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MODULAR STABILITY RESULTS OF GENERALIZED QUARTIC FUNCTIONAL EQUATIONS BY USING FATOU PROPERTY WITH Δ_2-CONDITION

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Abstract. In this research work, we introduce a new kind of finite variable quartic functional equation and examine the Hyers-Ulam stability of this quartic functional equation in modular spaces via fixed point technique with the help of Fatou property with Δ_2-condition. A counterexample is also provided to prove the non-stability for singular case.

1. Introduction

Ulam [29] initially proposed the concept of a stability problem for a functional equation in 1940 at the University of Wisconsin's Mathematics Club, in relation to the stability of group homomorphisms. Hyers [10] presented a partial response to Ulam's question for additive groups in the next year, assuming that groups are Banach spaces. The direct method, which was introduced by Hyers in [10], has been used to investigate the stability of numerous functional equations. Aoki [2] was the first to expand Hyers' theorem to the case of unbounded control functions. By reducing the criterion for the Cauchy difference, Rassias [23] was able to extend the outcome of Hyers' theorem.

Fixed point methods are the second most frequent method for showing the stability of functional equations. Baker [3] utilised a form of Banach's fixed point theorem to examine the stability of a functional equation in a single variable for the first time in 1991. Several researchers have used the fixed point method to investigate the stability of various functional equations in different normed spaces (see [5, 6, 22, 26]).

In 1950, Nakano [20] proposed the theory of linear modulars and the related notion of modular linear spaces. Since then, various mathematicians have worked on them extensively in [1, 13, 16, 18, 19, 21, 28, 32]. The concept of modular spaces has been extensively applied to the analysis of interpolation concept [14, 17] and as well as other Orlicz spaces [21] until now.

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Using the fixed point approach of a generalised Jensen functional equation with the Fatou condition fulfilling the Δ_2-condition of $0 < k \leq 2$, Sadeghi [24] established the Hyers-Ulam stability in convex modular spaces. In [31], Wongkum *et al.* presented the Hyers-Ulam stability of quadratic functional equations, which is based on enormous fixed point theory studies in the framework of modular spaces whose modulars are convex, lower semi-continuous, but do not fulfill any relatives of Δ_2-conditions. Many research papers on different generalizations and implications of the Hyers-Ulam stability to a variety of functional equations have been recently published (see [4, 7, 9, 11, 12, 25, 27, 30, 34]).

In [15], Lee *et al.* proposed the quartic functional equation as follows:

$$\phi(2u+v)+\phi(2u-v)=4\left[\phi(u+v)+\phi(u-v)\right]+24\phi(u)-6\phi(v) \tag{1}$$

and found its solution of (1) and demonstrated the Hyers-Ulam stability of the functional equation. It is simple to demonstrate that $\phi(v) = v^4$ satisfies the functional equation (1), which is known as a quartic functional equation and that every solution of the quartic functional equation is a quartic mapping.

We develop a new finite variable functional equation, in this paper, which has the form

$$\begin{aligned}\sum_{i=1}^{m}\phi\left(-v_i+\sum_{j=1}^{m}v_j\right) = {} & (m-8)\sum_{1\leq i<j<k<l\leq m}\phi\left(v_i+v_j+v_k+v_l\right) \\ & -\left(m^2-12m+28\right)\sum_{1\leq i<j<k\leq m}\phi\left(v_i+v_j+v_k\right) \\ & +\left(\frac{m^3-15m^2+60m-68}{2}\right)\sum_{1\leq i<j\leq m}\phi\left(v_i+v_j\right) \\ & +2\sum_{1\leq i<j\leq m}\phi\left(v_i-v_j\right)+\sum_{i=1}^{m}\phi\left(2v_i\right) \\ & -\left(\frac{m^4-17m^3+86m^2-148m+168}{6}\right)\sum_{i=1}^{m}\phi\left(v_i\right),\end{aligned} \tag{2}$$

where $m \geq 5$, and concentrate on demonstrating that this equation has the same solution as (1), i.e., $\phi(v) = v^4$. We also use fixed point theory with the help of the Fatou property with the Δ_2-condition to study the Hyers-Ulam stability of this quartic functional equation in modular spaces. A good example is given to demonstrate the non-stability of the single case.

2. Preliminaries

In this section, we recall some sufficient notions and concepts to achieve our results.

Definition 2.1. [33] Let A be a vector space over a field $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$). A generalized function $\rho : A \to [0, \infty]$ is called a modular if for $u, v \in A$,

(a) $\rho(u) = 0 \Leftrightarrow u = 0$,
(b) $\rho(\beta u) = \rho(u)$ for all scalars β with $|\beta| = 1$,
(c) $\rho(\beta u + \gamma v) \leq \rho(u) + \rho(v)$ for all scalars $\beta, \gamma \geq 0$ and $\beta + \gamma = 1$.

If (c) is replaced by

(c') $\rho(\beta u+\gamma v)\leq \beta\rho(u)+\gamma\rho(v)$ for every scalars $\beta,\gamma\geq 0$ and $\beta+\gamma=1$, then the function ρ is referred to as a convex modular.

The modular ρ defines an appropriate modular space A_ρ with

$$A_\rho=\{u\in A|\rho(\theta u)\to 0 \text{ as } \theta\to 0\}.$$

Definition 2.2. [33] Let A_ρ be a modular space and $\{v_n\}\in A_\rho$.

(i) $v_n \xrightarrow{\rho} v$ if $\rho(v_n-v)\to 0$ as $n\to\infty$.
(ii) $\{v_n\}$ is known as ρ-Cauchy if $\rho(v_l-v_n)\to 0$ as l,n tend to ∞.
(iii) A subset $T\subseteq A_\rho$ is known as ρ-complete if every ρ-Cauchy sequence is ρ-convergent in T.

Definition 2.3. [33] Let A_ρ be a modular space and $T\subseteq A_\rho$ be a nonempty subset. A mapping $J:T\to T$ is known as a quasicontraction if there is $k<1$ such that

$$\rho\left(Ju-Jv\right)\leq k\max\{\rho(u-v),\rho(u-Jv),\rho(v-Ju),\rho(u-Ju),\rho(v-Jv)\}$$

for all $u,v\in T$.

Definition 2.4. [33] For a modular space A_ρ, a nonempty subset $T\subseteq A_\rho$ and a mapping $J:T\to T$, the T orbit around a point v is defined by

$$O(J):=\{u,Ju,J^2u,\cdots\},$$

and the quantity

$$\Upsilon_\rho(J):=\sup\{\rho(p-q)|p,q\in O(J)\},$$

is called the orbital diameter of J at v. If $\Upsilon_\rho(J)<\infty$, in particular, one says that J has an orbit of v that is limited to v.

Fatou property: A modular ρ is said to have the Fatou property if $\rho(v)\leq \lim_{m\to\infty}\inf\rho(v_m)$ whenever $\{v_m\}\xrightarrow{\rho} v$. A modular ρ is said to fulfill the condition Δ_2 if there is $k>0$ such that $\rho(2v)\leq k\rho(v)$ for all $v\in A_\rho$.

Theorem 2.5. [33] Let B_ρ be a modular space such that ρ fulfills the Fatou property and $T\subseteq B_\rho$ be a ρ-complete subset. If $J:T\to T$ is a quasicontraction and J has a bounded orbit at v_0, then $\{J^n v_0\}\xrightarrow{\rho}\alpha$, where $\alpha\in T$.

3. Solution of the functional equation (2)

In this section, assume that A and B are two real vector spaces.

Theorem 3.1. *If $\chi:A\to B$ is an even mapping which fulfills the equation* (2) *for all $v_1,v_2,\cdots,v_m\in A$, then the mapping χ is quartic.*

Proof. By the evenness of χ, we have $\chi(-v)=\chi(v)$ for all $v\in A$. Now, setting $v_1=v_2=\cdots=v_m=0$ in (2), we have $\chi(0)=0$. Letting $(v_1,v_2,\cdots,v_m)=(v,0,\cdots,0)$ in (2), we get

$$\chi(2v)=2^4\chi(v),\ v\in A.$$

Thus, for any nonnegative integer $m\geq 0$, we obtain

$$\chi(2^m v)=16^m\chi(v),\ v\in A$$

and

$$\chi\left(\frac{v}{2^m}\right) = \frac{1}{14^m}\chi(v),\ v \in A.$$

Next, letting $(v_1, v_2, \cdots, v_m) = (u, u, v, 0, \cdots, 0)$ in (2), we obtain

$$\phi(2u+v) + \phi(2u-v) = 4[\phi(u+v) + \phi(u-v)] + 24\phi(u) - 6\phi(v)$$

for all $u, v \in A$. Hence the mapping χ is quartic. $\square$

4. Hyers-Ulam stability

In this section, assume that A is a linear space, B is a Banach space, B_ρ is a ρ-complete modular space and ρ is a convex modular on B_ρ with the Fatou property such that fulfills the Δ_2-condition with $0 < k \leq 2$.

For notational handiness, we can define the mapping $\chi : A \to B_\rho$ as follows:

$$\begin{aligned}
&\Theta\chi(v_1, v_2, \cdots, v_m) \\
&= \sum_{i=1}^{m}\chi\left(-v_i + \sum_{j=1}^{m} v_j\right) - (m-8)\sum_{1\leq i<j<k<l\leq m}\chi(v_i + v_j + v_k + v_l) \\
&+ \left(m^2 - 12m + 28\right)\sum_{1\leq i<j<k\leq m}\chi(v_i + v_j + v_k) \\
&- \left(\frac{m^3 - 15m^2 + 60m - 68}{2}\right)\sum_{1\leq i<j\leq m}\chi(v_i + v_j) \\
&-2\sum_{1\leq i<j\leq m}\chi(v_i - v_j) - \sum_{i=1}^{m}\chi(2v_i) \\
&+ \left(\frac{m^4 - 17m^3 + 86m^2 - 148m + 168}{6}\right)\sum_{i=1}^{m}\chi(v_i)
\end{aligned}$$

for all $v_1, v_2, \cdots, v_m \in A$.

THEOREM 4.1. *Let $\tau : A^m \to [0, +\infty)$ be a function such that*

$$\tau(2v_1, 2v_2, \cdots, 2v_m) \leq 2^4 L\tau(v_1, v_2, \cdots, v_m) \tag{3}$$

for all $v_1, v_2, \cdots, v_m \in A$ with $0 < L < 1$. If an even mapping $\tau : A \to B_\rho$ fulfills $\chi(0) = 0$ and

$$\rho(\Theta\chi(v_1, v_2, \cdots, v_m)) \leq \tau(v_1, v_2, \cdots, v_m), \tag{4}$$

for all $v_1, v_2, \cdots, v_m \in A$, then there is only one quartic solution $Q_4 : A \to B_\rho$ satisfying

$$\rho(Q_4(v) - \chi(v)) \leq \frac{1}{2^4(1-L)}\tau(v, 0, \cdots, 0), \tag{5}$$

for all $v \in A$.

Proof. Let us define the set

$$\Upsilon = \{s : A \to B_\rho\}$$

and the function $\overline{\rho}$ on Υ as

$$\overline{\rho}(s) =: \inf\{\lambda > 0 : \rho\left(s(v)\right) \leq \lambda\tau(v, 0, \cdots, 0),\ \forall\ v \in A\}.$$

We will prove that $\overline{\rho}$ is a convex modular on Υ. Clearly, $\overline{\rho}$ holds the axioms (a) and (b) of a modular. So it is enough to prove that the function $\overline{\rho}$ is convex and so $(c^{'})$ holds.

For any given $\delta > 0$, there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ which are real constants such that

$$\overline{\rho}(s) \leq \lambda_1 \leq \overline{\rho}(s) + \delta \text{ and } \overline{\rho}(t) \leq \lambda_2 \leq \overline{\rho}(t) + \delta.$$

Also

$$\rho(s(v)) \leq \lambda_1\tau(v, 0, \cdots, 0),\ \rho(t(v)) \leq \lambda_2\tau(v, 0, \cdots, 0),\ v \in A.$$

For any $\beta, \gamma \geq 0$ with $\beta + \gamma = 1$, we have

$$\rho\left(\beta s(v) + \gamma t(v)\right) \leq \beta\rho(s(v)) + \gamma\rho(t(v)) \leq \left(\lambda_1\beta + \lambda_2\gamma\right)\tau(v, 0, \cdots, 0)$$

and so we get

$$\overline{\rho}(\beta s + \gamma t) \leq \beta\overline{\rho}(s) + \gamma\overline{\rho}(t) + (\beta + \gamma)\delta.$$

Thus we conclude that the function $\overline{\rho}$ is a convex modular on Υ.

Next, we will verify that $\Upsilon_{\overline{\rho}}$ is $\overline{\rho}$-complete.

Suppose that $\{s_p\}$ is a $\overline{\rho}$-Cauchy sequence in $\Upsilon_{\overline{\rho}}$. Then for a given $\delta > 0$, there is a nonnegative integer n_0 satisfying

$$\overline{\rho}\left(s_p - s_q\right) < \delta,\ \forall\ p, q \geq n_0. \tag{6}$$

So we have

$$\rho\left(s_p(v) - s_q(v)\right) \leq \delta\tau(v, 0, \cdots, 0),\ v \in A,\ p, q \geq n_0. \tag{7}$$

Therefore, $\{s_p(v)\}$ is a ρ-Cauchy sequence in B_ρ. Since B_ρ is ρ-complete, $\{s_p(v)\}$ is convergent in B_ρ, for every $v \in A$. Now, let us define a mapping $s : A \to B_\rho$ by

$$s(v) := \lim_{m\to\infty} s_m(v),\ v \in A.$$

Since ρ has the Fatou property, by (7), we have

$$\rho(s_p(v) - s(v)) \leq \lim_{q\to\infty}\inf\rho\left(s_p(v) - s_q(v)\right) \leq \delta\tau(v, 0, \cdots, 0)$$

and so

$$\overline{\rho}\left(s_p - s\right) \leq \delta$$

for all $p \geq n_0$. Thus $\{s_p\}$ is $\overline{\rho}$-convergent. Hence $\Upsilon_{\overline{\rho}}$ is $\overline{\rho}$-complete.

Next, we will prove that $\overline{\rho}$ has the Fatou property. Suppose that a $\overline{\rho}$-convergent sequence $\{s_p\}$ converges to a point $s \in \Upsilon_{\overline{\rho}}$. For any given $\delta > 0$ and for every $p \in \mathbb{N}$, there is a real constant λ_p such that

$$\overline{\rho}(s_p) \leq \lambda_p \leq \overline{\rho}(s_p) + \delta.$$

So

$$\rho\left(s_p(v)\right) \leq \lambda_p\tau(v, 0, \cdots, 0),\ v \in A.$$

Since ρ hsa the Fatou property, we get

$$\begin{aligned}\rho(s(v)) &\leq \liminf_{p\to\infty} \rho(s_p(v)) \\ &\leq \liminf_{p\to\infty} \lambda_p \tau(v,0,\cdots,0) \\ &\leq \left[\liminf_{p\to\infty} \overline{\rho}(s_p)+\delta\right]\tau(v,0,\cdots,0).\end{aligned}$$

Thus we have

$$\overline{\rho}(s) \leq \liminf_{p\to\infty} \overline{\rho}(s_p)+\delta.$$

Hence $\overline{\rho}$ has the Fatou property. Let us define a mapping $\Phi : \Upsilon_{\overline{\rho}} \to \Upsilon_{\overline{\rho}}$ by

$$\Phi s(v) = \frac{1}{2^4}s(2v)$$

for all $v \in A$ and $s \in \Upsilon_{\overline{\rho}}$. For $s,t \in \Upsilon_{\overline{\rho}}$ and an arbitrary constant $\lambda \in [0,1]$ with $\overline{\rho}(s-t) < \lambda$, by the definition of $\overline{\rho}$, we obtain

$$\rho(s(v)-t(v)) \leq \lambda\tau(v,0,\cdots,0)$$

for all $v \in A$. By (3) and the above inequality, we get

$$\begin{aligned}\rho\left(\frac{s(2v)}{2^4}-\frac{t(2v)}{2^4}\right) &\leq \frac{1}{2^4}\rho\left(s(2v)-t(2v)\right) \\ &\leq \frac{1}{2^4}\lambda\tau\left(2v,0,\cdots,0\right) \\ &\leq \lambda L\tau(v,0,\cdots,0)\end{aligned}$$

for all $v \in A$. Hence

$$\overline{\rho}(\Phi s-\Phi t) \leq L\overline{\rho}(s-t),\ \forall\ s,t \in \Upsilon_{\overline{\rho}}.$$

That is, Φ is a $\overline{\rho}$-contraction.

Now, we will verify that Φ has a χ limited orbit. Letting $(v_1,v_2,\cdots,v_m) = (v,0,\cdots,0)$ in (4), we get

$$\rho\left(\frac{\chi(2v)}{2^4}-\chi(v)\right) \leq \frac{1}{2^4}\tau(v,0,\cdots,0) \tag{8}$$

for all $v \in A$. Replacing v by $2v$ in (8), we get

$$\rho\left(\frac{\chi(2^2v)}{2^4}-\chi(2v)\right) \leq \frac{1}{2^4}\tau(2v,0,\cdots,0). \tag{9}$$

By (8) and (9), we get

$$\begin{aligned}\rho\left(\frac{\chi(2^2v)}{2^{4(2)}}-\chi(v)\right) &\leq \rho\left(\frac{\chi(2^2v)}{2^{4(2)}}-\frac{\chi(2v)}{2^4}\right)+\rho\left(\frac{\chi(2v)}{2^4}-\chi(v)\right) \\ &\leq \frac{1}{2^{4(2)}}\tau(2v,0,\cdots,0)+\frac{1}{2^4}\tau(v,0,\cdots,0)\end{aligned}$$

for all $v \in A$. By induction, we can easily show that

$$\begin{aligned}
\rho\left(\frac{\chi\left(2^p v\right)}{2^{4p}}-\chi(v)\right) &\leq \sum_{j=1}^{p} \frac{1}{2^{4j}} \tau\left(2^{j-1} v, 0, \cdots, 0\right) \\
&\leq \frac{1}{L 2^4} \tau(v, 0, \cdots, 0) \sum_{j=1}^{p} L^j \\
&\leq \frac{1}{2^4(1-L)} \tau(v, 0, \cdots, 0) \qquad (10)
\end{aligned}$$

for all $v \in A$. By (10), we have

$$\begin{aligned}
\rho\left(\frac{\chi\left(2^p v\right)}{2^{4p}}-\frac{\chi\left(2^q v\right)}{2^{4q}}\right) &\leq \frac{1}{2} \rho\left(2 \frac{\chi\left(2^p v\right)}{2^{4p}}-2 \chi(v)\right)+\frac{1}{2} \rho\left(2 \frac{\chi\left(2^q v\right)}{2^{4q}}-2 \chi(v)\right) \\
&\leq \frac{k}{2} \rho\left(\frac{\chi\left(2^p v\right)}{2^{4p}}-\chi(v)\right)+\frac{k}{2} \rho\left(\frac{\chi\left(2^q v\right)}{2^{4q}}-\chi(v)\right) \\
&\leq \frac{k}{2^4(1-L)} \tau(v, 0, \cdots, 0)
\end{aligned}$$

for all $v \in A$ and $p, q \in \mathbb{N}$. We conclude that

$$\overline{\rho}\left(\Phi^p \chi-\Phi^q \chi\right) \leq \frac{k}{2^4(1-L)}.$$

This means that an orbit of Φ at χ is bounded. Thus the sequence $\{\Phi^p \chi\}$ is $\overline{\rho}$-convergent to $Q_4 \in \Upsilon_{\overline{\rho}}$, by Theorem 2.5. Now, by the $\overline{\rho}$-contractivity of Φ, we have

$$\overline{\rho}\left(\Phi^p \chi-\Phi Q_4\right) \leq L \overline{\rho}\left(\Phi^{p-1} \chi-Q_4\right).$$

If we take the limit as $p \rightarrow \infty$ and apply $\overline{\rho}$'s Fatou property, we get

$$\overline{\rho}\left(\Phi Q_4-Q_4\right) \leq \lim_{p \rightarrow \infty} \inf \overline{\rho}\left(\Phi Q_4-\Phi^p \chi\right) \leq L \lim_{p \rightarrow \infty} \inf \overline{\rho}\left(Q_4-\Phi^{p-1} \chi\right)=0.$$

Thus Q_4 is a fixed point of Φ. Replacing $(v_1, v_2, \cdots, v_m)$ by $(2^p v_1, 2^p v_2, \cdots, 2^p v_m)$ in (4), we get

$$\rho\left(\Theta \chi\left(2^p v_1, 2^p v_2, \cdots, 2^p v_m\right)\right) \leq \tau\left(2^p v_1, 2^p v_2, \cdots, 2^p v_m\right)$$

for all $v_1, v_2, \cdots, v_m \in A$. Therefore,

$$\rho\left(\frac{1}{2^{4p}} \Theta \chi\left(2^p v_1, 2^p v_2, \cdots, 2^p v_m\right)\right) \leq \frac{1}{2^{4p}} \tau\left(2^p v_1, 2^p v_2, \cdots, 2^p v_m\right).$$

Taking the limit as $p \rightarrow \infty$, we get

$$\Theta Q_4\left(v_1, v_2, \cdots, v_m\right)=0$$

for all $v_1, v_2, \cdots, v_m \in A$. It follows from Theorem 3 that Q_4 is quartic. By (10), we get (5).

Let $Q_4^{\prime}: A \rightarrow B_\rho$ be another quartic mapping that meets (5). Then

$$\overline{\rho}\left(Q_4-Q_4^{\prime}\right)=\overline{\rho}\left(\Phi Q_4-\Phi Q_4^{\prime}\right) \leq L \overline{\rho}\left(Q_4-Q_4^{\prime}\right),$$

which implies that

$$\overline{\rho}\left(Q_4-Q_4^{\prime}\right)=0 \Rightarrow Q_4=Q_4^{\prime}.$$

Therefore, the mapping Q_4 is the unique solution. This ends the proof. □

COROLLARY 4.2. *Let $\tau : A^m \to [0, +\infty)$ be a function such that*

$$\tau\left(2v_1, 2v_2, \cdots, 2v_m\right) \leq L2^4 \tau\left(v_1, v_2, \cdots, v_m\right)$$

for all $v_1, v_2, \cdots, v_m \in A$ with $0 < L < 1$. If an even mapping $\chi : A \to B$ fulfills $\chi(0) = 0$ and

$$\|\Theta\chi(v_1, v_2, \cdots, v_m)\| \leq \tau(v_1, v_2, \cdots, v_m)$$

for all $v_1, v_2, \cdots, v_m \in A$, then there is only one quartic solution $Q_4 : A \to B$ satisfying

$$\|Q_4(v) - \chi(v)\| \leq \frac{1}{2^4(1-L)} \tau(v, 0, \cdots, 0)$$

for all $v \in A$.

Proof. Each normed space is known to be a $\rho(v) := \|v\|$-modular space and to have the Δ_2-condition with $k = 2$. □

REMARK 4.3. If we replace $\tau(v_1, v_2, \cdots, v_m)$ by $\theta\left(\sum_{j=1}^{m} \|v_j\|^s\right)$ and take $L = 2^{s-4}$ in Corollary 4.2, then we have

$$\|Q_4(v) - \chi(v)\| \leq \frac{\theta\|v\|^s}{(2^4 - 2^s)}, \ v \in A,$$

where θ and s are constants with $s < 4$.

REMARK 4.4. If we replace $\tau(v_1, v_2, \cdots, v_m)$ by $\theta\left(\sum_{j=1}^{m} \|v_j\|^{ms} + \prod_{j=1}^{m} \|v_j\|^s\right)$ and take $L = 2^{ms-4}$ in Corollary 4.2, then we have

$$\|Q_4(v) - \chi(v)\| \leq \frac{\theta\|v\|^{ms}}{(2^4 - 2^{ms})}, \ v \in A,$$

where θ and s are constants with $ms < 4$.

THEOREM 4.5. *Let $\tau : A^m \to [0, +\infty)$ satisfy*

$$\tau\left(\frac{v_1}{2}, \frac{v_2}{2}, \cdots, \frac{v_m}{2}\right) \leq \frac{L}{2^4} \tau\left(v_1, v_2, \cdots, v_m\right)$$

for all $v_1, v_2, \cdots, v_m \in A$ with $0 < L < 1$. If an even mapping $\chi : A \to B_\rho$ fulfills $\chi(0) = 0$ and

$$\rho\left(\Theta\chi\left(v_1, v_2, \cdots, v_m\right)\right) \leq \tau(v_1, v_2, \cdots, v_m) \tag{11}$$

for all $v_1, v_2, \cdots, v_m \in A$, then there is only one quartic solution $Q_4 : A \to B_\rho$ satisfying

$$\rho\left(Q_4(v) - \chi(v)\right) \leq \frac{L}{2^4(1-L)} \tau(v, 0, \cdots, 0) \tag{12}$$

for all $v \in A$.

Proof. Let us define a set

$$\Upsilon = \{s : A \to B_\rho\}$$

and a function $\overline{\rho}$ on Υ as

$$\overline{\rho}(s) =: \inf\{\lambda > 0 : \rho(s(v)) \leq \lambda\tau(v, 0, \cdots, 0), \ \forall \ v \in A\}.$$

We have the same evidence as Theorem 4.1:

(1) The modular $\overline{\rho}$ is a convex on Υ.
(2) $\Upsilon_{\overline{\rho}}$ is $\overline{\rho}$-complete.

(3) $\overline{\rho}$ has the Fatou property.

Now, consider the mapping $\Phi : \Upsilon_{\overline{\rho}} \to \Upsilon_{\overline{\rho}}$ defined by:

$$\Phi s(v) = 2^4 s\left(\frac{v}{2}\right)$$

for all $v \in A$ and $s \in \Upsilon_{\overline{\rho}}$. For $s, t \in \Upsilon_{\overline{\rho}}$ and an arbitrary constant $\lambda \in [0,1]$ with $\overline{\rho}(s-t) < \lambda$, by the definition of $\overline{\rho}$, we have

$$\rho(s(v) - t(v)) \leq \lambda\tau(v, 0, \cdots, 0), \ v \in A.$$

It follows from the assumption and the above inequality that

$$\begin{aligned}
\rho\left(2^4 s\left(\frac{v}{2}\right) - 2^4 t\left(\frac{v}{2}\right)\right) &\leq k^4\rho\left(s\left(\frac{v}{2}\right) - t\left(\frac{v}{2}\right)\right) \\
&\leq k^4\lambda\tau\left(\frac{v}{2}, 0, \cdots, 0\right) \\
&\leq \lambda L\tau(v, 0, \cdots, 0), \ v \in A.
\end{aligned}$$

Hence

$$\overline{\rho}(\Phi s - \Phi t) \leq L\overline{\rho}(s-t), \ s, t \in \Upsilon_{\overline{\rho}}.$$

That is, Φ is a $\overline{\rho}$-contraction. We prove then that Φ has a bounded orbit at χ. Setting $(v_1, v_2, \cdots, v_m) = (v, 0, \cdots, 0)$ in (11), we get

$$\rho\left(2^4\chi(v) - \chi(2v)\right) \leq \psi(v, 0, \cdots, 0) \tag{13}$$

for all $v \in A$. Replacing v by $\frac{v}{2}$ in (13), we get

$$\rho\left(2^4\chi\left(\frac{v}{2}\right) - \chi(v)\right) \leq \tau\left(\frac{v}{2}, 0, \cdots, 0\right) \tag{14}$$

for all $v \in A$. Replacing v by $\frac{v}{2}$ in (14), we get

$$\rho\left(2^4\chi\left(\frac{v}{2^2}\right) - \chi\left(\frac{v}{2}\right)\right) \leq \tau\left(\frac{v}{2^2}, 0, \cdots, 0\right) \tag{15}$$

for all $v \in A$. By (13), (14) and (15), we get

$$\begin{aligned}
\rho\left(2^{4(2)}\chi\left(\frac{v}{2^2}\right) - \chi(v)\right) &\leq \rho\left(2^{4(2)}\chi\left(\frac{v}{2^2}\right) - 2^4\chi\left(\frac{v}{2}\right)\right) + \rho\left(2^4\chi\left(\frac{v}{2}\right) - \chi(v)\right) \\
&\leq k^3\rho\left(2^4\chi\left(\frac{v}{2^2}\right) - \chi\left(\frac{v}{2}\right)\right) + \rho\left(2^4\chi\left(\frac{v}{2}\right) - \chi(v)\right) \\
&\leq 2^4\tau\left(\frac{v}{2^2}, 0, \cdots, 0\right) + \tau\left(\frac{v}{2}, 0, \cdots, 0\right)
\end{aligned}$$

for all $v \in A$. By induction, we can easily show that

$$\begin{aligned}
\rho\left(2^{4p}\chi\left(\frac{v}{2^p}\right) - \phi(v)\right) &\leq \frac{1}{2^4}\sum_{j=1}^{p} 2^{4j}\tau\left(\frac{v}{2^j}, 0, \cdots, 0\right) \\
&\leq \frac{1}{2^4}\tau(v, 0, \cdots, 0)\sum_{j=1}^{p} L^j \\
&\leq \frac{L}{2^4(1-L)}\tau(v, 0, \cdots, 0) \qquad (16)
\end{aligned}$$

for all $v \in A$. It follows from (16) that

$$\begin{aligned}
&\rho\left(2^{4p}\chi\left(\frac{v}{2^p}\right) - 2^{4q}\chi\left(\frac{v}{2^q}\right)\right) \\
&\leq \frac{1}{2}\rho\left(2\cdot 2^{4p}\chi\left(\frac{v}{2^p}\right) - 2\chi(v)\right) + \frac{1}{2}\rho\left(2\cdot 2^{4q}\chi\left(\frac{v}{2^q}\right) - 2\chi(v)\right) \\
&\leq \frac{kL}{2^4(1-L)}\tau(v, 0, \cdots, 0)
\end{aligned}$$

for all $v \in A$ and all $p, q \in \mathbb{N}$. So we can conclude that

$$\overline{\rho}\left(\Phi^p\chi - \Phi^q\chi\right) \leq \frac{kL}{2^4(1-L)}.$$

This means that Φ orbit is limited to χ. The sequence $\{\Phi^p\chi\}$ is $\overline{\rho}$-convergent to $Q_4 \in \Upsilon_{\overline{\rho}}$ by Theorem 2.5.

Now, by the $\overline{\rho}$-contractivity of Φ, we have

$$\overline{\rho}\left(\Phi^p\chi - \Phi Q_4\right) \leq L\overline{\rho}\left(\Phi^{p-1}\chi - Q_4\right).$$

Taking the limit as $p \to \infty$ and using the Fatou property of $\overline{\rho}$, we get

$$\overline{\rho}\left(\Phi Q_4 - Q_4\right) \leq \lim_{p\to\infty}\inf \overline{\rho}\left(\Phi Q_4 - \Phi^p\chi\right) \leq L\lim_{p\to\infty}\inf \overline{\rho}\left(Q_4 - \Phi^{p-1}\chi\right) = 0.$$

Therefore, Q_4 is a fixed point of Φ. Replacing $(v_1, v_2, \cdots, v_m)$ by $\left(\frac{v_1}{2^p}, \frac{v_2}{2^p}, \cdots, \frac{v_m}{2^p}\right)$ in (11), we get

$$\rho\left(\Theta\chi\left(\frac{v_1}{2^p}, \frac{v_2}{2^p}, \cdots, \frac{v_m}{2^p}\right)\right) \leq \tau\left(\frac{v_1}{2^p}, \frac{v_2}{2^p}, \cdots, \frac{v_m}{2^p}\right)$$

for all $v_1, v_2, \cdots, v_m \in A$. Therefore,

$$\rho\left(2^{4p}\Theta\chi\left(\frac{v_1}{2^p}, \frac{v_2}{2^p}, \cdots, \frac{v_m}{2^p}\right)\right) \leq k^{4p}\tau\left(\frac{v_1}{2^p}, \frac{v_2}{2^p}, \cdots, \frac{v_m}{2^p}\right).$$

Taking to the limit as$p \to \infty$, we get

$$\Theta Q_4\left(v_1, v_2, \cdots, v_m\right) = 0$$

for all $v_1, v_2, \cdots, v_m \in A$. It follows from Theorem 3 that Q_4 is quartic. By (16), we get (12).

In order to prove the uniqueness of Q_4, consider another quartic solution $Q_4^{'} : A \to B_\rho$ satisfying (5). Then

$$\overline{\rho}\left(Q_4 - Q_4^{'}\right) = \overline{\rho}\left(\Phi Q_4 - \Phi Q_4^{'}\right) \leq L\overline{\rho}\left(Q_4 - Q_4^{'}\right)$$

which implies that $\overline{\rho}\left(Q_4 - Q_4^{'}\right) = 0$, i.e., $Q_4 = Q_4^{'}$. This ends the proof. □

Corollary 4.6. *Let $\tau : A^m \to [0, +\infty)$ satisfy*

$$\tau\left(\frac{v_1}{2}, \frac{v_2}{2}, \cdots, \frac{v_m}{2}\right) \leq \frac{L}{2^4}\tau\left(v_1, v_2, \cdots, v_m\right), \ v_1, v_2, \cdots, v_m \in A,$$

with $0 < L < 1$. If an even mapping $\chi : A \to B$ fulfills $\chi(0) = 0$ and

$$\|\Theta\chi(v_1, v_2, \cdots, v_m)\| \leq \tau(v_1, v_2, \cdots, v_m),$$

for all $v_1, v_2, \cdots, v_m \in A$, then there is only one quartic solution $Q_4 : A \to B$ fulfilling

$$\|Q_4(v) - \chi(v)\| \leq \frac{L}{2^4(1-L)}\tau(v, 0, \cdots, 0)$$

for all $v \in A$.

Proof. Each normed space is known to be a $\rho(v) := \|v\|$-modular space and to hve the Δ_2-condition with $k = 2$. $\square$

REMARK 4.7. If we replace $\tau(v_1, v_2, \cdots, v_m)$ by $\theta\left(\sum_{j=1}^{m} \|v_j\|^s\right)$ and take $L = 2^{4-s}$ in Corollary 4.6, then we have

$$\|Q_4(v) - \chi(v)\| \leq \frac{\theta\|v\|^s}{(2^s - 2^4)}, \ v \in A,$$

where θ and s are constants with $s > 4$.

REMARK 4.8. If we replace $\tau(v_1, v_2, \cdots, v_m)$ by $\theta\left(\sum_{j=1}^{m} \|v_j\|^{ms} + \prod_{j=1}^{m} \|v_j\|^s\right)$ and take $L = 2^{4-ms}$ in Corollary 4.6, then we have

$$\|Q_4(v) - \chi(v)\| \leq \frac{\theta\|v\|^{ms}}{(2^{ms} - 2^4)}, \ v \in A,$$

where θ and s are constants with $ms > 4$.

5. Counterexample

In this section, we examine an example to justify that the stability of the equation (2) fails for a singular case. Investigated by the excellent example provided by Gajda in [8], we present the upcoming counterexample which shows the instability in a particular condition $s = 4$ in Remark 4.3 of the equation (2).

REMARK 5.1. If a mapping $\chi : \mathbb{R} \to A$ fulfills (2), then the below assertions hold:

(1) $\chi(m^{c/4}v) = m^c\phi(v), \ v \in \mathbb{R}, m \in \mathbb{Q}$ and $c \in \mathbb{Z}$.
(2) $\chi(v) = v^4\chi(1), \ v \in \mathbb{R}$ if χ is continuous.

EXAMPLE 5.2. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\chi(v) = \sum_{p=0}^{\infty} \frac{\psi\left(2^p v\right)}{2^{4p}},$$

where

$$\psi(v) = \begin{cases} \lambda v^4, & -1 < v < 1 \\ \lambda, & \text{elsewhere.} \end{cases}$$

Then the function $\chi : \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{aligned} &|\Theta\chi(v_1, v_2, \cdots, v_m)| \\ &\leq \left(\frac{15m^5 - 238m^4 + 1117m^3 - 1862m^2 + 1376m}{24}\right)\left(\frac{2^{12}}{15}\right)\lambda\left(\sum_{j=1}^{m}|v_j|^4\right) \end{aligned} \tag{17}$$

for all $v_1, v_2, \cdots, v_m \in \mathbb{R}$, but there is not any quartic mapping $Q_4 : \mathbb{R} \to \mathbb{R}$ satisfying

$$|\chi(v) - Q_4(v)| \leq \delta|v|^4, \quad v \in \mathbb{R}, \tag{18}$$

where λ and δ are constants.

Proof. We can easily find that χ is bounded by $\frac{2^4}{15}\lambda$ on $\mathbb{R}$. If $\sum_{j=1}^{m}|v_j|^4 \geq \frac{1}{2^4}$ or 0, then

$$|\Theta\chi(v_1, v_2, \cdots, v_m)| < \left(\frac{15m^5 - 238m^4 + 1117m^3 - 1862m^2 + 1376m}{24}\right)\frac{2^4}{15}\lambda.$$

Thus (17) is valid. Next, suppose that

$$0 < \sum_{j=1}^{m}|v_j|^4 < \frac{1}{2^4}.$$

Then there exists an integer $l > 0$ satisfying

$$\frac{1}{2^{4(l+2)}} \leq \sum_{j=1}^{m}|v_j|^4 < \frac{1}{2^{4(l+1)}}. \tag{19}$$

So $2^{4l}|v_1| < \frac{1}{2^4}, 2^{4l}|v_2| < \frac{1}{2^4}, \cdots, 2^{4l}|v_m| < \frac{1}{2^4}$ and

$$\left.\begin{array}{r} 2^tv_1, 2^tv_2, \cdots, 2^tv_m \\ \sum_{i=1}^{m}\left(-2^tv_i + \sum_{j=1}^{m}2^tv_j\right) \\ \sum_{1\leq i<j<k<l\leq m}\left(2^t(v_i+v_j+v_k+v_l)\right) \\ \sum_{1\leq i<j<k\leq m}\left(2^t(v_i+v_j+v_k)\right) \\ \sum_{1\leq i<j\leq m}\left(2^t(v_i+v_j)\right) \\ \sum_{1\leq i<j\leq m}\left(2^t(v_i-v_j)\right) \end{array}\right\} \in]-1,1[, \quad t = 0, 1, \cdots, l-1.$$

Also, for $t = 0, 1, \cdots, l-1$,

$$\Theta\psi\left(v_1, v_2, \cdots, v_m\right) = 0.$$

Next, by (19), we obtain that

$$\begin{aligned} |\Theta\chi(v_1, v_2, \cdots, v_m)| &\leq \sum_{t=0}^{\infty}\frac{1}{2^{4t}}|\Theta\psi(2^tv_1, 2^tv_2, \cdots, 2^tv_m)| \\ &\leq \sum_{t=0}^{\infty}\frac{1}{2^{4t}}\Big|\sum_{i=1}^{m}\psi\left(-2^tv_i + \sum_{j=1}^{m}2^tv_j\right) \\ &\quad -(m-8)\sum_{1\leq i<j<k<l\leq m}\psi\left(2^t(v_i+v_j+v_k+v_l)\right) \\ &\quad +\left(m^2-12m+28\right)\sum_{1\leq i<j<k\leq m}\psi\left(2^t(v_i+v_j+v_k)\right) \\ &\quad -\left(\frac{m^3-15m^2+60m-68}{2}\right)\sum_{1\leq i<j\leq m}\psi\left(2^t(v_i+v_j)\right) \end{aligned}$$

$$-2\sum_{1\le i<j\le m}\psi\left(2^t(v_i-v_j)\right)-\sum_{i=1}^{m}\psi\left(2^t(2v_i)\right)$$
$$+\left(\frac{m^4-17m^3+86m^2-148m+168}{6}\right)\sum_{i=1}^{m}\psi\left(2^t(v_i)\right)\Bigg|$$
$$\le\sum_{t=l}^{\infty}\frac{1}{2^{4t}}\left(\frac{15m^5-238m^4+1117m^3-1862m^2+1376m}{24}\right)\lambda.$$

It follows from (19) that

$$|\Theta\chi(v_1,v_2,\cdots,v_m)|$$
$$\le\left(\frac{15m^5-238m^4+1117m^3-1862m^2+1376m}{24}\right)\frac{2^{12}}{15}\lambda\left(\sum_{j=1}^{m}|v_j|^4\right).$$

Thus the function χ fulfills the inequality (17).
Assume that a contrary that there exists a quartic solution $Q_4:\mathbb{R}\to\mathbb{R}$ which fulfills (18).

We know that χ is bounded and continuous.

In the perspective of Remark 5.1, Q_4 must be $Q_4(v)=cv^4,\ v\in\mathbb{R}.$ So we get

$$|\chi(v)|\le(\delta+|c|)\,|v|^4,\ v\in\mathbb{R}.$$

However, we can choose $l>0$ with $l\lambda>\delta+|c|$. If $v\in\left(0,\frac{1}{2^{l-1}}\right)$, then $2^tv\in(0,1)$ for all $t=0,1,\cdots,l-1$ and so we obtain

$$\chi(v)=\sum_{t=0}^{\infty}\frac{\psi(2^tv)}{2^{4t}}\ge\sum_{t=0}^{l-1}\frac{\lambda(2^tv)^4}{2^{4t}}=l\lambda v^4>(\delta+|c|)\ \ |v^4|,$$

which is a contradiction. □

6. Conclusion

A novel variety of quartic functional equation has been introduced and found a general solution for it in this research. The Hyers-Ulam stability results of this equation for fixed point problems in modular spaces employing the Fatou property fulfilling the Δ_2-condition were primarily examined. We have also included some comments to demonstrate the validity of our results' hypotheses.

Declarations

Availability of data and materials

No datasets were generated or analyzed during the current study.

Human and animal rights

We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest

The authors declare that they have no competing interests.

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