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SEQUENTIAL INTERVAL ESTIMATION FOR THE EXPONENTIAL HAZARD RATE WHEN THE LOSS FUNCTION IS STRICTLY CONVEX

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Abstract. Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables having common exponential density with unknown mean μ. In the sequential confidence interval estimation for the exponential hazard rate $\theta = 1/\mu$, when the loss function is strictly convex, the following stopping rule is proposed with the half length d of prescribed confidence interval I_n for the parameter θ;

$$\tau = \text{ smallest integer } n \text{ such that } n \geq z^2_{\alpha/2}\widehat{\theta}^2/d^2 + 2,$$

where $\widehat{\theta} = (n-1)\overline{X}_n^{\,-1}/n$ is the minimum risk estimator for θ and $z_{\alpha/2}$ is defined by $P(|Z| \leq \alpha/2) = 1 - \alpha$ $(\alpha \in (0,1))$ with $Z \sim N(0,1)$. For the confidence intervals I_n which is required to satisfy $P(\theta \in I_n) \geq 1 - \alpha$. These estimated intervals I_τ have the asymptotic consistency of the sequential procedure;

$$\lim_{d\to 0} P(\theta \in I_\tau) = 1 - \alpha,$$

where $\alpha \in (0,1)$ is given.

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1. Introduction

The exponential distribution is often used to model the time between independent events that happen at a constant average rate. In application, the exponential distribution can be used to model lifetimes of various practical situations including but not limited to lengths of times between successive catastrophic events and lengths of time between emergency arrivals at a hospital, to cite a few [7, 10].

Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed (IID) random variables having common exponential probability density function (PDF) with unknown mean μ which is given by

$$f_\mu(x) = \frac{1}{\mu} e^{-x/\mu} \times \mathrm{I}_{(0,\infty)}(x), \tag{1.1}$$

where $\mathrm{I}_A(\cdot)$ is the indicator function on the set A. The hazard function $h(t)$ of a random variable X at time t is defined by

$$h(t) = \lim_{\Delta t \to 0} \frac{P\{t \le X < t + \Delta t \mid X \ge t\}}{\Delta t}. \tag{1.2}$$

In the exponential case, the hazard function is represented by the inverse of the mean;

$$h(t) = \frac{f_\mu(t)}{\int_t^\infty f_\mu(x)dx} = \frac{1}{\mu} \equiv \theta, \quad \text{say}. \tag{1.3}$$

Models with constant hazard functions are unique and are often useful as baseline distributions to which other distributions are compared or as simple models for failure modes resulting in random failures. The exponential distribution can be characterized as the only distribution with a constant hazard rate [7, 10].

Estimation for the parameter is one of the most common forms of statistical inference. One measures a physical quantity in order to estimate its value [6]. The estimator $\widehat{\theta}$ is to be close to θ, we shall interpret this to mean that it will be close on the average. To make this requirement precise, it is necessary to specify a measure of the average closeness of an estimate to θ. The accuracy of an estimator $\widehat{\theta}$ is measured by the risk function

$$R(\theta, \widehat{\theta}) = E_\theta\left\{L(\theta, \widehat{\theta})\right\}, \tag{1.4}$$

where L is some loss function, for example, $L(\theta, \widehat{\theta}) = (\theta - \widehat{\theta})^2 + cn$ for the cost c per unit. The best estimator is $\widehat{\theta}$ which minimizes the risk for all θ. But there exists no uniformly best estimator. Sometimes the maximimu likelihood estimator (MLE) or the uniformly minimum variance unbiased estimator (UMVUE) is used as the good estimator of θ. When the loss function is strictly convex, the UMVUE for the parameter θ is the minimum risk estimator (MRE) by the Rao-Blackwell Theorem.

Takada [9] pointed out that fixed sample size procedure are not available for scale families. Thus, it is necessary to find a sequential sampling rule. Juhlin [4] studied the sequential estimation for the exponential mean parameter and Junvie [7] proposed the sequential confidence interval estimation for the exponential hazard rate using the MLE.

In this paper, estimating sequential confidence intervals for the exponential hazard rate $\theta = 1/\mu$, when the loss function is strictly convex, the following stopping rule is proposed with the half length d of prescribed confidence interval I_n for the parameter θ;

$$\tau = \text{ smallest integer } n \text{ such that } n \geq z_{\alpha/2}^2 \widehat{\theta}^2 / d^2 + 2, \tag{1.5}$$

where $\widehat{\theta}$ is the minimum risk estimator for the θ and $z_{\alpha/2}$ is defined by $P(|Z| \leq \alpha/2) = 1 - \alpha$ ($\alpha \in (0,1)$) with $Z \sim N(0,1)$. For the confidence intervals I_n which is required to satisfy $P(\theta \in I_n) \geq 1 - \alpha$, These estimated intervals I_τ have the asymptotic consistency of the sequential procedure;

$$\lim_{d \to 0} P(\theta \in I_\tau) = 1 - \alpha, \tag{1.6}$$

where $\alpha \in (0,1)$ is given.

2. Main Results

A sequence of random variables $\{X_n\}$ has asymptotically normal with mean μ_n and variance σ_n^2, briefly $X_n \sim AN(\mu_n, \sigma_n^2)$, if $\sigma_n^2 > 0$ for all n sufficiently large and

$$\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{\mathscr{D}} Z \sim N(0,1), \tag{2.1}$$

where $\xrightarrow{\mathscr{D}}$ stands for the convergence in distribution. The following lemma comes from Serfling [8].

LEMMA 2.1. *If $X_n \sim AN(\mu_n, \sigma_n^2)$, then also $a_n X_n + b_n \sim AN(\mu_n, \sigma_n^2)$ if and only if*

$$a_n \to 1, \ \frac{\mu_n(a_n - 1) + b_n}{\sigma_n} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.2}$$

In particular, if $\mu_n = 0, \sigma_n^2 = 1$, then $|a_n|$ instead $a_n \to 1$.

THEOREM 2.2. *Let $X_1, X_2, \cdots, X_n$ be* IID *exponential random variables with hazard rate θ and let $T_n = (n-1)\overline{X}_n^{-1}/n$. Then T_n is the* UMVUE *for θ and T_n has $AN(\theta, \theta^2/(n-2))$.*

Proof. First, we prove that T_n is the UMVUE for θ. Since the complete sufficient statistic $\overline{X}_n$ is unbiased, $\overline{X}_n$ is the UMVUE. From Lehmann-Sheffe's Theorem it suffices to show that T_n is an unbiased estimator of θ. Since $S = \sum_{i=1}^n X_i$ has the Erlang distribution, Erlang(n, θ), thus we have

$$E_\theta(T_n) = (n-1)\int_0^\infty \frac{1}{s\Gamma(n)}\theta^n s^{n-1} e^{-\theta s} ds = \theta \tag{2.3}$$

because $\theta^{n-1}s^{n-2}e^{-\theta s}/\Gamma(n-1)$ is the PDF of Erlang distribution, Erlang $(n-1, \theta)$. Next, we prove the asymptotic normality of T_n. Now, the variance of T_n is

$$Var(T_n) = (n-1)^2 \int_0^\infty \frac{1}{s^2\Gamma(n)}\theta^n s^{n-1} e^{-\theta s} ds - \theta^2 = \frac{\theta^2}{n-2}. \tag{2.4}$$

Since $\overline{X}_n^{-1}$ is MLE of θ, it follows asymptotic normality. From the Lemma 2.1 we obtain that

$$T_n = \frac{(n-1)}{n}\overline{X}_n^{-1} \sim AN\left(\theta, \frac{\theta^2}{n-2}\right). \tag{2.5}$$

This proof is complete. □

From Rao-Blackwell's Theorem we have the following corollary.

COROLLARY 2.3. *When the loss function is strictly convex, the* UMVUE *T_n for θ in Theorem 2.2 is the* MRE.

For a good estimator $\widehat{\theta}$, let $I_n = [\widehat{\theta} - d, \widehat{\theta} + d]$ be a confidence interval for θ with confidence coefficient $1 - \alpha$, where $d > 0$ and $0 < \alpha < 1$, that is,

$$P(\theta \in I_n) = 1 - \alpha. \tag{2.6}$$

Now, the coverage probability for θ with $I_n : T_n \pm d$ $(d > 0)$ as the confidence interval is given by $P\{\theta \in I_n\} \geq 1 - \alpha$.

Set

$$n^* = \frac{z^2_{\alpha/2}}{d^2}\theta^2 + 2. \tag{2.7}$$

Since for sufficiently large n

$$P\left\{\frac{|T_n - \theta|}{\theta/\sqrt{n-2}} \leq z_{\alpha/2}\right\} \approx 1 - \alpha, \tag{2.8}$$

we then have

$$P\{\theta \in I_n\} \geq P\left\{|T_n - \theta| \leq \frac{z_{\alpha/2}}{\sqrt{n-2}}\theta\right\} \approx 1 - \alpha. \tag{2.9}$$

Since θ is unknown, n^* is also unknown. In sequential estimating the confidence interval I_n, consider the following stopping rule:

$$\tau = \text{smallest integer } n \text{ such that } n \geq \frac{z^2_{\alpha/2}}{d^2}T_n^2 + 2, \tag{2.10}$$

where $T_n = (n-1)\overline{X}^{-1}/n$ is the UMVUE of θ.

From the Chow-Robbins procedure [1], one has the following results.

Lemma 2.4. *Let n^* and τ be defined as in* (2.7) *and* (2.10), *respectively. Then the following statements hold:*

(1) $P\{\tau < \infty\} = 1$ for all $d > 0$,

(2) $\tau \to \infty$ with probability 1 as $d \to 0$,

(3) $\dfrac{\tau}{n^*} \to 1$ with probability 1 as $d \to 0$.

Lemma 2.5. *Let τ be defined as in* (2.10). *Then*

$$\frac{T_\tau - \theta}{\theta/\sqrt{\tau - 2}} \xrightarrow{\mathscr{D}} N(0,1) \text{ as } d \to 0.$$

Proof. For all $\mu > 0$, by the Taylor expansion of $h(x) = (n-1)/nx$ at $x = \mu$, we have

$$T_n = \frac{n-1}{n}\theta - \frac{n-1}{n}\theta^2(\overline{X}_n - \mu) + R_n, \tag{2.11}$$

where $R_n = (n-1)(\overline{X}_n - \mu)^2/2n\xi^2$ and ξ is a random variable lying between $\overline{X}_n$ and μ. This yields

$$\frac{T_n - \theta}{\theta/\sqrt{n-2}} = a_n Z_n + b_n + R_n^*, \tag{2.12}$$

where $a_n = -(n-1)\sqrt{n-2}/n\sqrt{n}$, $b_n = -\sqrt{n-2}/n$,
$Z_n = \sqrt{n}\left(\overline{X}_n - \mu\right)/\mu$, and

$$\left|R^*(\overline{X}_n, \xi)\right| \le \frac{M\left|\overline{X}_n - \xi\right|^2}{2} \tag{2.13}$$

for some $M < \infty$. By the Central Limit Theorem, $Z_n \xrightarrow{\mathscr{D}} Z \sim N(0,1)$. From Lemma 1.1 we know that

$$a_n Z_n + b_n \xrightarrow{\mathscr{D}} N(0,1) \quad \text{as } n \to \infty. \tag{2.14}$$

Now, $\tau \to \infty$ as $d \to 0$. Since $a_n Z_n + b_n$ is uniform continuity in probability and stochastically bounded from Woodroofe [12], thus we have

$$a_\tau Z_\tau + b_\tau \xrightarrow{\mathscr{D}} Z \sim N(0,1) \quad \text{as} \quad d \to 0, \tag{2.15}$$

From the Strong Law of Large Numbers $\left|\overline{X}_\tau - \theta\right| \to 0$ with probability 1. As result, $R_\tau^* \to 0$ in probability as $d \to 0$. Consequently, from the Slutsky's Theorem the proof of this lemma is complete. □

THEOREM 2.6. *When the loss function is strictly convex, let τ be defined as in* (2.10)*. Then*

$$\lim_{d\to 0} P\left\{\theta \in I_\tau\right\} = 1 - \alpha.$$

Proof. From the definition of τ in (2.10), we have

$$d\sqrt{\tau - 2}/\theta \ge z_{\alpha/2}$$

and

$$P\left\{\theta \in I_\tau\right\} = P\left\{\frac{|T_\tau - \theta|}{\theta/\sqrt{\tau-2}} \le \frac{d\sqrt{\tau-2}}{\theta}\right\} \ge 1 - \alpha.$$

Set

$$Z_\tau = \frac{T_\tau - \theta}{\theta/\sqrt{\tau - 2}}. \tag{2.16}$$

For any fixed $\varepsilon > 0$,

$$\begin{aligned} P\left\{|Z_\tau| \le \frac{z_{\alpha/2}T_\tau}{\theta}\right\} =& P\left\{|Z_\tau| \le \frac{z_{\alpha/2}T_\tau}{\theta},\ \left|\frac{T_\tau}{\theta}-1\right| \le \varepsilon\right\} \\ &+ P\left\{|Z_\tau| \le \frac{z_{\alpha/2}T_\tau}{\theta},\ \left|\frac{T_\tau}{\theta}-1\right| > \varepsilon\right\} \\ \le& P\left\{|Z_\tau| \le \frac{z_{\alpha/2}T_\tau}{\theta},\ \left|\frac{T_\tau}{\theta}-1\right| \le \varepsilon\right\} \\ &+ P\left\{\left|\frac{T_\tau}{\theta}-1\right| > \varepsilon\right\} \\ \le& P\left\{|Z_\tau| \le z_{\alpha/2}(1+\varepsilon)\right\} + P\left\{\left|\frac{T_\tau}{\theta}-1\right| > \varepsilon\right\}. \end{aligned}$$

By the Strong Law of Large Numbers, $T_\tau \to \theta$ with probability 1 as $d \to 0$. Hence, $T_\tau/\theta \to 1$ in probability as $d \to 0$ and

$$\lim_{\varepsilon\to 0} P\left\{\left|\frac{T_\tau}{\theta}-1\right| > \varepsilon\right\} = 0.$$

Now, by Lemma 1.2, $Z_\tau \xrightarrow{\mathscr{D}} Z \sim N(0,1)$ as $d \to 0$. Thus, letting $d \to 0$ and taking $\varepsilon \to 0$ we have

$$\limsup_{d\to 0} P\left\{|Z_\tau| \le \frac{z_{\alpha/2}T_\tau}{\theta}\right\} \le 1-\varepsilon.$$

Similarly, we have

$$\begin{aligned} P\left\{|Z_\tau| \le \frac{z_{\alpha/2}T_\tau}{\theta}\right\} =& P\left\{|Z_\tau| \le \frac{z_{\alpha/2}T_\tau}{\theta},\ \left|\frac{T_\tau}{\theta}-1\right| \le \varepsilon\right\} \\ &- P\left\{\left|\frac{T_\tau}{\theta}-1\right| > \varepsilon\right\} \\ &+ P\left\{|Z_\tau| \le \frac{z_{\alpha/2}T_\tau}{\theta},\ \left|\frac{T_\tau}{\theta}-1\right| > \varepsilon\right\} \\ &+ P\left\{\left|\frac{T_\tau}{\theta}-1\right| > \varepsilon\right\} \\ \ge& P\left\{|Z_\tau| \le z_{\alpha/2}(1-\varepsilon)\right\} - P\left\{\left|\frac{T_\tau}{\theta}-1\right| > \varepsilon\right\}. \end{aligned}$$

Hence, taking $\varepsilon \to 0$ we get

$$\liminf_{d\to 0} P\left\{|Z_\tau| \le \frac{z_{\alpha/2}T_\tau}{\theta}\right\} \ge 1-\alpha$$

and thus we have

$$\lim_{d\to 0} P\left\{|Z_\tau| \le \frac{z_{\alpha/2}T_\tau}{\theta}\right\} \ge 1-\alpha.$$

As a result, $\liminf_{d\to 0} P\left\{\theta \in I_\tau\right\} \ge 1-\alpha$. Now, observe that

$$\tau - 1 < \frac{z_{\alpha/2}^2}{d^2}T_{\tau-1}^2 + 2$$

$$\frac{d}{\theta z_{\alpha/2}}\sqrt{\tau} < \sqrt{\frac{T_n^2}{\theta^2} + \frac{d^2}{\theta^2 z_{\alpha/2}^2}} \equiv k_d, \quad \text{say}$$

For any $\varepsilon > 0$,

$$\begin{aligned} P\left\{\theta \in I_\tau\right\} \le & P\left\{|Z_\tau| \le z_{\alpha/2}k_d,\ |k_d - 1| \le \varepsilon\right\} \\ & + P\left\{|Z_\tau| \le z_{\alpha/2}k_d,\ |k_d - 1| > \varepsilon\right\} \\ \le & P\left\{|Z_\tau| \le z_{\alpha/2}(1+\varepsilon)\right\} + P\left\{|k_d - 1| > \varepsilon\right\}. \end{aligned}$$

By the Strong Law of Large Number $k_d \to 1$ with probability 1 as $d \to 0$. Taking $\varepsilon \to 0$, it follows that

$$\limsup_{d\to 0} P\left\{\theta \in I_\tau\right\} \le 1-\alpha.$$

Therefore the proof is complete. □

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