma) = \left(\frac{\varsigma}{\varsigma + \|s\|}\right) 1_A$$

for each $\varsigma > 0$ introduces a C^*-algebra valued fuzzy norm and so the 4-tuple $(S, A, \mu_A, \star)$ is a C^*-algebra valued FNS.

4. Conclusion

We introduced and studied the notion of fuzzy inner product A-module, where A is an arbitrary unital C^*-algebra. Moreover, we constructed some examples of particular classes of C^*-algebras. As an application, we obtained some $M_n(A)$-valued fuzzy inner product, where $M_n(A)$ denotes the $n \times n$ matrix C^*-algebra of a unital C^*-algebra A. Moreover, we obtained some relations with the notion of C^*-valued fuzzy normed spaces.

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