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DIFFERENTIAL SUBORDINATION FOR STARLIKE FUNCTIONS

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Abstract. A normalized analytic function, f defined on the open unit disk, is starlike of order α if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$, and is said to be reciprocal starlike of order α if $\operatorname{Re}(f(z)/zf'(z)) > \alpha$. Such functions are univalent and, therefore we find sufficient conditions for functions to be starlike and reciprocal starlike. We prove a general differential subordination theorem and sufficient conditions in terms of $zf'(z)/f(z)$ and $1+zf''(z)/f'(z)$ for functions to be starlike. Further, we prove sufficient conditions for the reciprocal starlikeness of functions and integral operators.

1. Introduction

Let $\mathcal{H}[a,n]$ denote the class of all analytic functions f of the form $f(z) = a + \sum_{k=n}^{\infty} a_k z^k$ defined on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The class $\mathcal{A}_n$ consists of all functions f of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$. The class $\mathcal{A} := \mathcal{A}_1$ is the usual class of normalized analytic functions on $\mathbb{D}$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions univalent in $\mathbb{D}$. We shall be interested in the subclasses of $\mathcal{A}$ with specific geometric properties like starlikeness and convexity. A domain $D \subset \mathbb{C}$ is said to be starlike with respect to a point $z_0 \in D$ if the line segment joining z_0 to every other point $z \in D$ lies entirely in D. A function $f \in \mathcal{A}$ is starlike if $f \in \mathcal{S}$ and $f(\mathbb{D})$ is starlike with respect to the origin or, analytically, satisfies the condition $\operatorname{Re}(zf'(z)/f(z)) > 0$ for all $z \in \mathbb{D}$. The class $\mathcal{S}^*$ of starlike functions was introduced by Alexander [2]. A domain $D \subset \mathbb{C}$ is said to be convex if the line segment joining any two arbitrary points of D lies entirely in D, i.e. it is starlike with respect to each point of D. A function $f \in \mathcal{A}$ is said to be convex if $f(\mathbb{D})$ is a convex domain. The class of all convex univalent functions is denoted by $\mathcal{C}$ and is characterized by the condition $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ for all $z \in \mathbb{D}$. The classes of starlike and convex functions can be generalized by using the concept of order. For $0 \leq \alpha < 1$, a function $f \in \mathcal{S}^*(\alpha)$, the class of all starlike functions of order α, if and only if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$ for all $z \in \mathbb{D}$. Yet another way to generalize is to use reciprocal order. For $0 \leq \alpha < 1$, a function $f \in \mathcal{RS}^*(\alpha)$, the class of all reciprocal starlike functions of order α, if and only if $\operatorname{Re}(f(z)/(zf'(z)) > \alpha$ for all $z \in \mathbb{D}$. Note that $\mathcal{RS}^*(0) = \mathcal{S}^*(0) = \mathcal{S}^*$. Since starlike functions (of order $\alpha \geq 0$) are univalent and

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these functions are characterized by a simple analytic condition, several sufficient conditions for starlikeness were obtained in the literature [15, 18, 19, 22]. Nunokawa *et al.* [17] proved that every starlike function of reciprocal order $\alpha \geq 0$ is starlike and hence univalent. The class of starlike functions of reciprocal order were studied by various authors [4–6, 8, 16, 23].

For two functions f and g defined on $\mathbb{D}$, the function f is subordinate to the function g, written $f \prec g$, if there is an analytic function w in $\mathbb{D}$ with $|w(z)| \leq |z|$ such that $f = g \circ w$. For a univalent superordinate function g, the subordination $f \prec g$ holds if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. In terms of subordination, a function $f \in \mathcal{S}^*$ if and only if the subordination $zf'(z)/f(z) \prec (1+z)/(1-z)$ holds. Among other results, the theory of differential subordination provides sufficient condition on ψ for the function $p \in \mathcal{H}[a, n]$ to be subordinate to the function q when the function p satisfies the condition $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for all $z \in \mathbb{D}$. The class $\mathcal{Q}$ consits of all functions q that are analytic and univalent(one-to-one) on $\overline{\mathbb{D}} \backslash E(q)$, where $E(q)$ consists of all points $\zeta \in \partial\mathbb{D}$ for which $q(z) \to \infty$ as $z \to \zeta$. A function in $\mathcal{Q}$ is known as a function with nice boundary [13] . The following theorem is the fundamental theorem in the theory of differential subordination.

Theorem 1.1. *[13] For $\Omega \subset \mathbb{C}$ and $q \in \mathcal{Q}$, let the class of admissible functions $\Psi_n(\Omega, q)$ consists of all functions $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s; z) \notin \Omega$, when $r = q(\zeta)$, $s = m\zeta q'(\zeta)$,*

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \geq \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$\zeta \in \partial\mathbb{D} \backslash E(q)$, and $m \geq n$. If the function $p \in \mathcal{H}[a, n]$ satisfies the condition

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \quad (z \in \mathbb{D})$$

for some function $\psi \in \Psi(\Omega, q)$, then $p(z) \prec q(z)$.

The theory of differential subordination is very useful in obtaining sufficient conditions for starlikeness and convexity [1, 7, 9, 21]. As an application of Theorem 1.1, we find sufficient conditions for a function to be starlike of order α or reciprocal starlike of order α. In Section 2, we apply Theorem 1.1 with $q(z) = Mz$ to obtain a sufficient condition for a function $f \in \mathcal{A}$ to satisfy the inequality $|zf'(z)/f(z) - 1| < 1$. Note that this condition is sufficient for starlikeness of the function f. As further application, we present a general subordination theorem and a particular sufficient condition for a function f to satisfy the subordination $zf'(z)/f(z) \prec 1/(1 + Mz)$. In Section 3, sufficient conditions for functions $f \in \mathcal{A}$ to be starlike are presented and in Section 4, sufficient conditions for functions $f \in \mathcal{A}$, $f \in \mathcal{A}_2$ to be reciprocal starlike are presented. We shall be using the theory of second order differential subordination developed by Miller and Mocanu [11–13].

2. Subordination Theorems

Miller and Mocanu [13] proved that $|zf''(z)/f'(z)| \leq 3/2$ is sufficient for a function $f \in \mathcal{A}$ to be starlike. Generalizing the above result, we find radius of the disk centered at $-\alpha$ such that $|(zf''(z)/f'(z)) + \alpha| < \gamma(\alpha)$ is sufficient for $f \in \mathcal{A}$ to be starlike. We shall be using the following special case, when $q(z) = Mz$, of Theorem 1.1 to prove our result.

Theorem 2.1. *[13] Let Ω be any subset of $\mathbb{C}$ and n be any positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consist of those functions $\psi\colon \mathbb{C}^2 \times \mathbb{D} \longrightarrow \mathbb{C}$ such that*

$$\psi\left(Me^{i\theta}, Ke^{i\theta}; z\right) \notin \Omega,$$

whenever $K \geq nM$, $z \in \mathbb{D}$ and $\theta \in \mathbb{R}$. If the function $p \in \mathcal{H}[0, n]$ satisfies $\psi(p(z), zp'(z); z) \in \Omega$ for some $\psi \in \Psi_n(\Omega, M, 0)$, then $|p(z)| < M$.

Theorem 2.2. *For $\alpha \geq 0$, let $\gamma(\alpha)$ be defined by*

$$\gamma(\alpha) = \begin{cases} \alpha + \frac{3}{2}, & 0 \leq \alpha \leq \frac{1}{16}, \\ \sqrt{2 + 2\sqrt{\alpha} - \alpha + \alpha^2}, & \alpha \geq \frac{1}{16}. \end{cases}$$

If the function $f \in \mathcal{A}$ satisfies the condition

$$\left|\frac{zf''(z)}{f'(z)} + \alpha\right| < \gamma(\alpha) \quad (z \in \mathbb{D}),$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \quad (z \in \mathbb{D}),$$

and, therefore, the function f is starlike.

Proof. Define the function $p\colon \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{zf'(z)}{f(z)} - 1.$$

A computation shows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z) + 1} + p(z) + 1 = \Psi(p(z), zp'(z); z) \in \Omega$$

where $\Omega = \{w : |w| < \gamma(\alpha)\}$ and

$$\Psi(r, s; z) = \frac{s}{r + 1} + r + \alpha.$$

It is clear that $\Psi\left(e^{i\theta}, Ke^{i\theta}; z\right) \notin \Omega$ is equivalent to the inequality

$$\left|K + 1 + \alpha + e^{i\theta} + \alpha e^{-i\theta}\right|^2 \geq \gamma^2 |1 + e^{i\theta}|^2$$

or to, with $x = \cos\theta \in [-1, 1]$,

$$(K+1+\alpha)^2 + 1 + \alpha^2 + 2(K+1+\alpha)x + 2\alpha(2x^2 - 1) + 2(K+1+\alpha)\alpha x \geq \gamma^2(2 + 2x).$$

Since LHS of inequality is an increasing function of K and $K \geq 1$, the inequality holds if

$$4\alpha x^2 + 2((2+\alpha)(1+\alpha) - \gamma^2)x + (2+\alpha)^2 + (1-\alpha)^2 =: \phi(x) \geq 0$$

for all $-1 \leq x \leq 1$. We shall now show that $\phi(x) \geq 0$ for all x with $x \in [-1, 1]$.

Case (i): $\alpha \leq 1/16$ and $\gamma = \alpha + (3/2)$. In this case, we have

$$\phi(x) = 4\alpha x^2 - \frac{x}{2} - 4\alpha + \frac{1}{2} = \left(4\alpha(x+1) - \frac{1}{2}\right)(x - 1).$$

Since $x \leq 1$ and $\alpha \leq 1/16$, we have $4\alpha(x+1) - 1/2 \leq 0$ and hence $\phi(x) \geq 0$ for all x with $|x| \leq 1$.

Case (ii): $\alpha \geq 1/16$ and $\gamma = \sqrt{2+2\sqrt{\alpha}-\alpha+\alpha^2}$. In this case, we have for $|x| \leq 1$,

$$\begin{aligned}\phi(x) &= 4\alpha x^2 + (8\alpha - 4\sqrt{\alpha})x + 1 + 4\alpha - 4\sqrt{\alpha}\\ &= 4\alpha(x+1)^2 - 4\sqrt{\alpha}(x+1) + 1\\ &= 4\alpha y^2 - 4\sqrt{\alpha}y + 1 =: g(y),\end{aligned}$$

where $y = x + 1 \in [0, 2]$. For $0 \leq y \leq 2$, the function g attains the minimum at $y = 1/2\sqrt{\alpha} \in (0, 2)$. Hence, we have $\phi(x) \geq g(1/2\sqrt{\alpha}) = 0$, for all x with $|x| \leq 1$.

Therefore, in both the cases, $\phi(x) \geq 0$. Hence, by Theorem 2.1, the result follows. $\square$

For $\alpha = 0$, Theorem 2.2 reduces to the following corollary.

Corollary 2.3. *[13, Corollary 5.1.c.2] If the function $f \in \mathcal{A}$ satisfies the condition*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3}{2} \quad (z \in \mathbb{D}),$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \quad (z \in \mathbb{D}).$$

It is worth mentioning that Mocanu and Serb [14] obtained the sharp form of this result by proving that a function $f \in \mathcal{A}$ satisfies the inequality $|zf'(z)/f(z) - 1| < 1$ whenever $|f''(z)/f'(z)| < 1.5936$ and the bound is sharp. In view of this, it will be of interest to find the best possible $\gamma(\alpha)$ in Theorem 2.2.

For $\alpha = 1$, Theorem 2.2 reduces to the following corollary.

Corollary 2.4. *If the function $f \in \mathcal{A}$ satisfies the condition*

$$\left|\frac{zf''(z)}{f'(z)} + 1\right| < 2 \quad (z \in \mathbb{D}),$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \quad (z \in \mathbb{D}).$$

Robertson [20] combined $zf''(z)/f'(z)$ and $zf'(z)/f(z)$ to obtain sufficient conditions for a function in the class $\mathcal{A}$ to be starlike. He proved that if there exists a $k \in \mathbb{R}$ such that $0 < k \leq 2$ and $|zf''(z)/f'(z)| \leq k|zf'(z)/f(z)|$, then this is sufficient for $f \in \mathcal{A}$ to be starlike of order $2/(2 + k)$. This result was further sharpened by Mocanu [13, Theorem 5.3b.]. Motivated by these remarkable works, we generalize the result due to Robertson by finding suitable conditions on the parameters $\alpha, M, \gamma, \delta$ such that

$$\left|1 + \frac{zf''(z)}{f'(z)} + \alpha\right| < \gamma\left|\frac{zf'(z)}{f(z)} + \delta\right|$$

is sufficient for a function $f \in \mathcal{A}$ to be starlike. We shall first give a subordination theorem and use it to prove the generalized sufficient condition for starlikeness in Theorem 2.6.

Theorem 2.5. *Let Ω be a subset of $\mathbb{C}$. The class $\mathcal{S}(\Omega)$ consists of all admissible functions $\Psi\colon \mathbb{C}^2 \times \mathbb{D} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition*

$$\Psi\left(\frac{1}{1+Me^{i\theta}}, \frac{1-Ke^{i\theta}}{1+Me^{i\theta}}; z\right) \notin \Omega, \tag{1}$$

when $K, M \in \mathbb{R}, K \geq M$, $0 \leq \theta \leq 2\pi$ *and* $z \in \mathbb{D}$. *Let* $M \in \mathbb{R}$ *and the function* $f \in \mathcal{A}$ *satisfies the condition*

$$\Psi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}; z\right) \in \Omega \quad (z \in \mathbb{D}), \tag{2}$$

for some function $\Psi \in \mathcal{S}(\Omega)$, *then* $zf'(z)/f(z) \prec 1/(1 + Mz)$.

Proof. Define the function $p\colon \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{f(z)}{zf'(z)} - 1. \tag{3}$$

Since $f \in \mathcal{A}$, it follows that the function p is analytic and $p(0) = 0$. Also, from (3), we have

$$\frac{zf'(z)}{f(z)} = \frac{1}{p(z) + 1}$$

and

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 - zp'(z)}{p(z) + 1}.$$

We define the following transformations from $\mathbb{C}^2$ to $\mathbb{C}^2$ by

$$u = \frac{1}{1 + r} \quad \text{and} \quad v = \frac{1 - s}{1 + r}.$$

If we let

$$\psi(r, s; z) = \Psi(u, v; z) = \Psi\left(\frac{1}{1 + r}, \frac{1 - s}{1 + r}; z\right),$$

then, by (2), we get

$$\begin{aligned}\psi\left(p(z), zp'(z); z\right) &= \Psi\left(\frac{1}{1 + p(z)}, \frac{1 - zp'(z)}{1 + p(z)}; z\right)\\ &= \Psi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}; z\right) \in \Omega.\end{aligned}$$

For $r = Me^{i\theta}$ and $s = Ke^{i\theta}$, we have

$$u = \frac{1}{1 + Me^{i\theta}}, \quad \text{and} \quad v = \frac{1 - Ke^{i\theta}}{1 + Me^{i\theta}},$$

where $K \geq M$. Since $\Psi \in \mathcal{S}(\Omega)$, it follows that Ψ satisfies the admissibility condition

$$\Psi\left(\frac{1}{1 + Me^{i\theta}}, \frac{1 - Ke^{i\theta}}{1 + Me^{i\theta}}; z\right) \notin \Omega,$$

where $K, M \in \mathbb{R}$ and $K \geq M$. Therefore by Theorem 2.1, we have $|p(z)| < M$ or equivalently, $zf'(z)/f(z) \prec 1/(1 + Mz)$. □

THEOREM 2.6. *Let* $\delta \geq 0, \alpha \leq 1$ *and* $M \in [0, 1)$ *satisfy the conditions*

$$0 \leq \gamma \leq \frac{M(1 - \alpha) - |1 + \alpha|}{1 + \delta + M\delta}.$$

If the function $f \in \mathcal{A}$ *satisfies the condition*

$$\left|1 + \frac{zf''(z)}{f'(z)} + \alpha\right| < \gamma \left|\frac{zf'(z)}{f(z)} + \delta\right| \quad (z \in \mathbb{D}),$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1+Mz}.$$

Proof. Let $\Omega = (-\infty, 0)$ be a subset of $\mathbb{C}$. Define the function $\Psi\colon \mathbb{C}^2 \times \mathbb{D} \longrightarrow \mathbb{C}$ by

$$\Psi(u, v; z) = |v+\alpha| - \gamma|u+\delta|. \tag{4}$$

Then by the hypothesis, we have

$$\Psi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}; z\right) = \left|1+\frac{zf''(z)}{f'(z)}+\alpha\right| - \gamma\left|\frac{zf'(z)}{f(z)}+\delta\right| \in \Omega.$$

From (4), we have

$$\Psi\left(\frac{1}{1+Me^{i\theta}}, \frac{1-Ke^{i\theta}}{1+Me^{i\theta}}; z\right) = \left|\frac{1-Ke^{i\theta}}{1+Me^{i\theta}}+\alpha\right| - \gamma\left|\frac{1}{1+Me^{i\theta}}+\delta\right|.$$

We complete the proof by showing that for all $K, M \in \mathbb{R}$ with $K \geq M$,

$$\Psi\left(\frac{1}{1+Me^{i\theta}}, \frac{1-Ke^{i\theta}}{1+Me^{i\theta}}; z\right) \notin \Omega,$$

or equivalently,

$$\left|\frac{1-Ke^{i\theta}}{1+Me^{i\theta}}+\alpha\right| - \gamma\left|\frac{1}{1+Me^{i\theta}}+\delta\right| \geq 0.$$

This is equivalent to showing that

$$|1+\alpha+(M\alpha-K)e^{i\theta}| \geq \gamma|1+\delta+\delta Me^{i\theta}|. \tag{5}$$

Squaring (5), it is equivalent to show that

$$\begin{aligned}(1+\alpha)^2 + (M\alpha-K)^2 + 2(1+\alpha)(M\alpha-K)\cos\theta \\ - \gamma^2((1+\delta)^2+\delta^2M^2+2\delta M(1+\delta)\cos\theta) \geq 0.\end{aligned} \tag{6}$$

Since $|\cos\theta| \leq 1$, the inequality (6) follows if we show that, for all $K \geq M$,

$$(|M\alpha-K| - |1+\alpha|)^2 - \gamma^2(1+\delta+\delta M)^2 \geq 0. \tag{7}$$

For $\alpha \leq 1$ and $K \geq M$, we have $|M\alpha-K| \geq |M\alpha-M|$. Therefore, for γ given in the hypothesis, we have

$$\begin{aligned}(|M\alpha-K|-|1+\alpha|)^2 &\geq (|M\alpha-M|-|1+\alpha|)^2 \\ &\geq \gamma^2(1+\delta+\delta M)^2.\end{aligned}$$

Therefore, the inequality (7) holds and, hence by Theorem 2.5, the result follows. □

For $M = k/2$ and $\alpha = -1$, the above theorem reduces to the following corollary.

Corollary 2.7. *[20] If the function $f \in \mathcal{A}$ with $f(z)/z \neq 0$ and if there exists a $0 < k \leq 2$ such that*

$$\left|\frac{zf''(z)}{f'(z)}\right| < k\left|\frac{zf'(z)}{f(z)}\right| \quad (z \in \mathbb{D}),$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{2}{2+kz}.$$

3. Sufficient Conditons for Starlikeness

The ratios $zf''(z)/f'(z)$ and $zf'(z)/f(z)$ play fundamental roles in characterizing starlike and convex functions. Mocanu [13, Theorem 5.3d.] proved that if $f \in \mathcal{A}$ with $(f(z)f'(z))/z \neq 0$ then $|1 + zf''(z)| < \sqrt{2}|(zf'(z))/f(z)|$ is sufficient for f to be starlike. The following theorem gives γ such that

$$\left|1 + \frac{zf''(z)}{f'(z)} + \beta\right| < \gamma \left|\frac{zf'(z)}{f(z)} + \delta\right|,$$

is sufficient for a function $f \in \mathcal{A}$ to be starlike by making use of the following subordination theorem of Ravichandran *et al.* [3].

Theorem 3.1. *[3] Let Ω be a subset of $\mathbb{C}$ and let the function $\Psi\colon \mathbb{C}^2 \times \mathbb{D} \longrightarrow \mathbb{C}$ satisfy the admissibility condition*

$$\Psi(i\rho, i\tau; z) \notin \Omega$$

for all $z \in \mathbb{D}$ and for all real ρ, τ with

$$\rho\tau \geq \frac{1 + 3\rho^2}{2}.$$

If the function $f \in \mathcal{A}$ satisfies the conditions $f'(z)f(z)/z \neq 0$ and

$$\Psi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}; z\right) \in \Omega,$$

then the function f is starlike.

Theorem 3.2. *If the function $f \in \mathcal{A}$ satisfies the inequality*

$$\left|1 + \frac{zf''(z)}{f'(z)} + \beta\right| < \gamma \left|\frac{zf'(z)}{f(z)} + \delta\right| \quad (z \in \mathbb{D}),$$

where

$$\gamma = \begin{cases} \frac{3}{2}, & |\delta| \leq \sqrt{\frac{2}{3}}, \beta \in \mathbb{R} \\ \frac{3}{2}, & |\delta| > \sqrt{\frac{2}{3}}, |\beta| \geq \frac{\sqrt{3(3\delta^2-2)}}{2} \\ \frac{\sqrt{3\delta^2-1+2\beta^2\delta^2+\sqrt{(3\delta^2-1)^2-4\beta^2\delta^2}}}{\sqrt{2}\delta^2}, & |\delta| > \sqrt{\frac{2}{3}}, |\beta| \leq \frac{\sqrt{3(3\delta^2-2)}}{2} \end{cases}$$

then the function f is starlike.

Proof. Let $\Omega = (-\infty, 0)$ be a subset of $\mathbb{C}$. Define the function $\Psi\colon \mathbb{C}^2 \times \mathbb{D} \longrightarrow \mathbb{C}$ by

$$\Psi(u, v; z) = |v + \beta| - \gamma|u + \delta|.$$

Then, by the hypothesis, we have

$$\Psi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}; z\right) = \left|1 + \frac{zf''(z)}{f'(z)} + \beta\right| - \gamma\left|\frac{zf'(z)}{f(z)} + \delta\right| \in \Omega.$$

In view of Theorem 3.1, the proof is complete if we show that $\Psi(i\rho, i\tau; z) \notin \Omega$ for all $\rho, \tau \in \mathbb{R}$, with $\rho\tau \geq (1 + 3\rho^2)/2$. Since

$$\Psi(i\rho, i\tau; z) = |i\tau + \beta| - \gamma|i\rho + \delta| = \sqrt{\tau^2 + \beta^2} - \gamma\sqrt{\rho^2 + \delta^2},$$

it is enough to show that

$$(\tau^2+\beta^2)-\gamma^2(\rho^2+\delta^2)\geq 0 \tag{8}$$

for all $\rho,\tau\in\mathbb{R}$, with $\rho\tau\geq(1+3\rho^2)/2$. Multiplying (8) by ρ^2 and using $\rho\tau\geq(1+3\rho^2)/2$, we see that inequality (8) holds if the inequality

$$(9-4\gamma^2)\rho^4+(4\beta^2-4\gamma^2\delta^2+6)\rho^2+1\geq 0$$

holds for all $\rho\in\mathbb{R}$. Therefore, in order to complete the proof, it is enough to show that

$$(9-4\gamma^2)x^2+(4\beta^2-4\gamma^2\delta^2+6)x+1\geq 0 \tag{9}$$

for all $x\geq 0$.

Case (i): $|\delta|\leq\sqrt{2/3},\beta\in\mathbb{R}$. If $\gamma=3/2$, then

$$(9-4\gamma^2)x^2+(4\beta^2-4\gamma^2\delta^2+6)x+1\geq 4\beta^2+6\geq 0.$$

Therefore, the inequality (9) holds for all $x\geq 0$.

Case (ii): $|\delta|>\sqrt{2/3},|\beta|\geq\sqrt{3(3\delta^2-2)}/2$. Since $|\beta|\geq\sqrt{3(3\delta^2-2)}/2$, we have $4\beta^2+6\geq 9\delta^2$. Hence, for $\gamma=3/2$, we have $4\beta^2-4\gamma^2\delta^2+6\geq 0$. Therefore, the inequality (9) holds for all $x\geq 0$.

Case (iii): $|\delta|>\sqrt{2/3},|\beta|\leq\sqrt{3(3\delta^2-2)}/2$.

The number $\gamma=\sqrt{3\delta^2-1+2\beta^2\delta^2+\sqrt{(3\delta^2-1)^2-4\beta^2\delta^2}}/\sqrt{2}\delta^2$ is a solution of the equation

$$(4\beta^2-4\gamma^2\delta^2+6)^2-4(9-4\gamma^2)=0. \tag{10}$$

Also, we have

$$\begin{aligned}9-4\gamma^2&=\frac{9\delta^4-6\delta^2+2-4\beta^2\delta^2-2\sqrt{(3\delta^2-1)^2-4\beta^2\delta^2}}{\delta^4}\\&=\frac{\left(\sqrt{(3\delta^2-1)^2-4\beta^2\delta^2}-1\right)^2}{\delta^4}\geq 0.\end{aligned} \tag{11}$$

Note that the inequality $ax^2+bx+c\geq 0$ holds for all $x\in\mathbb{R}$ if $a>0$ and $b^2-4ac\leq 0$. Since (10) and (11) holds, it is clear that (9) holds for all $x\geq 0$. □

For $\delta=1,\beta=0$, Theorem 3.2 reduces to the following corollary.

Corollary 3.3. *[13, Theorem 5.3d.] The function $f\in\mathcal{A}$ with $f(z)f'(z)/z\neq 0$ satisfying the inequality*

$$\left|1+\frac{zf''(z)}{f'(z)}\right|<\sqrt{2}\left|\frac{zf'(z)}{f(z)}+1\right|\quad(z\in\mathbb{D}),$$

is starlike.

For $\delta=-1,\beta=0$, Theorem 3.2 reduces to the following corollary.

Corollary 3.4. *The function $f\in\mathcal{A}$ satisfying the inequality*

$$\left|1+\frac{zf''(z)}{f'(z)}\right|<\sqrt{2}\left|\frac{zf'(z)}{f(z)}-1\right|\quad(z\in\mathbb{D}),$$

is starlike.

For $\beta=-1,\delta=0$, Theorem 3.2 reduces to the following result.

Corollary 3.5. *The function $f \in \mathcal{A}$ satisfying the inequality*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3}{2}\left|\frac{zf'(z)}{f(z)}\right| \quad (z \in \mathbb{D}),$$

is starlike.

For $\beta = -1, \delta = -1$, Theorem 3.2 reduces to the following result.

Corollary 3.6. *The function $f \in \mathcal{A}$ satisfying the inequality*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3}{2}\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \mathbb{D}),$$

is starlike.

4. Sufficient Conditons for Reciprocal Starlikeness

In general, convex functions of an order α need not be reciprocal starlike of an order β. As an example, for $\alpha \neq 1/2$, consider

$$f(z) = \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1},$$

the function $f \in \mathcal{A}$ which is convex. However, it is not reciprocal starlike of any order. Thus, establishing sufficient conditions for a function to be reciprocal starlike is worth mentioning. Libera [10] proved that the integral operator,

$$F(z) = \frac{2}{z}\int_0^t f(t)\,dt \tag{12}$$

preserves some subclasses of univalent functions. Specifically, he has shown that the integral operator retains the properties of starlike functions, convex functions, and close to convex functions. Miller and Mocanu [13, Corollary2.6g.1.] established that for a function $f \in \mathcal{A}$, $\operatorname{Re}(zf'(z)/f(z)) > -1/2$ is sufficient for the integral operator F defined by (12) to be starlike. Generalizing this result to any $\alpha \in [0,1)$, we obtained a sufficient condition for F to be reciprocal starlike. We shall use the following subordination theorem proved by Madhumitha *et al.* [16] for establishing sufficient conditions for a function in $\mathcal{A}$ to exhibit reciprocal starlikeness.

Theorem 4.1. *[16] Let $\alpha \in [0,1)$ and $f \in \mathcal{A}$ with $f'(z) \neq 0$. For $\Omega \subset \mathbb{C}$, let $\Psi(\Omega)$ be the class of all functions $\Psi\colon \mathbb{C}^2 \times \mathbb{D} \longrightarrow \mathbb{C}$ satisfying*

$$\Psi\left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z\right) \notin \Omega,$$

$\tau \in \mathbb{R}$, $\zeta + i\eta \in \mathbb{C}$ with $(\alpha + i\tau)(\zeta + i\eta) \in \mathbb{R}$ and $(\alpha + i\tau)(\zeta + i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$. If $\psi \in \Psi(\Omega)$ and

$$\psi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}; z\right) \in \Omega \quad (z \in \mathbb{D}),$$

then the function f is reciprocal starlike of order α.

THEOREM 4.2. *For $0 \leq \alpha < 1$, let $\gamma(\alpha)$ be defined by*

$$\gamma(\alpha) = \begin{cases} \frac{1}{2}\left(\frac{\alpha}{\alpha-1} + \frac{\alpha-1}{\alpha+1}\right), & 0 \leq \alpha \leq \frac{3}{4} \\ \frac{1}{2}\left(\frac{\alpha-3}{\alpha} + \frac{\alpha-1}{\alpha+1}\right), & \frac{3}{4} \leq \alpha < 1, \end{cases}$$

If the function $f \in \mathcal{A}$ is starlike of order $\gamma(\alpha)$, then the function F defined by

$$F(z) = \frac{2}{z}\int_0^t f(t)dt$$

is reciprocal starlike of order α.

Proof. Define the function $p\colon \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{F(z)}{zF'(z)}.$$

Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w > \gamma(\alpha)\}$. Define the function $\Psi\colon \mathbb{C}^2 \times \mathbb{D} \longrightarrow \mathbb{C}$ by

$$\Psi(r, s; z) = \frac{1}{r} + \frac{s}{r+1} - \frac{s}{r}. \tag{13}$$

A computation shows that

$$\frac{zf'(z)}{f(z)} = \frac{1}{p(z)} + \frac{zp'(z)}{p(z)+1} - \frac{zp'(z)}{p(z)} = \Psi(p(z), zp'(z); z) \in \Omega. \tag{14}$$

We complete the proof by showing

$$\Psi\left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z\right) \notin \Omega,$$

where $\tau \in \mathbb{R}$, $\zeta + i\eta \in \mathbb{C}$ with $(\alpha + i\tau)(\zeta + i\eta) \in \mathbb{R}$ and $(\alpha + i\tau)(\zeta + i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$. From (13), we have

$$\Psi\left(\frac{zF'(z)}{F(z)}, 1 + \frac{zF''(z)}{F'(z)}; z\right) = 1 + \frac{zF''(z)}{F'(z)} + \frac{1}{1 + \frac{zF'(z)}{F(z)}}\left(\frac{zF'(z)}{F(z)} - \left(1 + \frac{zF''(z)}{F'(z)}\right)\right).$$

Then, we have

$$\begin{aligned} \Psi\left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z\right) &= \zeta + i\eta + \frac{1}{1 + \frac{1}{\alpha+i\tau}}\left(\frac{1}{\alpha + i\tau} - (\zeta + i\eta)\right) \\ &= \frac{(\zeta + i\eta)(\alpha + i\tau)}{\alpha + i\tau} + \frac{1 - (\alpha + i\tau)(\zeta + i\eta)}{1 + \alpha + i\tau} \\ &= \frac{(\zeta + i\eta)(\alpha + i\tau)}{\alpha + i\tau} + \frac{(1 + \alpha - i\tau) - (\zeta + i\eta)(\alpha + i\tau)(1 + \alpha - i\tau)}{(1+\alpha)^2 + \tau^2}. \end{aligned}$$

Therefore, for $\tau \in \mathbb{R}$ and $\zeta + i\eta \in \mathbb{C}$ with $(\alpha + i\tau)(\zeta + i\eta) \in \mathbb{R}$ and $(\alpha + i\tau)(\zeta + i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$, the real part of $\Psi(1/(\alpha + i\tau), \zeta + i\eta; z)$ is given by

$$\begin{aligned}\operatorname{Re}\Psi\left(\frac{1}{\alpha+i\tau},\zeta+i\eta;z\right) &= \frac{\alpha(\alpha+i\tau)(\zeta+i\eta)}{\alpha^2+\tau^2} + \frac{1+\alpha}{(1+\alpha)^2+\tau^2} \\ &\quad - \frac{(\alpha+i\tau)(\zeta+i\eta)(1+\alpha)}{(1+\alpha)^2+\tau^2} \\ &\leq -\frac{\alpha}{2(1-\alpha)}\left(\frac{(3-\alpha)(1-\alpha)+\tau^2}{\alpha^2+\tau^2}\right) \\ &\quad - \frac{1+\alpha}{2(1-\alpha)}\left(\frac{(1-\alpha)^2+\tau^2}{(1+\alpha)^2+\tau^2}\right).\end{aligned}$$

Let $\tau^2 = t$ and the function $g : [0, \infty) \to \mathbb{R}$ be defined by

$$g(t) = -\frac{1+\alpha}{2(1-\alpha)}\left(\frac{(1-\alpha)^2+t}{(1+\alpha)^2+t}\right). \tag{15}$$

Define the function $k : [0, \infty) \to \mathbb{R}$ by

$$k(t) = -\frac{\alpha}{2(1-\alpha)}\left(\frac{(3-\alpha)(1-\alpha)+t}{\alpha^2+t}\right). \tag{16}$$

For $0 \leq \alpha < 1$ and $t \geq 0$, from (15), we have

$$g'(t) = -\frac{1+\alpha}{2(1-\alpha)}\left(\frac{4\alpha}{t+(1+\alpha)^2}\right) < 0.$$

Therefore the function g is a decreasing function of t and hence

$$g(t) \leq g(0) = \frac{\alpha-1}{2(\alpha+1)}.$$

Differentiating (16), we have

$$k'(t) = -\frac{\alpha}{2(1-\alpha)}\left(\frac{4\alpha-3}{(t+\alpha^2)^2}\right).$$

Case (i): $0 \leq \alpha \leq 3/4$, $t \geq 0$. For all $\tau \in \mathbb{R}$, $\zeta + i\eta \in \mathbb{C}$ with $(\alpha + i\tau)(\zeta + i\eta) \in \mathbb{R}$ and $(\alpha + i\tau)(\zeta + i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$, $k'(t) > 0$. Therefore the function $k(t) \leq k(\infty) = \alpha/(2(\alpha-1))$ and hence

$$\operatorname{Re}\Psi\left(\frac{1}{\alpha+i\tau},\zeta+i\eta;z\right) \leq \frac{1}{2}\left(\frac{\alpha}{\alpha-1}+\frac{\alpha-1}{\alpha+1}\right).$$

It then follows that $\Psi\left(1/(\alpha + i\tau), \zeta + i\eta; z\right) \notin \Omega$. Hence by Theorem 4.1, the result follows.

Case (ii): $3/4 < \alpha < 1$, $t \geq 0$. For all $\tau \in \mathbb{R}$, $\zeta + i\eta \in \mathbb{C}$ with $(\alpha + i\tau)(\zeta + i\eta) \in \mathbb{R}$ and $(\alpha + i\tau)(\zeta + i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$, we have $k'(t) < 0$ and therefore $k(t) \leq k(0) = (\alpha - 3)/2\alpha$ and hence

$$\operatorname{Re}\Psi\left(\frac{1}{\alpha+i\tau},\zeta+i\eta;z\right) \leq \frac{1}{2}\left(\frac{\alpha-3}{\alpha}+\frac{\alpha-1}{\alpha+1}\right).$$

By case (i) and case (ii), it is clear that $\Psi\left(1/(\alpha + i\tau), \zeta + i\eta; z\right) \notin \Omega$. Hence the result follows from an application of Theorem 4.1. $\square$

For $\gamma(\alpha)$ defined in Theorem 4.2, we have $\gamma(0) = -1/2$. Therefore, in the case $\alpha = 0$, Theorem 4.2 reduces to the following corollary.

Corollary 4.3. *[13, Corollary2.6g.1.] If the function $f \in \mathcal{A}$ and satisfies the following condition*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > -\frac{1}{2} \quad (z \in \mathbb{D}),$$

then the function F defined by

$$F(z) = \frac{2}{z}\int_0^t f(t)dt$$

is reciprocal starlike.

We know that every convex function $f \in \mathcal{A}$ is starlike of order $1/2$. Miller and Mocanu [13] proved that the convexity can be weakened by restricting the class of functions to $\mathcal{A}_2$. They proved that $\operatorname{Re}(1 + zf''(z)/f'(z)) > -1/2$ is sufficient for a function $f \in \mathcal{A}_2$ to be starlike of order $1/2$. The following theorem gives sufficient condition for a function in the class $\mathcal{A}_2$ to be starlike of reciprocal order $\alpha \in [0, 1)$. For $q(z) = (1+z)/(1-z)$, Theorem 1.1 reduces to the following theorem and we shall be using this to prove a sufficient condition for reciprocal starlikeness.

Theorem 4.4. *[13] Let Ω be a subset of $\mathbb{C}$. The class $P_n(\Omega)$ consists of all admissible functions $\psi\colon \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(\rho i, \sigma, \mu + i\nu; z) \notin \Omega$$

when $\rho, \sigma, \mu, \nu \in \mathbb{R}$ and $\sigma \leq -n(1 + \rho^2)/2$, $\sigma + \mu \leq 0$ and $z \in \mathbb{D}$. Let the function $\psi \in \mathcal{P}_n(\Omega)$. If the analytic function p with $p(0) = 1$ satisfies the condition

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \quad (z \in \mathbb{D}),$$

then $\operatorname{Re} p(z) > 0$.

Theorem 4.5. *If the function $f \in \mathcal{A}_2$ and satisfies the following condition*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \gamma(\alpha) \quad (z \in \mathbb{D}),$$

where $\gamma(\alpha)$ is defined by the function,

$$\gamma(\alpha) = \begin{cases} \frac{\alpha}{1-\alpha}, & \frac{1}{2} < \alpha \leq \frac{2}{3} \\ \frac{2-\alpha}{\alpha}, & \frac{2}{3} \leq \alpha < 1, \end{cases}$$

then the function f is reciprocal starlike of order α.

Proof. Define the function $p\colon \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{\frac{f(z)}{zf'(z)} - \alpha}{1 - \alpha}.$$

A simple computation then shows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 - (1-\alpha)zp'(z)}{\alpha + (1-\alpha)p(z)}.$$

Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w < \gamma(\alpha)\}$. Define the function $\Psi \colon \mathbb{C}^2 \times \mathbb{D} \longrightarrow \mathbb{C}$ by

$$\Psi(r, s; z) = \frac{1 - (1-\alpha)s}{\alpha + (1-\alpha)r}. \tag{17}$$

Then, by the hypothesis, we have

$$\Psi(p(z), zp'(z); z) = 1 + \frac{zf''(z)}{f'(z)} \in \Omega.$$

We complete the proof by showing $\Psi(i\rho, \sigma; z) \notin \Omega$ where $\rho, \sigma \in \mathbb{R}$ and $\sigma \leq -(1+\rho^2)$. From (17), we have

$$\Psi(i\rho, \sigma; z) = \frac{1 - (1-\alpha)\sigma}{\alpha + (1-\alpha)i\rho}.$$

Therefore, the real part of the function Ψ is given by

$$\operatorname{Re} \Psi(i\rho, \sigma; z) = \frac{\alpha(1 - (1-\alpha)\sigma)}{\alpha^2 + (1-\alpha)^2\rho^2} \geq \frac{\alpha + \alpha(1-\alpha)(1+\rho^2)}{\alpha^2 + (1-\alpha)^2\rho^2} = g(\rho^2)$$

where the function $g : [0, \infty) \to \mathbb{R}$ is defined by

$$g(t) = \frac{\alpha + \alpha(1-\alpha)(1+t)}{\alpha^2 + (1-\alpha)^2 t}.$$

Case (i): $1/2 < \alpha \leq 2/3$. In this case, we have

$$g'(t) = -\frac{\alpha(3\alpha - 2)(\alpha - 1)}{((\alpha-1)^2 t + \alpha)^2} < 0.$$

Therefore, the function g is decreasing in $[0, \infty)$ and hence, the function g attains its minima at ∞. Thus, we have $g(t) \geq g(\infty) = \alpha/(1-\alpha)$ and hence $\Psi(i\rho, \sigma; z) \notin \Omega$ for all $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1+\rho^2)$.

Case (ii): $2/3 \leq \alpha < 1$. In this case, we have $g'(t) > 0$. Therefore, the function g is increasing in $[0, \infty)$ and hence the function g attains its minima at $t = 0$. Thus, we have $g(t) \geq g(0) = (2-\alpha)/\alpha$ and hence $\Psi(i\rho, \sigma; z) \notin \Omega$ for all $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1+\rho^2)$.
By case (i) and case (ii), it is clear that $\Psi(i\rho, \sigma; z) \notin \Omega$ for all $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1+\rho^2)$. Hence, the result follows from Theorem 4.4. $\square$

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