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A PRIORI ESTIMATES FOR SOLUTIONS TO ELLIPTIC EQUATIONS IN LONG DOMAINS

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Abstract. We consider a second-order linear uniformly elliptic partial differential operator in non-divergence form. For the operator, there is a well-known Aleksandrov-Bakel'man-Pucci estimate(ABP estimate, in short). Following the proof of the original ABP estimate, using a rectangular cone than a circular cone, we obtain a smaller constant than the original estimate for the upper bound. Also, we show that our improved result implies the original ABP estimate and is more useful for long domains than the original one.

1. Introduction

In this paper, we treat a second-order linear uniformly elliptic partial differential operator. In particular, we will consider the following operator of the nondivergence form:

$$L = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} + \sum_{i=1}^{n} b_i(x) D_i + c(x). \tag{L}$$

Here, D_{ij} represents a partial derivative in the x_i, x_j-direction, namely,

$$D_{ij} = D_i D_j = \frac{\partial^2}{\partial x_j \partial x_i}, \quad \text{and } D_i = \frac{\partial}{\partial x_i}$$

for $i, j = 1, \ldots, n$ in the considered domain lying in $\mathbb{R}^n$ for $n \in \mathbb{N}$.

The coefficients are assumed to be measurable but not necessarily continuous. Uniform ellipticity means that the second-order coefficients a_{ij} satisfy, for some strictly positive constants c_0, C_0,

$$c_0|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le C_0|\xi|^2, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \tag{UE}$$

for any $x \in \Omega(\subset \mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. Without loss of generality, we will assume

$$a_{ij}(x) = a_{ji}(x) \quad \text{for each } i, j = 1, 2, ..., n.$$

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For a given domain, we assume it is an open and connected subset of $\mathbb{R}^n$, $n \geq 1$.

The classical Aleksandrov-Bakel'man-Pucci estimate (the ABP estimate, in short) states the following

THEOREM 1.1. *Let Ω be a bounded and open set in $\mathbb{R}^n$, and let L be a second-order uniformly elliptic* (UE) *differential operator in nondivergence form* (L). *In addition, let u be a function in $W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ such that $Lu \geq f$ in Ω, and $b_i, f \in L^n(\Omega)$, $c(x) \leq 0$. Then,*

$$\sup_{\Omega} u := \sup_{x\in\Omega} u(x) \leq \sup_{\partial\Omega} u_+ + C_1 \cdot diam(\Omega) \cdot e^{C_1 S} \|f_-\|_{L^n(\Omega)},$$

where $\partial\Omega$ denotes its topological boundary, $u_+ := max\{u, 0\}$, $f_- := max\{-f, 0\}$,

$$S := S(\Omega) := \sum_{i=1}^{n} \int_{\Omega} |b_i|^n dx,$$

$$\|f_-\|_{L^n(\Omega)} := \left(\int_{\Omega} |f_-|^n dx\right)^{1/n}, \text{ the norm of the function } f_- \text{ in } L^n(\Omega),$$

$diam(\Omega)$ is a diameter of Ω, and C_1 is a positive constant depending only on c_0, C_0 and n.

Originally, ABP estimate is proved by Aleksandrov and other mathematicians [1–3,14], and an important tool in elliptic PDE. See [8,10,12] for example. For a detailed history of the ABP estimate, one may refer to [9, 11] and references therein. For a proof, one may refer to [2], [9, (9.14)]. The above version is modified from [13, Theorem 1.1] to our purpose.

For the case of $f \geq 0$, it easily reduces to a maximum principle of the following form:

COROLLARY 1.1. *Let Ω be a open and bounded set in $\mathbb{R}^n$, and let u be a function belong to $W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ such that $Lu \geq 0$ in Ω. Then, the positive maximum of u is attained on $\partial\Omega$.*

Similarly, if $Lu \leq 0$ in Ω, then the negative minimum of u is attained on $\partial\Omega$.

In this paper, we are interested in replacing the constant of $diam(\Omega)$ with some other constants which is a character of the given domains. This result gives us a better estimate for long domains. See Theorem 3.1, Theorem 3.3 and Remark 3.6. Also, we need to point out that Cabre obtained a different kind of geometric constant. See [4, 5].

In section 2, we prepare some preliminaries, especially ABP estimate for L without lower order terms. In section 3, we present the main result, Theorem 3.1. The estimate from the main result is useful for the long domain. See Remark 3.6.

For simplicity, we will treat an operator with the second-order terms only, without lower-order terms. But one can generalize to the general elliptic operator (L).

Concluding the introductory section, we enlist some notations which will be used later. The set of first n natural numbers is denoted by I_n. Namely, $I_n := \{1, 2, ..., n\}$.

In this paper, we are interested in a rectangular domain:

$$Q_{(r_i,s_i)} := \{x \in \mathbb{R}^n \,|\, \forall i \in I_n, r_i < x_i < s_i, x = (x_1, ..., x_n)\}.$$

$B_r(x)$ is a open ball of radius r centered at x, namely,

$$B_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}.$$

For simplicity, we use B_r for $B_r(0)$. ω_n denotes the n-dimensional measure of a unit ball, $\omega_n := |B_1|$.

The diameter of any given domain is denoted by $diam(\Omega)$ which means $\sup_{x,y\in\Omega}\{|x-y|\}$. The topological boundary of a given domain Ω is denoted by $\partial\Omega$.

2. Preliminaries

In this section, we provide a proof of ABP estimate, Theorem 1.1 for a simplified case L_0 of L, where

$$L_0 = \sum_{i,j=1}^{n} a_{ij}(x)D_{ij}. \tag{L_0}$$

Note that L turn into L_0 if $b_i = c = 0$. The idea of the proof will be adapted to the proof of the main results. For more general cases with low-order terms, one may refer to [4–7, 9, 15].

Firstly, we introduce two definitions, the upper contact set and normal mapping.

Definition 2.1. If $u \in C(\Omega)$, we can define the upper contact set of u with a notation of Γ^+ or Γ_u^+, to be a subset of Ω,

$$\Gamma^+ = \{y \in \Omega \,|\, \exists p = p(y) \in \mathbb{R}^n, \forall x \in \Omega, u(x) \le u(y) + p \cdot (x - y)\}.$$

When $u \in C^1(\Omega)$, then it necessarily $p(y) = Du(y)$. For the next, we define the normal mapping:

Definition 2.2. For $u \in C(\Omega)$, the normal mapping, denoted by $\chi(y) = \chi_u(y)$, means the following:

$$\chi(y) = \{p \in \mathbb{R}^n \,|\, \forall x \in \Omega, u(x) \le u(y) + p \cdot (x - y)\}.$$

In particular, if $u \in C^1(\Omega)$, then χ is the gradient vector field of u on Γ^+.

Example 2.3. We present the circular cone centered at the origin, radius of $r > 0$, maximum of $M > 0$ at 0.

$$u(x) = M\left(1 - \frac{|x|}{r}\right).$$

By direct computation, $\chi_u(x) = \begin{cases} \frac{Mx}{r|x|} \text{ for } x \neq 0, \\ B_{M/r}(0) \text{ for } x = 0. \end{cases}$

Thus, we have $|\chi_u(\Omega)| = |B_{M/r}|$.

For the next, we consider a rectangular cone.

Example 2.4. For $M, r_i, s_i > 0$, $i = 1, ..., n$, let

$$\Omega := \{x \in \mathbb{R}^n \,|\, -r_i < x_i < s_i, i = 1, ..., n, x = (x_1, ..., x_n)\}$$

and

$$v(x) = v(x_1, ..., x_n) = \min_{i\in I_n} \min\{\frac{M}{r_i}(r_i + x_i), \frac{M}{s_i}(s_i - x_i)\}.$$

Then $\chi_v(\Omega) = [-\frac{M}{s_1}, \frac{M}{r_1}] \times \cdots \times [-\frac{M}{s_n}, \frac{M}{r_n}]$. Thus,

$$|\chi_v(\Omega)| = M^n \prod_{i\in I_n} \frac{s_i + r_i}{s_i r_i}.$$

The following two lemmas are basically from [9].

LEMMA 2.5. *For $u \in C^2(\Omega) \cap C^0(\Omega)$ we have*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{d}{\omega_n^{1/n}} \left(\int_{\Gamma^+} |detD^2 u| \right)^{1/n}$$

where $d = diam(\Omega)$.

Proof. Considering $u - \sup_{\partial\Omega} u$, without loss of generality, we may assume that $u \leq 0$ on $\partial\Omega$.

Now we show that the maximum value of u can be estimated in terms of $|\chi(\Omega)|$. Let $\max_\Omega u = u(y)$, and k be the function whose graph is the cone K with vertex $(y, u(y))$ and base $\partial\Omega$. Then $\chi_k(\Omega) \subset \chi_u(\Omega)$. Since each supporting hyperplane of K, there exists a parallel hyperplane tangent to the graph of u. Now let $\tilde{k}$ be the function whose graph is the cone $\tilde{K}$ with vertex $(y, u(y))$ and base $B_d(y)$. Here, $B_d(y)$ is an open ball in $\mathbb{R}^n$ centered at y of radius d, and d is the diameter of the domain Ω. It is clear that $\chi_{\tilde{k}}(\Omega) \subset \chi_k(\Omega)$. By Example 2.3,

$$\omega_n \left(\frac{u(y)}{d} \right)^n = |\chi_{\tilde{k}}(\Omega)| \leq |\chi_k(\Omega)| \leq |\chi_u(\Omega)|.$$

By definition of χ,

$$|\chi_u(\Omega)| = |\chi(\Gamma^+)| = |Du(\Gamma^+)| \tag{1}$$

Since the Jacobian matrix of $Du : \Gamma_u^+ \to \mathbb{R}^n$, D^2u, is negative semi-definite, $Du - \epsilon Id_n$ has a maximal rank for an arbitrary $\epsilon > 0$, where Id_n is the $n \times n$ identity matrix. From the transformation formula for multiple integrals, we have

$$|(Du - \epsilon Id_n)(\Gamma_u^+)| \leq \int_{\Gamma_u^+} |det(D^2u - \epsilon Id_n)|. \tag{2}$$

Letting $\epsilon \to 0^+$, combining (1), (2), we have

$$|\chi_u(\Omega)| = |Du(\Gamma^+)| \leq \int_{\Gamma_u^+} |detD^2u|.$$

Now we have

$$\omega_n \left(\frac{u(y)}{d} \right)^n \leq \int_{\Gamma^+} |detD^2u|,$$

which leads to the completion of the proof. □

LEMMA 2.6. *For $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, we have*

$$|detD^2u| \leq \frac{1}{det(a_{ij})} \cdot \left(\frac{-a_{ij}D_{ij}u}{n} \right)^n \tag{3}$$

on Γ^+. Thus, we have

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{d}{n\omega_n^{1/n}} \left\| \frac{-a_{ij}D_{ij}u}{\mathcal{D}^{1/n}} \right\|_{L^n(\Gamma^+)}, \tag{4}$$

where $d = diam(\Omega)$, $\mathcal{D} = det(a_{ij})$.

Proof. Note that D^2u is nonpositive on Γ^+. Thus, $-D^2u$ becomes a positive symmetric.

For a proof, considering $A = -D^2u$, $B = (a_{ij})$, it is enough to show that the following claim: for positive symmetric matrix A and B,

$$detAdetB \le \left(\frac{tr(AB)}{n}\right)^n,$$

where $tr(\cdot)$ denotes the trace operator.

Note that $B = OD_BO^T$ for some orthonormal matrix O, diagonal matrix D_B. Then $B = O\sqrt{D_B}\sqrt{D_B}O^T = O\sqrt{D_B}O^TO\sqrt{D_B}O^T$. Let $\sqrt{B} = O\sqrt{D_B}O^T$. Then

$$\begin{aligned} tr(AB) &= tr(A\sqrt{B}\sqrt{B}) \\ &= tr(\sqrt{B}A\sqrt{B}) \text{ (since } tr(XYZ) = tr(ZXY)) \\ &\ge n\sqrt[n]{det\sqrt{B}A\sqrt{B}} \text{ (using } \sqrt{B}A\sqrt{B} \text{ is symmetric.)} \\ &= n\sqrt[n]{detAB} = n\sqrt[n]{detAdetB}, \end{aligned}$$

which leads to (3). By Lemma 2.5, we have (4). □

Note that $c_0 \le \mathcal{D}^{1/n} \le C_0$ from the uniform elliptic condition, (UE). Thus, in our setting, the denominator of (4) is bounded.

Now we present the simplified ABP estimate with its proof. For a proof of ABP estimate of Theorem 1.1 with lower order terms, one may refer to [9].

THEOREM 2.1. *Let Ω be a bounded and open set in $\mathbb{R}^n$, and let L_0 be a second-order uniformly elliptic* (UE) *differential operator in the simplified nondivergence form* (L_0). *In addition, let u be a function in $W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ such that $Lu \ge f$ in Ω, and $f \in L^n(\Omega)$. Then,*

$$\sup_{\Omega} u := \sup_{x\in\Omega} u(x) \le \sup_{\partial\Omega} u_+ + \frac{diam(\Omega)}{n\omega^{1/n}} \left\| \frac{f_-}{\mathcal{D}^{1/n}} \right\|_{L^n(\Omega)}, \tag{5}$$

where $\partial\Omega$ denotes its topological boundary, $u_+ := max\{u, 0\}$, $f_- := max\{-f, 0\}$,

$$\|f_-\|_{L^n(\Omega)} := \left(\int_{\Omega} |f_-|^n dx\right)^{1/n}, \text{ the norm of the function } f_- \text{ in } L^n(\Omega),$$

$diam(\Omega)$ is a diameter of Ω, and $\mathcal{D} = det(a_{ij})$.

Proof. Let $a_{ij}D_{ij}u \ge f$ in Ω. Thus, we have $0 \le -a_{ij}D_{ij}u \le -f \le f_-$ in Γ^+_u, where $f_- := \max\{-f, 0\}$.

Combining Lemma 2.5 and Lemma 2.6, we obtain (5) for $u \in C^2(\Omega)$. Now, we generalize these results to functions $u \in C^0(\Omega)\cap W^{2,n}_{loc}(\Omega)$ by a standard approximation argument.

Let $\{u_m\}$ be a sequence of functions in $C^2(\Omega)$ converging in $W^{2,n}_{loc}(\Omega)$ to u. For arbitrary $\epsilon > 0$, assume that u_m converges to u in $W^{2,n}(\Omega_\epsilon)$ and $u_m \le \epsilon + \sup_{\partial\Omega} u_+ = \epsilon$ on $\partial\Omega_\epsilon$, where

$$\Omega_\epsilon \subset\subset \Omega, \quad \Omega_{\epsilon_1} \subset \Omega_{\epsilon_2} \text{ for } \epsilon_2 \le \epsilon_1, \quad \cup_\epsilon \Omega_\epsilon = \Omega.$$

Applying Lemma 2.6 to u_m in Ω_ϵ, we have

$$\sup_{\Omega_\epsilon} u_m \le \sup_{\partial\Omega_\epsilon} u_m + \frac{diam(\Omega)}{n\omega_n^{1/n}} \left\| \frac{-a_{ij}D_{ij}u_m}{\mathcal{D}^{1/n}} \right\|_{L^n(\Gamma^+_{u_m}\cap\Omega_\epsilon)}.$$

Note that, on $\Gamma^+_{u_m}$,

$$0 \le -a_{ij}D_{ij}u_m \le -a_{ij}D_{ij}(u_m - u) - a_{ij}D_{ij}u \le -a_{ij}D_{ij}(u_m - u) + f_-.$$

Also,

$$\|D_{ij}(u_m - u)\|_{L^n(\Omega_\epsilon)} \to 0^+ \text{ in } \Omega, \quad \|f_-\|_{L^n(\Gamma^+_{u_m}\cap\Omega_\epsilon)} \le \|f_-\|_{L^n(\Omega_\epsilon)},$$

$u_m \to u$ uniformly on Ω_ϵ. Thus in all,

$$\sup_{\Omega_\epsilon} u \le \epsilon + \sup_{\partial\Omega} u_+ + \frac{diam(\Omega)}{n\omega_n^{1/n}} \left\| \frac{f_-}{\mathcal{D}^{1/n}} \right\|_{L^n(\Omega_\epsilon)}.$$

Now the desired estimate follows after taking $\epsilon \to 0^+$. □

3. The main results

In this section, we present the main result, Theorem 3.1, and generalize it to Theorem 3.3. The idea is to follow the idea of the original ABP estimate, Theorem 2.1, using a rectangular cone from Example 2.4 than a circular cone from Example 2.3.

3.1. Rectangular domains. In this subsection, we consider the case of rectangular domain, $\Omega = Q_{(r_i,s_i)}$.

THEOREM 3.1. *Let $u \in W^{2,n}_{loc}(Q_{(r_i,s_i)}) \cap C^0(\overline{Q_{(r_i,s_i)}})$, where*

$$Q_{(r_i,s_i)} := \{x \in \mathbb{R}^n \,|\, r_i < x_i < s_i, i = 1, ..., n, x = (x_1, ..., x_n)\}$$

for $r_i, s_i \in \mathbb{R}$, $r_i < s_i$, $i = 1, ..., n$. Also, $L_0 u \ge f$ in $Q_{(r_i,s_i)}$ for some $f \in L^n(Q_{(r_i,s_i)})$. Then, we have

$$\sup_{Q_{(r_i,s_i)}} u \le \sup_{\partial Q_{(r_i,s_i)}} u + \frac{1}{n} \cdot \left(\prod_{i\in I_n} \frac{d_i^- d_i^+}{(d_i^- + d_i^+)} \right)^{1/n} \left\| \frac{f_-}{\mathcal{D}^{1/n}} \right\|_{L^n(\Gamma^+)}, \tag{6}$$

where $u(y) = \sup_{Q_{(r_i,s_i)}} u$ for some $y \in Q_{(r_i,s_i)}$, $y = (y_1, ..., y_n)$, $d_i^- = d_i^-(y) := y_i - r_i$, $d_i^+ = d_i^+(y) := s_i - y_i$, $\mathcal{D} = det(a_{ij})$.

Also, we have

$$\sup_{Q_{(r_i,s_i)}} u \le \sup_{\partial Q_{(r_i,s_i)}} u + \frac{1}{n} \cdot \left(\prod_{i\in I_n} \frac{(s_i - r_i)}{4} \right)^{1/n} \left\| \frac{f_-}{\mathcal{D}^{1/n}} \right\|_{L^n(\Gamma^+)}. \tag{7}$$

Proof. Basically, we follow the proof of Lemma 2.5 and Lemma 2.6 with a condition of $\Omega = Q_{(r_i,s_i)}$ and $u(y) = \max_\Omega u$.

First, we assume that $u \in C^2(Q_{(r_i,s_i)})$. Considering $u - \sup_{\partial Q_{(r_i,s_i)}} u$, we can assume that $u \le 0$ on $\partial Q_{(r_i,s_i)}$.

Let k be the function whose graph is the rectangular cone K with vertex $(y, u(y))$ and base $\partial Q_{(r_i,s_i)}$. Similar to Lemma 2.5, $\chi_k(\Omega) \subset \chi_u(\Omega)$.

By Example 2.4,

$$(u(y))^n \prod_{i\in I_n} \frac{(d_i^- + d_i^+)}{d_i^- d_i^+} = |\chi_k(\Omega)| \le |\chi_u(\Omega)|.$$

By definition of χ and the change of variables formula,

$$|\chi_u(\Omega)| = |\chi(\Gamma^+)| = |Du(\Gamma^+)| \le \int_{\Gamma^+} |detD^2u|.$$

By Lemma 2.6 and the fact that $a_{ij}D_{ij}u \le 0$ on Γ^+, we finished the proof of (6). For (7), note that

$$\prod_{i\in I_n} \frac{d_i^- d_i^+}{(d_i^- + d_i^+)} \le \prod_{i\in I_n} \frac{\frac{(s_i-r_i)^2}{4}}{(s_i - r_i)} \le \prod_{i\in I_n} \frac{(s_i - r_i)}{4}.$$

Here we used the fact that

$$(y_i - r_i)(s_i - y_i) \le \left(\frac{s_i - r_i}{2}\right)^2$$

for each $y_i \in (r_i, s_i)$, $i \in I_n$.

For a generalization to $u \in W^{2,n}_{loc}$, we follow the proof of Theorem 2.1. We omit details. □

For an application, we show a distance weighted estimate:

COROLLARY 3.2. *Let $d(y) = dist(y, \partial Q_{(r_i,s_i)}) = \min\{s_j - y_j, y_j - r_j\}$ for some $j \in I_n$. Then*

$$\sup_{Q_{(r_i,s_i)}} u \le \sup_{\partial Q_{(r_i,s_i)}} u + \frac{d(y)^{1/n}}{n} \cdot \left(\prod_{i\in I_n\setminus\{j\}} \frac{d_i}{4}\right)^{1/n} \left\| \frac{a_{ij}D_{ij}u}{\mathcal{D}^{1/n}} \right\|_{L^n(\Gamma^+)},$$

where $d_i = s_i - r_i$.

Proof. Note that $d(y) = \min_{i\in I_n} \min\{d_i^-, d_i^+\}$. Without loss of generality, let $d(y) = d_j^-$ for some $j \in I_n$. Note that

$$\frac{d_j^- d_j^+}{(d_j^- + d_j^+)} \le d(y)\frac{d_j}{d_j} = d(y), \qquad \frac{d_i^- d_i^+}{(d_i^- + d_i^+)} \le \frac{(d_i/2)^2}{d_i} = \frac{d_i}{4}$$

for $j \ne i$. This completes the proof from (6). □

3.2. General domains. In this subsection, we consider the general bounded domain of $\Omega \subset \mathbb{R}^n$. We consider the case of $\Omega \subset Q_{r_i,s_i}$, Theorem 3.3, and $\Omega \subset B_{R/2}$, Corollary 3.4.

THEOREM 3.3. *Let $u \in W^{2,n}_{loc}(\Omega) \cap C^0(\overline{\Omega})$, and*

$$\Omega \subset Q_{(r_i,s_i)} := \{x \in \mathbb{R}^n \,|\, r_i < x_i < s_i, i = 1, ..., n, x = (x_1, ..., x_n)\}$$

for $r_i, s_i \in \mathbb{R}$, $r_i < s_i$, $i = 1, ..., n$. Also, $L_0 u \ge f$ in Ω for some $f \in L^n(\Omega)$. Then, we have

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u + \frac{1}{n} \cdot \left(\prod_{i\in I_n} \frac{d_i^- d_i^+}{(d_i^- + d_i^+)}\right)^{1/n} \left\| \frac{f_-}{\mathcal{D}^{1/n}} \right\|_{L^n(\Gamma^+)},$$

where $u(y) = \sup_\Omega u$ for some $y \in \Omega$, $y = (y_1, ..., y_n)$, $d_i^- = d_i^-(y) := y_i - r_i$, $d_i^+ = d_i^+(y) := s_i - y_i$.

Also, we have

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u + \frac{1}{n} \cdot \left(\prod_{i\in I_n} \frac{(s_i - r_i)}{4}\right)^{1/n} \left\| \frac{f_-}{\mathcal{D}^{1/n}} \right\|_{L^n(\Gamma^+)}.$$

Proof. Basically, we follow the proof of Theorem 3.1 with a condition of $\Omega \subset Q_{(r_i,s_i)}$. First, we assume that $u \in C^2(Q_{(r_i,s_i)})$.

Considering $u - \sup_{\partial\Omega} u$, we can assume that $u \leq 0$ on $\partial\Omega$. If $u \leq 0$ in Ω, then there is nothing to prove. Let $\max_\Omega u = u(y) > 0$, and k be the function whose graph is the cone K with vertex $(y, u(y))$ and base $\partial\Omega$. Similar to Lemma 2.5, $\chi_k(\Omega) \subset \chi_u(\Omega)$.

Now let $\tilde{k}$ be the function whose graph is the cone $\tilde{K}$ with vertex $(y, u(y))$ and base $\partial Q_{(r_i,s_i)}$. It is clear that $\chi_{\tilde{k}}(\Omega) \subset \chi_k(\Omega)$. By Example 2.4,

$$(u(y))^n \prod_{i\in I_n} \frac{d_i^- + d_i^+}{d_i^- d_i^+} \leq |\chi_{\tilde{k}}(\Omega)| \leq |\chi_k(\Omega)| \leq |\chi_u(\Omega)|.$$

By definition of χ and the change of variables formula,

$$|\chi_u(\Omega)| = |\chi(\Gamma^+)| = |Du(\Gamma^+)| \leq \int_{\Gamma^+} |detD^2u|.$$

Now we have

$$(u(y))^n \prod_{i\in I_n} \frac{d_i^- + d_i^+}{d_i^- d_i^+} \leq \int_{\Gamma^+} |detD^2u|.$$

The remaining part of the proof is the same as Theorem 3.1. □

Corollary 3.4. *Let $u \in W^{2,n}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ and the domain Ω satisfies that $\Omega \subset B_{R/2}$. Also, $L_0 u \geq f$ in Ω for some $f \in L^n(\Omega)$. Then we have*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u + \frac{R}{2n} \left\| \frac{f_-}{\mathcal{D}^{1/n}} \right\|_{L^n(\Gamma^+)}.$$

Proof. After translation, one may assume that $u(0) = \sup_\Omega u$, $\Omega \subset B_R(0) = B_R$. Since $\Omega \subset B_R$, $\Omega \subset Q_{(-R_i,R_i)}$ for $R_i = R$ for each $i \in I_n$. By Theorem 3.3, (3.3),

$$\sup_\Omega u \leq \sup_{\partial\Omega} u + \frac{R}{2n} \left\| \frac{f_-}{\mathcal{D}^{1/n}} \right\|_{L^n(\Gamma^+)}.$$

□

Remark 3.5. Corollary 3.4 corresponds to the usual ABP estimates. Thus, our result, Theorem 3.1, implies the standard ABP estimate without lower-order terms, Theorem 2.1. Also, see [9, Problems 9.1].

Remark 3.6. Here, we compare the result of ABP estimate with Theorem 3.3. For the right-hand side term, we have

$$\frac{diam(\Omega)}{n\omega_n^{1/n}} \text{ and } \frac{1}{n} \cdot \left(\prod_{i\in I_n} \frac{(s_i - r_i)}{4} \right)^{1/n}$$

from Theorem 2.1 and Theorem 3.3, respectively.

Consider the domain of $\Omega := (-r, r) \times (-\frac{1}{r}, \frac{1}{r})$ lying in $\mathbb{R}^n$. Note that

$$\frac{diam(\Omega)}{n\omega_n^{1/n}} \geq \frac{2r}{n\omega_n^{1/n}} \to \infty,$$

but

$$\frac{1}{n} \cdot \left(\prod_{i\in I_n} \frac{(s_i - r_i)}{4} \right)^{1/n} = \frac{1}{2} \cdot \frac{1}{4} \cdot \sqrt{2r \times \frac{2}{r}} = \frac{1}{4}.$$

Thus, the estimate from Theorem 3.3 does hold with a smaller constant than one from Theorem 2.1 even when the domain is long enough.

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