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SOME LOWER BOUND ESTIMATES FOR THE GENERALIZED DERIVATIVE OF A POLYNOMIAL

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Abstract. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, then Rather et al. (Some inequalities for polynomials with restricted zeros, Ann. Univ. Ferrara, 67 (2021), 183-189.) proved that for all z on $|z| = 1$ for which $P(z) \neq 0$,

$$Re\left(z\frac{P'(z)}{P(z)}\right) \geq \frac{n}{1+k}\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}.$$

In this paper, we extend this inequality to the generalised derivative by taking s-folded zeros at origin. As an application, we obtain some lower bound estimates for the generalized derivative and generalized polar derivative of a polynomial with restricted zeros, which include various results due to Turán, Malik, Dubinin, Aziz, Rather and Govil as special cases.

1. Introduction

The issue of the extremal properties of polynomials gained attention in the latter half of the 19th century, partly due to the research of the renowned chemist Mendeleev, who sought to determine the limits of the derivative of a particular type of polynomial. Serge Bernstein later established a key result (for details, see [20]) concerning the estimation of the upper bound of the maximum modulus of the derivative of a polynomial based on the maximum modulus of the polynomial itself. He demonstrated that if P is a polynomial of degree less than or equal to n, then

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{1}$$

This insightful introduction to polynomial inequalities draws many researchers into the field and inspires them to seek refinements and generalisations of the existing findings for various types of polynomials. Let $\mathcal{P}_n$ represent the set of all polynomials of degree n over the field $\mathbb{C}$ of complex numbers. Regarding the estimation of the lower bound for the maximum modulus of the derivative of a polynomial based on the maximum modulus of the original polynomial, Paul Turán [21] demonstrated that if $P \in \mathcal{P}_n$ has all its roots within $|z| \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{2}$$

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The inequality (2) is sharp and equality holds for the polynomials having all its zeros on the unit disc. Now regarding the estimation of the lower bounds of $Re\left(\frac{zP'(z)}{P(z)}\right)$ on $|z|=1$, Dubinin [5] proved that:

THEOREM 1.1. *If $P \in \mathcal{P}_n$ having all its zeros in $|z| \leq 1$ then for all z on $|z|=1$ for which $P(z) \neq 0$,*

$$Re\left(z\frac{P'(z)}{P(z)}\right) \geq \frac{n}{2}\left\{1+\frac{1}{n}\left(\frac{|a_n|-|a_0|}{|a_n|+|a_0|}\right)\right\}. \tag{3}$$

As an application of this, Dubinin [5] in the same paper obtained an interesting refinement of (2) by involving the coefficients of the leading term and the constant term of the polynomial $P(z)$. He proved that if all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{1}{2}\left(n+\frac{|a_n|-|a_0|}{|a_n|+|a_0|}\right)\max_{|z|=1}|P(z)|. \tag{4}$$

The inequality (2) has many applications and so is of considerable interest. Thus one would seek to generalise this for polynomials having all zeros in the disc $|z| \leq k$, $k > 0$. The case for $0 < k \leq 1$ was done by Malik [9] who proved that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k}\max_{|z|=1}|P(z)|. \tag{5}$$

Equality in (5) holds for the polynomial $P(z)=(z+k)^n$.
While the case for $k \geq 1$ was done by Govil [6] who showed that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, $k \geq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k^n}\max_{|z|=1}|P(z)|. \tag{6}$$

It can be easily verified that equality in (6) holds for polynomials of the type $P(z) = z^n+k^n$. Rather et al. [15] generalised Theorem 1.1 without invoking boundary Schwarz lemma by proving the following result:

THEOREM 1.2. *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k, k \leq 1$, then for all z on $|z|=1$ for which $P(z) \neq 0$,*

$$Re\left(z\frac{P'(z)}{P(z)}\right) \geq \frac{n}{1+k}\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}. \tag{7}$$

REMARK 1.3. For $k=1$, Theorem 1.1 is a special case of Theorem 1.2.

As a consequence of Theorem 1.2, Rather et al. [15] proved the following:

THEOREM 1.4. *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k,\ k \leq 1$, then for $|z|=1$,*

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k}\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}\max_{|z|=1}|P(z)|. \tag{8}$$

Pertaining to the refinement of inequality (5) and generalisation of inequality (4) Rather et al. [15] proved the following result by taking s-folded zeros at origin:

THEOREM 1.5. *If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}) \in \mathcal{P}_n,\ 0 \le s \le n$ has all zeros in $|z| \le k \le 1$, then*

$$Re\left(z\frac{P'(z)}{P(z)}\right) \ge \frac{n}{1+k}\left\{1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right\}. \tag{9}$$

Moreover, Rather et al. [15] proved the following theorem for the s-folded zeros at origin:

THEOREM 1.6. *If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}) \in \mathcal{P}_n,\ 0 \le s \le n$ has all zeros in $|z| \le k \le 1$, then*

$$\max_{|z|=1}|P'(z)| \ge \frac{n}{1+k}\left\{1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right\}\max_{|z|=1}|P(z)|. \tag{10}$$

Equality in (10) *holds for $P(z) = z^s(z+k)^{n-s}, s < n$.*

DEFINITION 1.7 (Polar Derivative [14]). If $P \in \mathcal{P}_n$, then the *polar derivative* of $P(z)$ with respect to a complex number $\alpha \in \mathbb{C}$ is defined as:

$$D_\alpha[P](z) = nP(z) + (\alpha - z)P'(z).$$

REMARK 1.8. The polar derivative operator $D_\alpha[P](z)$ has the following key properties:

1. For any polynomial $P \in \mathcal{P}_n$, the polar derivative $D_\alpha[P](z)$ is a polynomial of degree at most $n-1$.
2. This operator generalizes the ordinary derivative in the following precise sense:
$$\lim_{\alpha\to\infty}\frac{D_\alpha[P](z)}{\alpha} = P'(z),$$
where the convergence is uniform on compact subsets of $\mathbb{C}$. Specifically, for any $R > 0$,
$$\sup_{|z|\le R}\left|\frac{D_\alpha[P](z)}{\alpha} - P'(z)\right| \to 0 \quad \text{as} \quad \alpha \to \infty.$$
3. The generalized polar derivative $D_\alpha^\gamma[P](z)$ (see definition 1.15) further extends this concept, reducing to the classical polar derivative when $\gamma = (1, 1, \ldots, 1)$.

Bernstein-type inequalities involving polar derivatives instead of ordinary derivatives have attracted considerable interest in polynomial analysis. In this direction, Aziz [2] pioneered the study of polar derivative inequalities by establishing bounds based on the polynomial's behavior in the unit disk. Building upon the classical inequality (1), Aziz [2] proved the following fundamental result for polar derivatives:

$$\max_{|z|=1}|D_\alpha[P](z)| \ge n\left(\frac{|\alpha|-1}{2}\right)\max_{|z|=1}|P(z)|, \tag{11}$$

where $P(z)$ is a polynomial of degree n and $|\alpha| \ge 1$. This inequality represents a direct polar analogue of Bernstein's inequality and serves as the foundation for subsequent generalizations, including our work on the generalized polar derivative $D_\alpha^\gamma[P](z)$. The result (11) is sharp, with equality holding for polynomials $P(z)$ having all their zeros at the origin, i.e., $P(z) = cz^n$, $c \ne 0$. For further details on the polar derivative, we refer to [8, 10–14].

For the class of polynomials with all zeros contained in $|z| \leq k$, Aziz and Rather [3] obtained several sharp bounds for the maximum modulus of the polar derivative $D_\alpha[P](z)$ on the unit circle. Their work includes the following significant extension of inequality (5) to the polar derivative:

THEOREM 1.9. *For any polynomial $P \in \mathcal{P}_n$ with all zeros in $|z| \leq k$ and any $\alpha \in \mathbb{C}$ satisfying $|\alpha| \geq k \leq 1$,*

$$\max_{|z|=1} |D_\alpha[P](z)| \geq \frac{n}{1+k}(|\alpha| - k) \max_{|z|=1} |P(z)|. \tag{12}$$

Equality in (12) holds for $P(z) = (z-k)^n$ and α is real with $\alpha \geq k$.

As an application of Theorem 1.2, Rather et al. [15] obtained the following refinement of inequality (12) for polynomials with s-fold zeros at the origin:

THEOREM 1.10. *Let $P(z) = z^s f(z) \in \mathcal{P}_n$, where*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-s} z^{n-s}$$

with $0 \leq s \leq n$, be a polynomial having all its zeros in $|z| \leq k \leq 1$. Then for any $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,

$$\max_{|z|=1} |D_\alpha[P](z)| \geq n\left(\frac{|\alpha| - k}{1+k}\right)\left[1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right] \max_{|z|=1} |P(z)|. \tag{13}$$

Equality in (13) holds for $P(z) = (z-k)^n$ and α is real with $\alpha \geq k$.

Numerous refinements and generalizations of these inequalities have been extensively studied in the literature (see, e.g., [4, 17]).

We now present a result of Govil and Kumar [7] concerning polynomials with all zeros lying in the closed disc $|z| \leq k$, where $k \geq 1$, and exactly s zeros (counting multiplicities) at the origin.

THEOREM 1.11 (Govil and Kumar [7]). *Let $P(z) = z^s f(z) \in \mathcal{P}_n$ be a polynomial with:*

- *s-fold zeros at the origin ($0 \leq s \leq n$),*
- *all remaining zeros in $|z| \leq k$ where $k \geq 1$.*

Then for any $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, the polar derivative $D_\alpha[P](z)$ satisfies:

$$\max_{|z|=1} |D_\alpha[P](z)| \geq \frac{n(|\alpha| - k)}{1+k^n}\left[1 + \frac{1}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right] \max_{|z|=1} |P(z)|. \tag{14}$$

REMARK 1.12. This result extends previous work on polar derivatives to the case of $k \geq 1$ while incorporating the effect of s-fold zeros at the origin. The inequality is sharp, with equality holding for $P(z) = (z+k)^n$ when α is real with $\alpha \geq k$.

DEFINITION 1.13 (Generalized Derivative [19]). Let $\mathcal{P}_n$ denote the space of complex polynomials of degree at most n. For $P(z) = a\prod_{j=1}^{n}(z - z_j) \in \mathcal{P}_n$ and a weight vector $\gamma = (\gamma_1, \ldots, \gamma_n) \in S$, where

$$S = \{\gamma \in \mathbb{R}^n : \gamma_j \geq 0 \text{ for } 1 \leq j \leq n\},$$

the generalized derivative of $P(z)$ with respect to γ is defined as

$$P^\gamma(z) := P(z)\sum_{j=1}^{n}\frac{\gamma_j}{z - z_j} = \sum_{j=1}^{n}\gamma_j P_j(z), \tag{15}$$

where $P_j(z) = a\prod_{\substack{i=1\\ i\neq j}}^{n}(z-z_i)$ is the polynomial obtained by removing the j-th root from $P(z)$.

Observe that for $\gamma = (1,1,1,\dots,1)$, the operator $P^{\gamma}(z)$ reduces to the ordinary derivative $P'(z)$. This motivates the terminology "generalized derivative" for $P^{\gamma}(z)$. This generalised derivative was introduced by Sz-Nagy (see [19]).

Recently, Bhat et al. [4] extended inequality (5) to generalised derivative by proving that:

Theorem 1.14. *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k \le 1$, then*

$$\max_{|z|=1}|P^{\gamma}(z)| \ge \frac{\wedge}{(1+k)}\max_{|z|=1}|P(z)|\,. \tag{16}$$

Definition 1.15 (Generalized Polar Derivative [4]). Let $\mathcal{P}_n$ denote the space of polynomials of degree at most n. For $P \in \mathcal{P}_n$, $\alpha \in \mathbb{C}$, and $\gamma \in S$, the generalized polar derivative of P is defined as:

$$D_{\alpha}^{\gamma}[P](z) := \Lambda P(z) + (\alpha - z)P^{\gamma}(z), \tag{17}$$

where $\Lambda = \sum_{j=1}^{n}\gamma_j$ and $P^{\gamma}(z)$ denotes the generalized derivative of $P(z)$.

This definition extends the classical notion of polar derivative, as when $\gamma = (1,1,\dots,1)$, we recover the standard polar derivative:

$$\begin{aligned} D_{\alpha}^{(1,\dots,1)}P(z) &= nP(z) + (\alpha - z)P'(z)\\ &= D_{\alpha}P(z), \end{aligned}$$

where $D_{\alpha}P(z)$ denotes the classical polar derivative.

Recently, Rather et al. [17] extended Theorem 1.14 to the generalised polar derivatives and proved the following:

Theorem 1.16. *If all the zeros of a polynomial $P \in \mathcal{P}_n$ lie in $|z| \le k$, where $k \le 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$,*

$$\max_{|z|=1}|D_{\alpha}^{\gamma}[P](z)| \ge \frac{\wedge}{1+k}(|\alpha| - k)\max_{|z|=1}|P(z)|. \tag{18}$$

Now concerning the estimate of lower bound of $|D_{\alpha}^{\gamma}[P](z)|$ on $|z| = 1$, Rather et al. [18] established the following result:

Theorem 1.17. *If all the zeros of a polynomial $P \in \mathcal{P}_n$ lie in $|z| \le k$, where $k \le 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$,*

$$\max_{|z|=1}|D_{\alpha}^{\gamma}[P](z)| \ge \frac{(|\alpha| - k)}{1+k}\left[\wedge + k\gamma_m\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\right]\max_{|z|=1}|P(z)|. \tag{19}$$

Remark 1.18. It is easy to verify that Theorem 1.17 refines Theorem 1.9.

2. Main Results

In this section, we present our principal findings which extend Theorems 1.5 and 1.6 to the Nagy derivative. Additionally, we establish generalized polar derivative analogues of Theorems 1.10 and 1.11, considering in all cases polynomials with s-fold zeros at the origin.

The section is organized as follows: Subsection 2.1 contains the precise statements and other special cases of our main theorems, Subsection 2.2 collects the necessary lemmas, and Subsection 2.3 presents the detailed proofs of the main results.

2.1. Statements and Implications of Main Theorems.

THEOREM 2.1. *If* $P(z) = z^s f(z) \in \mathcal{P}_n$, *where* $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-s} z^{n-s}$ *and* $0 \le s \le n$ *has all its zeros in the disc* $|z| \le k \le 1$, *then for all* z *on* $|z| = 1$ *for which* $P(z) \neq 0$,

$$Re\left(\frac{zP^{\gamma}(z)}{P(z)}\right) \ge \frac{k}{1+k}\left\{\frac{\wedge}{k} + \gamma_m\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right\}, \tag{20}$$

where $\gamma_m = \min\{\gamma_1, \gamma_2, ..., \gamma_n\}$.

REMARK 2.2. For $\gamma = (1, 1, \cdots, 1)$, the above theorem reduces to Theorem 1.5, which for $k = 1, s = 0$ yields Theorem 1.1.

Now concerning the estimation of the lower bounds of $|P^{\gamma}(z)|$ on $|z| = 1$, for the polynomials having all its zeros in the disc $|z| \le k,\ k \le 1$, we prove the following extension of Theorem 1.6 to generalised derivative.

THEOREM 2.3. *If* $P(z) = z^s f(z) \in \mathcal{P}_n$, *where* $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-s} z^{n-s}$ *and* $0 \le s \le n$ *has all its zeros in the disc* $|z| \le k \le 1$, *then*

$$\max_{|z|=1}|P^{\gamma}(z)| \ge \frac{k}{1+k}\left\{\frac{\wedge}{k} + \gamma_m\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right\}\max_{|z|=1}|P(z)|, \tag{21}$$

where $\gamma_m = \min\{\gamma_1, \gamma_2, ..., \gamma_n\}$ *and the result is best possible as shown by the polynomial* $P(z) = (z + k)^n$.

REMARK 2.4. If we take $\gamma = (1, 1, \cdots, 1)$ the above theorem reduces to Theorem 1.6.

Next, we extend Theorem 1.10 to the generalised polar derivative of a polynomial by proving the following result:

THEOREM 2.5. *If* $P(z) = z^s f(z) \in \mathcal{P}_n$, *where* $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-s} z^{n-s}$ *and* $0 \le s \le n$ *having all zeros in the disc* $|z| \le k \le 1$, *then for* $\alpha \in \mathbb{C}$ *with* $|\alpha| \ge k$,

$$\max_{|z|=1}|D_{\alpha}^{\gamma}[P](z)| \ge \frac{|\alpha| - k}{1+k}\left[\wedge + k\gamma_m\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right]\max_{|z|=1}|P(z)|, \tag{22}$$

where $\gamma_m = \min\{\gamma_1, \gamma_2, ..., \gamma_n\}$.
If we put $k = 1$ in inequality (22), we get the following result:

COROLLARY 2.6. *If all the zeros of a polynomial* $P \in \mathcal{P}_n$ *lie in* $|z| \le 1$, *then for* $\alpha \in \mathbb{C}$ *with* $|\alpha| \ge 1$,

$$\max_{|z|=1}|D_{\alpha}^{\gamma}[P](z)| \ge \frac{|\alpha| - 1}{2}\left\{\wedge + \gamma_m\left(s + \frac{|a_{n-s}| - |a_0|}{|a_{n-s}| + |a_0|}\right)\right\}\max_{|z|=1}|P(z)|, \tag{23}$$

where $\gamma_m = \min\{\gamma_1, \gamma_2, ..., \gamma_n\}$ *and the result is best possible as shown by the polynomial* $P(z) = (z + 1)^n$.

REMARK 2.7. By taking $\gamma = (1, 1, 1, ..., 1), s = 0$ and dividing both sides of inequality (23) by $|\alpha|$ and letting $|\alpha| \to \infty$, the inequality (23) reduces to the inequality (4).

Remark 2.8. For $\gamma = (1, 1, \ldots, 1)$, inequality (22) reduces to inequality (13).

Remark 2.9. If we divide both sides of inequality (22) by $|\alpha|$ and let $|\alpha| \to \infty$, it reduces to inequality (21).

We now extend Theorem 1.11 to the generalised polar derivative of a polynomial having all zeros in $|z| \leq k$, $k \geq 1$. The result thus obtained generalises many inequalities of the type (6) to the generalised polar derivative.

Theorem 2.10. *If $P(z) = z^s f(z) \in \mathcal{P}_n$ where $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-s} z^{n-s}$ and $0 \leq s \leq n$ having all zeros in the disc $|z| \leq k$, $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$*

$$\max_{|z|=1} |D_\alpha^\gamma[P](z)| \geq \frac{|\alpha| - k}{1 + k^n} \left[\wedge + \gamma_m \left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|} \right) \right] \max_{|z|=1} |P(z)|. \tag{24}$$

Remark 2.11. If we divide inequality (24) on both sides by $|\alpha|$ and then let $|\alpha| \to \infty$, we get

$$\max_{|z|=1} |P^\gamma(z)| \geq \frac{1}{1 + k^n} \left\{ \wedge + \gamma_m \left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|} \right) \right\} \max_{|z|=1} |P(z)|.$$

2.2. Preliminary Lemmas. The proofs rely on the following technical results. The first lemma is due to Rather et al. [15].

Lemma 2.12 ([15]). *For $0 \leq x_j \leq 1$, $j = 1, \ldots, n$,*

$$\sum_{j=1}^{n} \frac{1 - x_j}{1 + x_j} \geq \frac{1 - \prod_{j=1}^{n} x_j}{1 + \prod_{j=1}^{n} x_j}, \quad \forall\, n \in \mathbb{N}. \tag{25}$$

Next two lemmas are due to Rather et al. [17].

Lemma 2.13 ([17]). *For $P \in \mathcal{P}_n$ and $Q(z) = z^n \overline{P(1/\overline{z})}$, we have for $|z| = 1$,*

$$|Q^\gamma(z)| = |\Lambda P(z) - z P^\gamma(z)| \quad \text{and} \quad |P^\gamma(z)| = |\Lambda Q(z) - z Q^\gamma(z)|.$$

Lemma 2.14 ([17]). *If $P \in \mathcal{P}_n$ has all zeros in $|z| \leq k \leq 1$, then*

$$k|P^\gamma(z)| \geq |Q^\gamma(z)| \quad \text{for} \quad |z| = 1,$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Next lemma is due to Aziz [1].

Lemma 2.15 ([1]). *For $P \in \mathcal{P}_n$ with zeros in $|z| \leq k \geq 1$,*

$$\max_{|z|=k} |P(z)| \geq \frac{2k^n}{1 + k^n} \max_{|z|=1} |P(z)|.$$

Next lemma is a simple consequence of Maximium Modulus Theorem.

Lemma 2.16. *For $P \in \mathcal{P}_n$ and $k \geq 1$,*

$$\max_{|z|=k} |P(z)| \leq k^n \max_{|z|=1} |P(z)|. \tag{26}$$

2.3. Proofs of the Main Theorems.

Proof of Theorem 2.1. Let $z_j,\ j = s+1, s+2, \cdots, n$ be the $n-s$ zeros of $f(z)$, then

$$P^{\gamma}(z) = P(z)\sum_{j=1}^{n}\frac{\gamma_j}{z-z_j}$$
$$= P(z)\left\{\frac{1}{z}\sum_{j=1}^{s}\gamma_j + \sum_{j=s+1}^{n}\frac{\gamma_j}{z-z_j}\right\}.$$

Therefore, for all z on $|z|=1$ for which $P(z)\neq 0$, we have

$$\frac{P^{\gamma}(z)}{P(z)} = \frac{1}{z}\sum_{j=1}^{s}\gamma_j + \sum_{j=s+1}^{n}\frac{\gamma_j}{z-z_j}.$$

Now

$$f^{\gamma}(z) = f(z)\sum_{j=s+1}^{n}\frac{\gamma_j}{z-z_j},$$

so

$$\frac{f^{\gamma}(z)}{f(z)} = \sum_{j=s+1}^{n}\frac{\gamma_j}{z-z_j}.$$

Therefore, for all z on $|z|=1$ for which $P(z)\neq 0$, we obtain

$$\frac{P^{\gamma}(z)}{P(z)} = \frac{1}{z}\sum_{j=1}^{s}\gamma_j + \frac{f^{\gamma}(z)}{f(z)}.$$

Hence, for $|z|=1$,

$$Re\left(\frac{zP^{\gamma}(z)}{P(z)}\right) = Re\left(\frac{zf^{\gamma}(z)}{f(z)}\right) + \sum_{j=1}^{s}\gamma_j$$
$$\geq \sum_{j=s+1}^{n}\frac{\gamma_j}{1+|z_j|} + \sum_{j=1}^{s}\gamma_j$$
$$= \sum_{j=s+1}^{n}\frac{\gamma_j}{1+k}\left(\frac{k-|z_j|}{1+|z_j|}+1\right) + \sum_{j=1}^{s}\gamma_j$$
$$= \frac{1}{1+k}\sum_{j=s+1}^{n}\gamma_j\left(\frac{k-|z_j|}{1+|z_j|}\right) + \frac{1}{1+k}\sum_{j=s+1}^{n}\gamma_j + \sum_{j=1}^{s}\gamma_j$$
$$= \frac{1}{1+k}\sum_{j=s+1}^{n}\gamma_j\left(\frac{k-|z_j|}{1+|z_j|}\right) + \frac{1}{1+k}\sum_{j=1}^{n}\gamma_j + \frac{k}{k+1}\sum_{j=1}^{s}\gamma_j.$$

Let $\gamma_m = \min(\gamma_1, \gamma_2, \cdots, \gamma_n)$, then for $|z|=1$,

$$Re\left(\frac{zP^{\gamma}(z)}{P(z)}\right) \geq \frac{1}{1+k}\left\{\wedge + k\gamma_m\sum_{j=s+1}^{n}\left(\frac{k-|z_j|}{k+k|z_j|}\right) + k\gamma_m s\right\}.$$

Since $k \leq 1 \implies k|z_j| \leq |z_j|$,

$$k + k|z_j| \leq k + |z_j| \implies \frac{1}{k + k|z_j|} \geq \frac{1}{k + |z_j|}.$$

Hence for $|z| = 1$,

$$\begin{aligned} Re\left(\frac{zP^{\gamma}(z)}{P(z)}\right) &\geq \frac{1}{1+k}\left\{\wedge + k\gamma_m\left(\sum_{j=s+1}^{n}\left(\frac{k-|z_j|}{k+|z_j|}\right)+s\right)\right\} \\ &= \frac{1}{1+k}\left\{\wedge + k\gamma_m\left(s+\sum_{j=s+1}^{n}\left(\frac{1-\frac{|z_j|}{k}}{1+\frac{|z_j|}{k}}\right)\right)\right\}. \end{aligned}$$

Since $|z_j| \leq k \implies \frac{|z_j|}{k} \leq 1$, so by Lemma 2.12, we have for $|z| = 1$,

$$\begin{aligned} Re\left(\frac{zP^{\gamma}(z)}{P(z)}\right) &\geq \frac{1}{1+k}\left\{\wedge + k\gamma_m\left(s+\frac{1-\prod_{j=s+1}^{n}\frac{|z_j|}{k}}{1+\prod_{j=s+1}^{n}\frac{|z_j|}{k}}\right)\right\} \\ &= \frac{1}{1+k}\left\{\wedge + k\gamma_m\left(s+\frac{1-\frac{1}{k^{n-s}}\left|\frac{a_o}{a_{n-s}}\right|}{1+\frac{1}{k^{n-s}}\left|\frac{a_o}{a_{n-s}}\right|}\right)\right\} \\ &= \frac{1}{1+k}\left\{\wedge + k\gamma_m\left(s+\frac{k^{n-s}|a_{n-s}|-|a_o|}{k^{n-s}|a_{n-s}|+|a_o|}\right)\right\} \\ &= \frac{k}{1+k}\left\{\frac{\wedge}{k} + \gamma_m\left(s+\frac{k^{n-s}|a_{n-s}|-|a_o|}{k^{n-s}|a_{n-s}|+|a_o|}\right)\right\}. \end{aligned}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.3. We know that

$$\left|z\frac{P^{\gamma}(z)}{P(z)}\right| \geq Re\left(\frac{zP^{\gamma}(z)}{P(z)}\right).$$

Using inequality (20), we have for $|z| = 1$,

$$\left|z\frac{P^{\gamma}(z)}{P(z)}\right| \geq \frac{k}{1+k}\left\{\frac{\wedge}{k} + \gamma_m\left(s+\frac{k^{n-s}|a_{n-s}|-|a_o|}{k^{n-s}|a_{n-s}|+|a_o|}\right)\right\}.$$

This implies,

$$|P^{\gamma}(z)| \geq \frac{k}{1+k}\left\{\frac{\wedge}{k} + \gamma_m\left(s+\frac{k^{n-s}|a_{n-s}|-|a_o|}{k^{n-s}|a_{n-s}|+|a_o|}\right)\right\}|P(z)|$$

or

$$\max_{|z|=1}|P^{\gamma}(z)| \geq \frac{k}{1+k}\left\{\frac{\wedge}{k} + \gamma_m\left(s+\frac{k^{n-s}|a_{n-s}|-|a_o|}{k^{n-s}|a_{n-s}|+|a_o|}\right)\right\}\max_{|z|=1}|P(z)|. \tag{27}$$

This completes the proof of Theorem 2.3. □

Proof of Theorem 2.5. We have by Lemma 2.13 and Lemma 2.14

$$\begin{aligned}|D_\alpha^\gamma[P](z)| &= |\wedge P(z) + (\alpha - z)P^\gamma(z)| \\ &= |\wedge P(z) + \alpha P^\gamma(z) - zP^\gamma(z)| \\ &\geq |\alpha||P^\gamma(z)| - |\wedge P(z) - zP^\gamma(z)| \\ &= |\alpha||P^\gamma(z)| - |Q^\gamma(z)| \\ &\geq |\alpha||P^\gamma(z)| - k|P^\gamma(z)|.\end{aligned}$$

Thus,

$$|D_\alpha^\gamma[P](z)| \geq (|\alpha| - k)|P^\gamma(z)|.$$

So by using inequality (27), we have

$$\max_{|z|=1} |D_\alpha^\gamma[P](z)| \geq \frac{(|\alpha| - k)}{1+k} \left[\wedge + k\gamma_m \left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right] \max_{|z|=1} |P(z)|. \tag{28}$$

This completes the proof of Theorem 2.5. □

Proof of Theorem 2.10. Consider the scaled polynomial $F(z) = P(kz)$, then all the zeros of $F(z)$ lie in $|z| \leq 1$, as $P(z)$ has all zeros in $|z| \leq k$, $k \geq 1$. Let $\delta = \frac{\alpha}{k}$, then $|\delta| \geq 1$. Thus from inequality (23) when applied to scaled polynomial $F(z)$ we have,

$$\begin{aligned}\max_{|z|=1} |D_\delta^\gamma[F](z)| &\geq \left(\frac{|\delta| - 1}{2}\right) \left\{\wedge + \gamma_m \left(s + \frac{|k^{n-s}a_{n-s}| - |a_o|}{|k^{n-s}a_{n-s}| + |a_o|}\right)\right\} \max_{|z|=1} |F(z)| \\ &= \left(\frac{|\alpha| - k}{2k}\right) \left\{\wedge + \gamma_m \left(s + \frac{k^{n-s}|a_{n-s}| - |a_o|}{k^{n-s}|a_{n-s}| + |a_o|}\right)\right\} \max_{|z|=1} |F(z)|.\end{aligned}$$

Now by invoking Lemma 2.15 and replacing $F(z)$ by $P(kz)$ we have

$$\max_{|z|=1} |D_\delta^\gamma[P](kz)| \geq \left(\frac{|\alpha| - k}{2k}\right) \left\{\wedge + \gamma_m \left(s + \frac{k^{n-s}|a_{n-s}| - |a_o|}{k^{n-s}|a_{n-s}| + |a_o|}\right)\right\} \frac{2k^n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{29}$$

Also by the definition

$$\begin{aligned}\max_{|z|=1} |D_\delta^\gamma[P](kz)| &= |\wedge P(kz) + (\delta - z)[P(kz)]^\gamma| \\ &= |\wedge P(kz) + (\delta - z)kP^\gamma(kz)| \\ &= \left|\wedge P(kz) + (\frac{\alpha}{k} - z)kP^\gamma(kz)\right| \\ &= \max_{|z|=k} |D_\alpha^\gamma[P](z)|.\end{aligned}$$

Hence from inequality (29) we get

$$\max_{|z|=k} |D_\alpha^\gamma[P](z)| \geq \left(\frac{|\alpha| - k}{k(1+k^n)}\right) k^n \left\{\wedge + \gamma_m \left(s + \frac{k^{n-s}|a_{n-s}| - |a_o|}{k^{n-s}|a_{n-s}| + |a_o|}\right)\right\} \max_{|z|=1} |P(z)|.$$

Since $D_\alpha^\gamma[P](z)$ is of degree $n-1$, hence by Lemma 2.16 we have

$$k^{n-1} \max_{|z|=1} |D_\alpha^\gamma[P](z)| \geq \max_{|z|=k} |D_\alpha^\gamma[P](z)|.$$

Therefore

$$k^{n-1} \max_{|z|=1} |D_\alpha^\gamma[P](z)| \geq \left(\frac{|\alpha| - k}{1+k^n}\right) k^{n-1} \left\{\wedge + \gamma_m \left(s + \frac{k^{n-s}|a_{n-s}| - |a_o|}{k^{n-s}|a_{n-s}| + |a_o|}\right)\right\} \max_{|z|=1} |P(z)|.$$

This implies that

$$\max_{|z|=1} |D_\alpha^\gamma[P](z)| \geq \left(\frac{|\alpha| - k}{1 + k^n}\right) \left\{ \wedge + \gamma_m \left(s + \frac{k^{n-s}|a_{n-s}| - |a_o|}{k^{n-s}|a_{n-s}| + |a_o|} \right) \right\} \max_{|z|=1} |P(z)|.$$

This completes the proof of Theorem 2.10. □

3. Conclusion

In this work, we established several new inequalities for the generalized polar derivative $D_\alpha^\gamma[P](z)$ and the generalized derivative $P^\gamma(z)$ of complex polynomials $P \in \mathcal{P}_n$ with restricted zeros. Our main results extend classical theorems on polar and ordinary derivatives to these generalized operators, while incorporating the case of s-folded zeros at the origin.

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