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GENERALIZED LUKASIEWICZ FUZZY SUBALGEBRAS OF BCI-ALGEBRAS AND BCK-ALGEBRAS

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Abstract. The aim of this paper is to generalize Łukasiewicz fuzzy subalgebras in BCK/BCI-algebras. First, the concept of (α, ε)-Łukasiewicz fuzzy subalgebras using fuzzy points is defined and examples to explain it are given, and then several properties are investigated. The relationship between Łukasiewicz fuzzy subalgebras and (α, ε)-Łukasiewicz fuzzy subalgebras is discussed, and the conditions under which the ε-Łukasiewicz fuzzy set to be an (α, ε)-Łukasiewicz fuzzy subalgebra are explored. The characterizations of (α, ε)-Łukasiewicz fuzzy subalgebras are examined. Conditions under which Łukasiewicz $\in$-set, Łukasiewicz q-set and Łukasiewicz O-set can be subalgebras are handled.

1. Introduction

Łukasiewicz (fuzzy) logic, which is named by the Polish logician Jan Łukasiewicz, is a foundational system in the realm of fuzzy logic and multi-valued logic. This logic extends the classical two-valued logic (true or false) by allowing the degree of truth represented as real numbers in the unit interval $[0, 1]$. Łukasiewicz (fuzzy) logic uses a t-norm and t-co-norm, ensuring the operations are well-suited for reasoning with uncertainty. Using the Łukasiewicz t-norm, Jun [6] constructed the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and applied it to BCK-algebras and BCI-algebras. The concept of Łukasiewicz fuzzy sets is also applied to BCK/BCI-algebras, BE-algebras, Hilbert algebras, hoops, and Sheffer stroke Hilbert algebras, etc. (see [1, 2, 7–9, 11, 13, 14]).

In this paper, we consider the generalized version of Łukasiewicz fuzzy subalgebras in BCK/BCI-algebras. We define the concept of (α, ε)-Łukasiewicz fuzzy subalgebras using fuzzy points, and provide examples to explain it. We investigate several properties of (α, ε)-Łukasiewicz fuzzy subalgebras, and discuss the relationship between Łukasiewicz fuzzy subalgebras and (α, ε)-Łukasiewicz fuzzy subalgebras. We explore the conditions under which the ε-Łukasiewicz fuzzy set to be an (α, ε)-Łukasiewicz fuzzy subalgebra, and examine the characterizations of (α, ε)-Łukasiewicz fuzzy subalgebras. We find conditions under which the Łukasiewicz $\in$-set, Łukasiewicz q-set and Łukasiewicz O-set are subalgebras.

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2. Preliminaries

This section lists the known default content that will be used later. See the books [3, 10] for further information regarding BCK-algebras and BCI-algebras.

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [4] and [5]) and was extensively investigated by several researchers.

If a set X has a special element "0" and a binary operation "$*$" satisfying the conditions:

(I_1) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\ (((\mathfrak{a} * \mathfrak{b}) * (\mathfrak{a} * \mathfrak{c})) * (\mathfrak{c} * \mathfrak{b}) = 0)$,
(I_2) $(\forall \mathfrak{a}, \mathfrak{b} \in X)\ ((\mathfrak{a} * (\mathfrak{a} * \mathfrak{b})) * \mathfrak{b} = 0)$,
(I_3) $(\forall \mathfrak{a} \in X)\ (\mathfrak{a} * \mathfrak{a} = 0)$,
(I_4) $(\forall \mathfrak{a}, \mathfrak{b} \in X)\ (\mathfrak{a} * \mathfrak{b} = 0,\ \mathfrak{b} * \mathfrak{a} = 0\ \Rightarrow\ \mathfrak{a} = \mathfrak{b})$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(K) $(\forall \mathfrak{a} \in X)\ (0 * \mathfrak{a} = 0)$,

then X is called a *BCK-algebra*. In what follows, BCK/BCI-algebra is expressed as $(X, 0)_*$.

The order relation "$\leq$" in a BCK/BCI-algebra $(X, 0)_*$ is defined as follows:

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \leq \mathfrak{b}\ \Leftrightarrow\ \mathfrak{a} * \mathfrak{b} = 0).$$

Every BCK/BCI-algebra $(X, 0)_*$ satisfies the following conditions (see [3, 10]):

$$(\forall \mathfrak{a} \in X)\,(\mathfrak{a} * 0 = \mathfrak{a})\,,$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\,(\mathfrak{a} \leq \mathfrak{b}\ \Rightarrow\ \mathfrak{a} * \mathfrak{c} \leq \mathfrak{b} * \mathfrak{c},\ \mathfrak{c} * \mathfrak{b} \leq \mathfrak{c} * \mathfrak{a})\,,$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\,((\mathfrak{a} * \mathfrak{b}) * \mathfrak{c} = (\mathfrak{a} * \mathfrak{c}) * \mathfrak{b})\,.$$

Every BCI-algebra $(X, 0)_*$ satisfies (see [3])

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)\,(\mathfrak{a} * (\mathfrak{a} * (\mathfrak{a} * \mathfrak{b})) = \mathfrak{a} * \mathfrak{b})\,,$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in X)\,(0 * (\mathfrak{a} * \mathfrak{b}) = (0 * \mathfrak{a}) * (0 * \mathfrak{b}))\,.$$

A subset K of a BCK/BCI-algebra $(X, 0)_*$ is called a *subalgebra* of X (see [3, 10]) if it satisfies

$$(\forall \mathfrak{a}, \mathfrak{b} \in K)(\mathfrak{a} * \mathfrak{b} \in K).$$

A fuzzy set h in a set X of the form

$$h(\mathfrak{b}) := \begin{cases} t \in (0, 1] & \text{if } \mathfrak{b} = \mathfrak{a}, \\ 0 & \text{if } \mathfrak{b} \neq \mathfrak{a}, \end{cases}$$

is said to be a *fuzzy point* with support $\mathfrak{a}$ and value t and is denoted by $\langle \mathfrak{a}/t \rangle$.

For a fuzzy set h in a set X, we say that a fuzzy point $\langle \mathfrak{a}/t \rangle$ is

(i) *contained* in h, denoted by $\langle \mathfrak{a}/t \rangle \in h$, (see [12]) if $h(\mathfrak{a}) \geq t$.
(ii) *quasi-coincident* with h, denoted by $\langle \mathfrak{a}/t \rangle\, q\, h$, (see [12]) if $h(\mathfrak{a}) + t > 1$.

Definition 2.1 (see [6]). Let h be a fuzzy set in a set X and let $\varepsilon \in [0, 1]$. A function

$$Ł_h^{\varepsilon} : X \to [0, 1],\ x \mapsto \max\{0, h(x) + \varepsilon - 1\}$$

is called an *ε-Lukasiewicz fuzzy set* (of h) in X.

Let $Ł_h^\varepsilon$ be an ε-Łukasiewicz fuzzy set of a fuzzy set h in X. If $\varepsilon = 1$, then $Ł_h^\varepsilon(x) = \max\{0, h(x) + 1 - 1\} = \max\{0, h(x)\} = h(x)$ for all $x \in X$. This shows that if $\varepsilon = 1$, then the ε-Łukasiewicz fuzzy set of a fuzzy set h in X is the classisical fuzzy set h itself in X. If $\varepsilon = 0$, then $Ł_h^\varepsilon(x) = \max\{0, h(x) + 0 - 1\} = \max\{0, h(x) - 1\} = 0$ for all $x \in X$, that is, if $\varepsilon = 0$, then the ε-Łukasiewicz fuzzy set is the zero fuzzy set. Therefore, in handling the ε-Łukasiewicz fuzzy set, the value of ε can always be considered to be in $(0, 1)$.

Let h be a fuzzy set in a set X and $\varepsilon \in (0, 1)$. If $h(x) + \varepsilon \leq 1$ for all $x \in X$, then the ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ of h in X is the 0-constant function, that is, $Ł_h^\varepsilon(x) = 0$ for all $x \in X$. Therefore, in order for the ε-Łukasiewicz fuzzy set to have a meaningful form, a fuzzy set h in X and $\varepsilon \in (0, 1)$ must be set to satisfy the following condition:

$$(\exists x \in X)(h(x) + \varepsilon > 1).$$

For the Łukasiewicz fuzzy set $Ł_h^\varepsilon$ (of h) in X and $t \in (0, 1]$, consider the sets

$$(Ł_h^\varepsilon, t)_\in := \{x \in X \mid \langle x/t\rangle \in Ł_h^\varepsilon\},$$

$$(Ł_h^\varepsilon, t)_q := \{x \in X \mid \langle x/t\rangle \, q \, Ł_h^\varepsilon\},$$

which are called the *Łukasiewicz ∈-set* and *Łukasiewicz q-set*, respectively, of $Ł_h^\varepsilon$ (with value t). Also, consider a set:

$$O(Ł_h^\varepsilon) := \{x \in X \mid Ł_h^\varepsilon(x) > 0\}$$

which is called the *Łukasiewicz O-set* of $Ł_h^\varepsilon$. It is observed that

$$O(Ł_h^\varepsilon) = \{x \in X \mid h(x) + \varepsilon - 1 > 0\}.$$

Definition 2.2 (see [6]). Let h be a fuzzy set in a BCK/BCI-algebra $(X, 0)_*$ and ε an element of $(0, 1)$. Then its ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ in X is called an *ε-Łukasiewicz fuzzy subalgebra* of $(X, 0)_*$ if it satisfies:

$$\langle x/t_{\mathfrak{a}}\rangle \in Ł_h^\varepsilon,\ \langle y/t_{\mathfrak{b}}\rangle \in Ł_h^\varepsilon \ \Rightarrow \langle (x * y)/\min\{t_{\mathfrak{a}}, t_{\mathfrak{b}}\}\rangle \in Ł_h^\varepsilon \tag{1}$$

for all $x, y \in X$ and $t_{\mathfrak{a}}, t_{\mathfrak{b}} \in (0, 1]$.

Lemma 2.3. (see [6, Theorem 3.6]) *If h is a fuzzy subalgebra of a BCK/BCI-algebra $(X, 0)_*$ and ε is an element of $(0, 1)$, then its ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ in X is an ε-Łukasiewicz fuzzy subalgebra of $(X, 0)_*$.*

3. Generalized Łukasiewicz fuzzy subalgebras

In this section, let h and ε be a fuzzy set in X and an element of $(0, 1)$, respectively, unless otherwise specified. In addition, BCK-algebra or BCI-algebra is expressed as $(X, 0)_*$.

Definition 3.1. An ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ in X is called an *(α, ε)-Łukasiewicz fuzzy subalgebra* (briefly, *(α, ε)-Łf-subalgebra*) of $(X, 0)_*$ if the following assertion is valid.

$$\langle x/t_{\mathfrak{a}}\rangle \in Ł_h^\varepsilon,\ \langle y/t_{\mathfrak{b}}\rangle \in Ł_h^\varepsilon,\ \alpha \in \mathbb{R}^+ \ \vdash \langle (x * y)/\min\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}, \alpha\}\rangle \in Ł_h^\varepsilon \tag{2}$$

for all $x, y \in X$ and $t_{\mathfrak{a}}, t_{\mathfrak{b}} \in (0, 1]$.

Remark 3.2. Let $Ł_h^\varepsilon$ be an (α, ε)-Łf-subalgebra of $(X, 0)_*$. If $\alpha \geq 1$, then (2) goes back to (1). Hence if $\alpha \geq 1$, then every (α, ε)-Łf-subalgebra is an ε-Łukasiewicz fuzzy subalgebra. Also, it is clear that every ε-Łukasiewicz fuzzy subalgebra is an (α, ε)-Łf-subalgebra for all $\alpha \geq 1$. So when $\alpha \in (0, 1]$, (α, ε)-Łf-subalgebra has an independent meaning.

Example 3.3. Consider a BCK-algebra $(X, 0)_*$ where $X = \{e_0, e_1, e_2, e_3, e_4\}$ and a binary operation "$*$" is given by Table 1 (see [10])).

Table 1. Tabular representation for the operation "$*$"

$*$	e_0	e_1	e_2	e_3	e_4
e_0	e_0	e_0	e_0	e_0	e_0
e_1	e_1	e_0	e_1	e_0	e_0
e_2	e_2	e_2	e_0	e_0	e_0
e_3	e_3	e_3	e_3	e_0	e_0
e_4	e_4	e_3	e_4	e_1	e_0

Define a fuzzy set h in X as follows:

$$h : X \to [0, 1],\ y \mapsto \begin{cases} 0.76 & \text{if } y = e_0, \\ 0.69 & \text{if } y = e_1, \\ 0.63 & \text{if } y = e_2, \\ 0.57 & \text{if } y = e_3, \\ 0.42 & \text{if } y = e_4. \end{cases}$$

Given $\varepsilon := 0.59$, the ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ of h in X is given as follows:

$$Ł_h^\varepsilon : X \to [0, 1],\ y \mapsto \begin{cases} 0.35 & \text{if } y = e_0, \\ 0.28 & \text{if } y = e_1, \\ 0.22 & \text{if } y = e_2, \\ 0.16 & \text{if } y = e_3, \\ 0.01 & \text{if } y = e_4. \end{cases}$$

It is routine to verify that $Ł_h^\varepsilon$ is an (α, ε)-Łf-subalgebra of $(X, 0)_*$ for all $\alpha \in (0, 0.01]$.

Theorem 3.4. *Every ε-Łukasiewicz fuzzy subalgebra is an (α, ε)-Łf-subalgebra for all $\alpha \in \mathbb{R}^+$.*

Proof. The proof is straightforward. □

Corollary 3.5. *If h is a fuzzy subalgebra of $(X, 0)_*$, then its ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ is an (α, ε)-Łf-subalgebra of $(X, 0)_*$ for all $\alpha \in \mathbb{R}^+$.*

From the perspective of Theorem 3.4, we can say that the (α, ε)-Łf-subalgebra is a more generalized concept than the ε-Łukasiewicz fuzzy subalgebra.

The following example shows that there exists $\alpha \in \mathbb{R}^+$ such that an (α, ε)-Łf-subalgebra may not be an ε-Łukasiewicz fuzzy subalgebra. Also, the converse of Corollary 3.5 may not be true.

Example 3.6. Let $X = \{0, e_1, e_2, e_3, e_4\}$ be a set in which a binary operation "$*$" is given by Table 2.

TABLE 2. Tabular representation for the operation “$*$”

$*$	e_0	e_1	e_2	e_3	e_4
e_0	e_0	e_0	e_2	e_3	e_4
e_1	e_1	e_0	e_2	e_3	e_4
e_2	e_2	e_2	e_0	e_4	e_3
e_3	e_3	e_3	e_4	e_0	e_2
e_4	e_4	e_4	e_3	e_2	e_0

Then $(X,0)_*$ is a BCI-algebra (see [3]). Define a fuzzy set h in X as follows:

$$h : X \to [0,1],\ x \mapsto \begin{cases} 0.73 & \text{if } x = e_0, \\ 0.69 & \text{if } x = e_1, \\ 0.62 & \text{if } x = e_2, \\ 0.58 & \text{if } x = e_3, \\ 0.56 & \text{if } x = e_4. \end{cases}$$

Given $\varepsilon := 0.64$, the ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ of h in X is given as follows:

$$Ł_h^\varepsilon : X \to [0,1],\ x \mapsto \begin{cases} 0.37 & \text{if } x = e_0, \\ 0.33 & \text{if } x = e_1, \\ 0.26 & \text{if } x = e_2, \\ 0.22 & \text{if } x = e_3, \\ 0.20 & \text{if } x = e_4. \end{cases}$$

It is routine to verify that $Ł_h^\varepsilon$ is an (α,ε)-Łf-subalgebra of $(X,0)_*$ for all $\alpha \in (0, 0.2]$. But $Ł_h^\varepsilon$ is not an ε-Łukasiewicz fuzzy subalgebra of $(X,0)_*$ because of

$$Ł_h^\varepsilon(e_2 * e_3) = Ł_h^\varepsilon(e_4) = 0.20 \ngeq 0.22 = \min\{Ł_h^\varepsilon(e_2), Ł_h^\varepsilon(e_3)\}.$$

Also, h is not a fuzzy subalgebra of $(X,0)_*$ because of

$$h(e_2 * e_3) = h(e_4) = 0.56 \ngeq 0.58 = \min\{h(e_2), h(e_3)\}.$$

THEOREM 3.7. *Consider $\alpha, \beta \in \mathbb{R}^+$. If $\alpha \geq \beta$, then every (α,ε)-Łf-subalgebra of $(X,0)_*$ is a (β,ε)-Łf-subalgebra of $(X,0)_*$.*

Proof. Let $\alpha, \beta \in \mathbb{R}^+$ be such that $\alpha \geq \beta$, and let $Ł_h^\varepsilon$ be an (α,ε)-Łf-subalgebra of $(X,0)_*$. Let $x, y \in X$ and $t_\mathfrak{a}, t_\mathfrak{b} \in (0,1]$ be such that $\langle x/t_\mathfrak{a}\rangle \in Ł_h^\varepsilon$ and $\langle y/t_\mathfrak{b}\rangle \in Ł_h^\varepsilon$. Then $\langle (x*y)/\min\{t_\mathfrak{a},\, t_\mathfrak{b}, \alpha\}\rangle \in Ł_h^\varepsilon$ by Definition 3.1, and thus

$$Ł_h^\varepsilon(x*y) \geq \min\{t_\mathfrak{a},\, t_\mathfrak{b}, \alpha\} \geq \min\{t_\mathfrak{a},\, t_\mathfrak{b}, \beta\},$$

that is, $\langle (x*y)/\min\{t_\mathfrak{a},\, t_\mathfrak{b}, \beta\}\rangle \in Ł_h^\varepsilon$. This shows that

$$\langle x/t_\mathfrak{a}\rangle \in Ł_h^\varepsilon,\ \langle y/t_\mathfrak{b}\rangle \in Ł_h^\varepsilon,\ \beta \in \mathbb{R}^+ \ \vdash \langle (x*y)/\min\{t_\mathfrak{a},\, t_\mathfrak{b}, \beta\}\rangle \in Ł_h^\varepsilon$$

for all $x, y \in X$ and $t_\mathfrak{a}, t_\mathfrak{b} \in (0,1]$. Therefore $Ł_h^\varepsilon$ is a (β,ε)-Łf-subalgebra of $(X,0)_*$. $\square$

In Example 3.6, the ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ is an (α,ε)-Łf-subalgebra of $(X,0)_*$ for all $\alpha \in (0, 0.2]$. If we take $\beta = 0.3$, then $\alpha < \beta$. Note that $\langle e_2/0.26\rangle \in Ł_h^\varepsilon$, $\langle e_3/0.22\rangle \in Ł_h^\varepsilon$ and $\beta \in (0.2, 1] \subseteq \mathbb{R}^+$. But $\langle (e_2 * e_3)/\min\{0.26,\, 0.22, \beta\}\rangle \,\overline{\in}\, Ł_h^\varepsilon$, that is,

$$\begin{aligned} &\langle e_2/0.26\rangle \in Ł_h^\varepsilon,\ \langle e_3/0.22\rangle \in Ł_h^\varepsilon,\ \beta \in (0.2, 1] \subseteq \mathbb{R}^+ \\ &\nvdash \langle (e_2 * e_3)/\min\{0.26,\, 0.22, \beta\}\rangle \in Ł_h^\varepsilon \end{aligned}$$

Hence $Ł_h^\varepsilon$ is not a (β,ε)-Łf-subalgebra of $(X,0)_*$. This shows that if $\alpha<\beta$, then any (α,ε)-Łf-subalgebra may not be a (β,ε)-Łf-subalgebra.

Lemma 3.8. *Every (α,ε)-Łf-subalgebra $Ł_h^\varepsilon$ of $(X,0)_*$ satisfies*

$$\langle x/t\rangle \in Ł_h^\varepsilon,\ \alpha\in(0,1] \vdash \langle 0/\min\{t,\alpha\}\rangle \in Ł_h^\varepsilon \tag{3}$$

for all $x\in X$ and $t\in(0,1]$.

Proof. The combination of (I_3) and (2) induces (3). □

Proposition 3.9. *Every (α,ε)-Łf-subalgebra $Ł_h^\varepsilon$ of a BCI-algebra $(X,0)_*$ satisfies*

$$\langle x/t\rangle \in Ł_h^\varepsilon,\ \alpha\in(0,1] \vdash \langle (0*x)/\min\{t,\alpha\}\rangle \in Ł_h^\varepsilon. \tag{4}$$

for all $x,y\in X$ and $t\in(0,1]$.

Proof. Let $x\in X$ and $\alpha,t\in(0,1]$ be such that $\langle x/t\rangle\in Ł_h^\varepsilon$. Since $\langle 0/Ł_h^\varepsilon(0)\rangle\in Ł_h^\varepsilon$, we have

$$\langle (0*x)/\min\{t,\ \alpha\}\rangle = \langle (0*x)/\min\{t,\ Ł_h^\varepsilon(0),\alpha\}\rangle \in Ł_h^\varepsilon$$

by Definition 3.1 and Lemma 3.8. Hence (4) is verified. □

The combination of Lemma 2.3 and Theorem 3.4 induces the following corollary.

Corollary 3.10. *If h is a fuzzy subalgebra of a BCI-algebra $(X,0)_*$, then its ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ in X satisfies (4).*

Proposition 3.11. *If h is a fuzzy subalgebra of a BCI-algebra $(X,0)_*$, then its ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ in X satisfies*

$$\begin{aligned}&\langle x/t_{\mathfrak a}\rangle \in Ł_h^\varepsilon,\ \langle y/t_{\mathfrak b}\rangle \in Ł_h^\varepsilon,\ \alpha\in(0,1]\\ &\vdash \langle (x*(0*y))/\min\{\alpha,\ t_{\mathfrak a},\ t_{\mathfrak b}\}\rangle \in Ł_h^\varepsilon\end{aligned}$$

for all $x,y\in X$ and $t_{\mathfrak a},t_{\mathfrak b}\in(0,1]$.

Proof. Assume that h is a fuzzy subalgebra of a BCI-algebra $(X,0)_*$. Let $x,y\in X$ and $t_{\mathfrak a},t_{\mathfrak b},\alpha\in(0,1]$ be such that $\langle x/t_{\mathfrak a}\rangle\in Ł_h^\varepsilon$ and $\langle y/t_{\mathfrak b}\rangle\in Ł_h^\varepsilon$. Then $Ł_h^\varepsilon(x)\geq t_{\mathfrak a}$ and $Ł_h^\varepsilon(y)\geq t_{\mathfrak b}$. Hence

$$\begin{aligned}Ł_h^\varepsilon(x*(0*y)) &= \max\{0, h(x*(0*y))+\varepsilon-1\}\\ &\geq \max\{0,\min\{h(x),h(0*y)\}+\varepsilon-1\}\\ &= \max\{0,\min\{h(x),h(y)\}+\varepsilon-1\}\\ &= \max\{0,\min\{h(x)+\varepsilon-1,h(y)+\varepsilon-1\}\}\\ &= \min\{\max\{0,h(x)+\varepsilon-1\},\max\{0,h(y)+\varepsilon-1\}\}\\ &= \min\{Ł_h^\varepsilon(x),Ł_h^\varepsilon(y)\}\\ &\geq \min\{t_{\mathfrak a},t_{\mathfrak b}\}\geq \min\{\alpha,t_{\mathfrak a},t_{\mathfrak b}\},\end{aligned}$$

that is, $\langle (x*(0*y))/\min\{\alpha,\ t_{\mathfrak a},\ t_{\mathfrak b}\}\rangle\in Ł_h^\varepsilon$. This completes the proof. □

We provide conditions for an ε-Łukasiewicz fuzzy set to be an (α,ε)-Łf-subalgebra.

Theorem 3.12. *For every fuzzy set h in X, if its ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ satisfies*

$$\langle y/t_{\mathfrak b}\rangle \in Ł_h^\varepsilon,\ \langle z/t_{\mathfrak c}\rangle \in Ł_h^\varepsilon,\ \alpha\in\mathbb{R}^+ \vdash \langle (x*y)/\min\{t_{\mathfrak b},\ t_{\mathfrak c},\alpha\}\rangle \in Ł_h^\varepsilon \tag{5}$$

for all $t_{\mathfrak b},t_{\mathfrak c}\in(0,1]$ and $x,y,z\in X$ with $z\leq x$, then $Ł_h^\varepsilon$ is an (α,ε)-Łf-subalgebra of $(X,0)_*$.

Proof. Let $\alpha \in \mathbb{R}^+$, $x, y \in X$ and $t_{\mathfrak{a}}, t_{\mathfrak{b}} \in (0,1]$ be such that $\langle x/t_{\mathfrak{a}} \rangle \in Ł_h^\varepsilon$ and $\langle y/t_{\mathfrak{b}} \rangle \in Ł_h^\varepsilon$. Since $x \leq x$ for all $x \in X$, it follows from (5) that

$$\langle (x * y)/\min\{t_{\mathfrak{a}},\, t_{\mathfrak{b}}, \alpha\} \rangle \in Ł_h^\varepsilon.$$

Therefore, $Ł_h^\varepsilon$ is an (α, ε)-Łf-subalgebra of $(X, 0)_*$. □

We discuss the characterization of an (α, ε)-Łf-subalgebra.

THEOREM 3.13. *An ε-Lukasiewicz fuzzy set $Ł_h^\varepsilon$ in X is an (α, ε)-Łf-subalgebra of $(X, 0)_*$ if and only if it satisfies*

$$Ł_h^\varepsilon(x * y) \geq \min\{Ł_h^\varepsilon(x), Ł_h^\varepsilon(y), \alpha\} \tag{6}$$

for all $x, y \in X$ and $\alpha \in \mathbb{R}^+$.

Proof. Assume that $Ł_h^\varepsilon$ is an (α, ε)-Łf-subalgebra of $(X, 0)_*$. Let $x, y \in X$ and $\alpha \in \mathbb{R}^+$. Note that $\langle x/Ł_h^\varepsilon(x) \rangle \in Ł_h^\varepsilon$ and $\langle y/Ł_h^\varepsilon(y) \rangle \in Ł_h^\varepsilon$. Hence

$$\langle (x * y)/\min\{Ł_h^\varepsilon(x),\, Ł_h^\varepsilon(y), \alpha\} \rangle \in Ł_h^\varepsilon$$

by Definition 3.1, and so $Ł_h^\varepsilon(x * y) \geq \min\{Ł_h^\varepsilon(x), Ł_h^\varepsilon(y), \alpha\}$.

Conversely, suppose that $Ł_h^\varepsilon$ satisfies (6). Let $\alpha \in \mathbb{R}^+$, $x, y \in X$ and $t_{\mathfrak{a}}, t_{\mathfrak{b}} \in (0,1]$ be such that $\langle x/t_{\mathfrak{a}} \rangle \in Ł_h^\varepsilon$ and $\langle y/t_{\mathfrak{b}} \rangle \in Ł_h^\varepsilon$. Then $Ł_h^\varepsilon(x) \geq t_{\mathfrak{a}}$ and $Ł_h^\varepsilon(y) \geq t_{\mathfrak{b}}$, and thus

$$Ł_h^\varepsilon(x * y) \geq \min\{Ł_h^\varepsilon(x), Ł_h^\varepsilon(y), \alpha\} \geq \min\{t_{\mathfrak{a}}, t_{\mathfrak{b}}, \alpha\}$$

by (6). Hence $\langle (x * y)/\min\{t_{\mathfrak{a}}, t_{\mathfrak{b}}, \alpha\} \rangle \in Ł_h^\varepsilon$, which shows that (2) is valid. Thereore, $Ł_h^\varepsilon$ is an (α, ε)-Łf-subalgebra of $(X, 0)_*$. □

THEOREM 3.14. *If an ε-Lukasiewicz fuzzy set $Ł_h^\varepsilon$ in X is an (α, ε)-Łf-subalgebra of $(X, 0)_*$, then the Łukasiewicz $\in$-set $(Ł_h^\varepsilon, t)_\in$ of $Ł_h^\varepsilon$ is a subalgebra of $(X, 0)_*$ for all $\alpha \in \mathbb{R}^+$ and $t \in (0,1]$ with $t \leq \alpha$.*

Proof. Let $\alpha \in \mathbb{R}^+$ and and $t \in (0,1]$ be such that $t \leq \alpha$. If $x, y \in (Ł_h^\varepsilon, t)_\in$, then $\langle x/t \rangle \in Ł_h^\varepsilon$ and $\langle y/t \rangle \in Ł_h^\varepsilon$. It follows from (2) that

$$\langle (x * y)/t \rangle = \langle (x * y)/\min\{t, \alpha\} \rangle \in Ł_h^\varepsilon,$$

i.e., $x * y \in (Ł_h^\varepsilon, t)_\in$. Thus $(Ł_h^\varepsilon, t)_\in$ is a subalgebra of $(X, 0)_*$. □

We now consider the converse of Theorem 3.14.

THEOREM 3.15. *Let $\alpha \in \mathbb{R}^+$. Given an ε-Lukasiewicz fuzzy set $Ł_h^\varepsilon$ in X, if its Łukasiewicz $\in$-set $(Ł_h^\varepsilon, t)_\in$ is a subalgebra of $(X, 0)_*$ for all $t \in (0,1]$ with $t \leq \alpha$, then $Ł_h^\varepsilon$ is an (α, ε)-Łf-subalgebra of $(X, 0)_*$.*

Proof. Let $x, y \in X$ and $t_{\mathfrak{a}}, t_{\mathfrak{b}} \in (0,1]$ be such that $\langle x/t_{\mathfrak{a}} \rangle \in Ł_h^\varepsilon$ and $\langle y/t_{\mathfrak{b}} \rangle \in Ł_h^\varepsilon$. If $t_{\mathfrak{a}} \leq \alpha$ and $t_{\mathfrak{b}} \leq \alpha$, then $x \in (Ł_h^\varepsilon, t_{\mathfrak{a}})_\in \subseteq (Ł_h^\varepsilon, \min\{t_{\mathfrak{a}}, t_{\mathfrak{b}}\})_\in$ and $y \in (Ł_h^\varepsilon, t_{\mathfrak{b}})_\in \subseteq (Ł_h^\varepsilon, \min\{t_{\mathfrak{a}}, t_{\mathfrak{b}}\})_\in$. Since $\min\{t_{\mathfrak{a}}, t_{\mathfrak{b}}\} \leq \alpha$, $(Ł_h^\varepsilon, \min\{t_{\mathfrak{a}}, t_{\mathfrak{b}}\})_\in$ is a subalgebra of $(X, 0)_*$ by assumption. Hence

$$x * y \in (Ł_h^\varepsilon, \min\{t_{\mathfrak{a}}, t_{\mathfrak{b}}\})_\in \subseteq (Ł_h^\varepsilon, \min\{t_{\mathfrak{a}}, t_{\mathfrak{b}}, \alpha\})_\in,$$

and so $\langle (x * y)/\min\{t_{\mathfrak{a}},\, t_{\mathfrak{b}}, \alpha\} \rangle \in Ł_h^\varepsilon$. Therefore, $Ł_h^\varepsilon$ is an (α, ε)-Łf-subalgebra of $(X, 0)_*$. □

THEOREM 3.16. *If an ε-Lukasiewicz fuzzy set $Ł_h^\varepsilon$ in X is an (α, ε)-Łf-subalgebra of $(X, 0)_*$, then the Łukasiewicz q-set $(Ł_h^\varepsilon, t)_q$ of $Ł_h^\varepsilon$ is a subalgebra of $(X, 0)_*$ for all $\alpha \in \mathbb{R}^+$ and $t \in (0,1]$ with $t + \alpha \geq 1$.*

Proof. Let $\alpha \in \mathbb{R}^+$ and and $t \in (0,1]$ be such that $t+\alpha \geq 1$. If $x, y \in (Ł_h^\varepsilon, t)_q$, then $\langle x/t \rangle\, q\, Ł_h^\varepsilon$ and $\langle y/t \rangle\, q\, Ł_h^\varepsilon$, that is, $Ł_h^\varepsilon(x) + t > 1$ and $Ł_h^\varepsilon(y) + t > 1$. It follows from Theorem 3.13 that

$$Ł_h^\varepsilon(x * y) \geq \min\{Ł_h^\varepsilon(x), Ł_h^\varepsilon(y), \alpha\} \geq \min\{1-t, \alpha\} = 1-t.$$

Hence $\langle (x*y)/t \rangle\, q\, Ł_h^\varepsilon$, and so $x * y \in (Ł_h^\varepsilon, t)_q$. Therefore, $(Ł_h^\varepsilon, t)_q$ is a subalgebra of $(X, 0)_*$. □

Proposition 3.17. *Given an (α, ε)-Lf-subalgebra $Ł_h^\varepsilon$ of $(X,0)_*$, if its Łukasiewicz $\in$-set $(Ł_h^\varepsilon, t)_\in$ is a subalgebra of $(X,0)_*$ for all $\alpha \in \mathbb{R}^+$ and $t \in (0.5, 1]$ with $t \leq \alpha$, then the following inequality is valid:*

$$\max\{Ł_h^\varepsilon(x * y), 0.5\} \geq \min\{Ł_h^\varepsilon(x), Ł_h^\varepsilon(y), \alpha\} \tag{7}$$

for all $x, y \in X$.

Proof. Assume that $(Ł_h^\varepsilon, t)_\in$ is a subalgebra of $(X,0)_*$ for all $\alpha \in \mathbb{R}^+$ and $t \in (0.5, 1]$ with $t \leq \alpha$. If (7) is not valid, then there exist $t \in (0,1]$ and $\mathfrak{a}, \mathfrak{b} \in X$ such that

$$\max\{Ł_h^\varepsilon(\mathfrak{a} * \mathfrak{b}), 0.5\} < t \leq \min\{Ł_h^\varepsilon(\mathfrak{a}), Ł_h^\varepsilon(\mathfrak{b}), \alpha\}.$$

Then $t \in (0.5, 1]$, $t \leq \alpha$, $\langle \mathfrak{a}/t \rangle \in Ł_h^\varepsilon$ and $\langle \mathfrak{b}/t \rangle \in Ł_h^\varepsilon$. Hence $\mathfrak{a}, \mathfrak{b} \in (Ł_h^\varepsilon, t)_\in$, and so $\mathfrak{a} * \mathfrak{b} \in (Ł_h^\varepsilon, t)_\in$. It follows that $t \leq Ł_h^\varepsilon(\mathfrak{a} * \mathfrak{b}) = \max\{Ł_h^\varepsilon(\mathfrak{a} * \mathfrak{b}), 0.5\}$. This is a contradiction, and therefore $Ł_h^\varepsilon$ satisfies the inequality (7). □

The combination of Theorems 3.14 and 3.15 induces the following corollary.

Corollary 3.18. *If an ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ in X is an (α, ε)-Lf-subalgebra of $(X,0)_*$, then it satisfies the inequality* (7).

Now, we discuss the converse of Proposition 3.17.

Theorem 3.19. *If an ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ in X satisfies the inequality* (7) *for all $\alpha \in \mathbb{R}^+$ and $x, y \in X$, then its Łukasiewicz $\in$-set $(Ł_h^\varepsilon, t)_\in$ is a subalgebra of $(X,0)_*$ for all $t \in (0.5, 1]$ with $t \leq \alpha$.*

Proof. Let $t \in (0.5, 1]$ and $x, y \in X$ be such that $t \leq \alpha$ and $x, y \in (Ł_h^\varepsilon, t)_\in$. Then $Ł_h^\varepsilon(x) \geq t$ and $Ł_h^\varepsilon(y) \geq t$. Thus (7) induces

$$\max\{Ł_h^\varepsilon(x * y), 0.5\} \geq \min\{Ł_h^\varepsilon(x), Ł_h^\varepsilon(y), \alpha\} \geq \min\{t, \alpha\} = t$$

and so $Ł_h^\varepsilon(x * y) \geq t$ since $t > 0.5$. Hence $x * y \in (Ł_h^\varepsilon, t)_\in$, and thus $(Ł_h^\varepsilon, t)_\in$ is a subalgebra of $(X, 0)_*$. □

Theorem 3.20. *If an ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ in X is an (α, ε)-Lf-subalgebra of $(X,0)_*$ for all $\alpha (\neq 0) \in \mathbb{R}^+$, then its Łukasiewicz O-set $O(Ł_h^\varepsilon)$ is a subalgebra of $(X,0)_*$.*

Proof. Assume that $Ł_h^\varepsilon$ is an (α, ε)-Lf-subalgebra of $(X,0)_*$ for all $\alpha(\neq 0) \in \mathbb{R}^+$. If $x, y \in O(Ł_h^\varepsilon)$, then $h(x) + \varepsilon - 1 > 0$ and $h(y) + \varepsilon - 1 > 0$. Hence Theorem 3.13 induces

$$\begin{aligned} h(x*y) + \varepsilon - 1 &= \max\{0, h(x*y) + \varepsilon - 1\} = Ł_h^\varepsilon(x * y) \\ &\geq \min\{Ł_h^\varepsilon(x), Ł_h^\varepsilon(y), \alpha\} \\ &= \min\{h(x) + \varepsilon - 1, h(y) + \varepsilon - 1, \alpha\} > 0, \end{aligned}$$

and so $x * y \in O(Ł_h^\varepsilon)$. Therefore, $O(Ł_h^\varepsilon)$ is a subalgebra of $(X,0)_*$. □

THEOREM 3.21. *Let $\alpha \in \mathbb{R}^+$ be such that $\alpha + \varepsilon \leq 1$. If an ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ in X satisfies*

$$\text{(8)} \qquad \langle x/t_{\mathfrak{a}} \rangle \in Ł_h^\varepsilon,\ \langle y/t_{\mathfrak{b}} \rangle \in Ł_h^\varepsilon,\ \alpha \in \mathbb{R}^+ \ \vdash \langle (x * y)/\max\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}, \alpha\} \rangle \, q \, Ł_h^\varepsilon$$

for all $x, y \in X$ and $t_{\mathfrak{a}}, t_{\mathfrak{b}} \in (0, 1]$, then its Łukasiewicz O-set $O(Ł_h^\varepsilon)$ is a subalgebra of $(X, 0)_*$.

Proof. Assume that $Ł_h^\varepsilon$ satisfies (8) for all $x, y \in X$ and $t_{\mathfrak{a}}, t_{\mathfrak{b}} \in (0, 1]$. Let $x, y \in O(Ł_h^\varepsilon)$. Then $h(x) + \varepsilon - 1 > 0$ and $h(y) + \varepsilon - 1 > 0$. If we take $t_{\mathfrak{a}} := Ł_h^\varepsilon(x)$ and $t_{\mathfrak{b}} := Ł_h^\varepsilon(y)$, then $\langle x/t_{\mathfrak{a}} \rangle \in Ł_h^\varepsilon$ and $\langle y/t_{\mathfrak{b}} \rangle \in Ł_h^\varepsilon$. Thus

$$\langle (x * y)/\max\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}, \alpha\} \rangle \, q \, Ł_h^\varepsilon$$

by (8). If $x * y \notin O(Ł_h^\varepsilon)$, then $Ł_h^\varepsilon(x * y) = 0$ and so

$$\begin{aligned} &Ł_h^\varepsilon(x * y) + \max\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}, \alpha\} = \max\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}, \alpha\} \\ &= \max\{Ł_h^\varepsilon(x),\ Ł_h^\varepsilon(y), \alpha\} \\ &\leq \max\{h(x) + \varepsilon - 1, h(y) + \varepsilon - 1, 1 - \varepsilon\}. \end{aligned}$$

It follows that $Ł_h^\varepsilon(x * y) + \max\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}, \alpha\} \leq 1 - \varepsilon \leq 1$ or

$$\begin{aligned} &Ł_h^\varepsilon(x * y) + \max\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}, \alpha\} \leq \max\{h(x) + \varepsilon - 1, h(y) + \varepsilon - 1\} \\ &= \max\{h(x), h(y)\} + \varepsilon - 1 \leq 1 + \varepsilon - 1 = \varepsilon \leq 1. \end{aligned}$$

This is a contradiction, and thus $x * y \in O(Ł_h^\varepsilon)$. Therefore, $O(Ł_h^\varepsilon)$ is a subalgebra of $(X, 0)_*$. □

COROLLARY 3.22. (see [6, Theorem 3.21]) *If an ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ in X satisfies* (8) *for all $\alpha \leq \max\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}\}$, then its Łukasiewicz O-set $O(Ł_h^\varepsilon)$ is a subalgebra of $(X, 0)_*$.*

THEOREM 3.23. *If an ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ in X satisfies*

$$\text{(9)} \qquad \langle x/t_{\mathfrak{a}} \rangle \, q \, Ł_h^\varepsilon,\ \langle y/t_{\mathfrak{b}} \rangle \, q \, Ł_h^\varepsilon,\ \alpha \in \mathbb{R}^+ \ \vdash \langle (x * y)/\max\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}, \alpha\} \rangle \in Ł_h^\varepsilon$$

for all $x, y \in X$ and $t_{\mathfrak{a}}, t_{\mathfrak{b}} \in (0, 1]$, then its Łukasiewicz O-set $O(Ł_h^\varepsilon)$ is a subalgebra of $(X, 0)_*$.

Proof. Let $\alpha \in \mathbb{R}^+$ and $x, y \in O(Ł_h^\varepsilon)$. Then $h(x) + \varepsilon - 1 > 0$ and $h(y) + \varepsilon - 1 > 0$. Thus $Ł_h^\varepsilon(x) + 1 = h(x) + \varepsilon - 1 + 1 = h(x) + \varepsilon > 1$ and $Ł_h^\varepsilon(y) + 1 = h(x) + \varepsilon - 1 + 1 = h(y) + \varepsilon > 1$, that is, $\langle x/1 \rangle \, q \, Ł_h^\varepsilon$ and $\langle y/1 \rangle \, q \, Ł_h^\varepsilon$. It follows from (9) that $\langle (x * y)/\max\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}, \alpha\} \rangle \in Ł_h^\varepsilon$. Hence $Ł_h(x * y) \geq \max\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}, \alpha\}$, and so $Ł_h(x * y) \geq \max\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}\} > 0$ or $Ł_h(x * y) \geq \alpha > 0$. Thus $x * y \in O(Ł_h^\varepsilon)$ and $O(Ł_h^\varepsilon)$ is a subalgebra of $(X, 0)_*$. □

COROLLARY 3.24. (see [6, Theorem 3.22]) *If an ε-Łukasiewicz fuzzy set $Ł_h^\varepsilon$ in X satisfies* (9) *for all $\alpha \leq \max\{t_{\mathfrak{a}},\ t_{\mathfrak{b}}\}$, then its Łukasiewicz O-set $O(Ł_h^\varepsilon)$ is a subalgebra of $(X, 0)_*$.*

4. Conclusion

Łukasiewicz (fuzzy) logic, which is the logic of the Łukasiewicz t-norm, is a non-classical and many-valued logic. It was originally defined in the early 20th century by Jan Łukasiewicz as a three-valued logic BCK/BCI-algebras originally defined by K. Iséki and S. Tanaka in [5] to generalize the set difference in set theory. Using the idea of Łukasiewicz t-norm, Jun constructed the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and applied it to BCK-algebras and BCI-algebras. For the purpose of considering the generalization of Łukasiewicz fuzzy subalgebras in BCK/BCI-algebras, we defined (α, ε)-Łukasiewicz fuzzy subalgebras using fuzzy points and provided examples to illustrate it. We investigated several properties arising from (α, ε)-Łukasiewicz fuzzy subalgebras, and discussed the relation between Łukasiewicz fuzzy subalgebras and (α, ε)-Łukasiewicz fuzzy subalgebras. We explored the characterizations of (α, ε)-Łukasiewicz fuzzy subalgebras, and examined the conditions under which the ε-Łukasiewicz fuzzy set to be an (α, ε)-Łukasiewicz fuzzy subalgebra. We found conditions for the Łukasiewicz $\in$-set, Łukasiewicz q-set and Łukasiewicz O-set to be subalgebras.

The ideas and results obtained in this paper can be applied to various forms of logical algebras in the future.

References

[1] S. S. Ahn, E. H. Roh and Y. B. Jun, *Ideals in BE-algebras based on Łukasiewicz fuzzy set*, European Journal of Pure and Applied Mathematics, **15** (3) (2022), 1307–1320. https://doi.org/10.29020/nybg.ejpam.v15i3.4467

[2] R. A. Borzooei, S. S. Ahn and Y. B. Jun, *Łukasiewicz fuzzy filters of Sheffer stroke Hilbert algebras*, Journal of Intelligent & Fuzzy Systems, **46** (2024), 8231–8243. https://doi.org/10.3233/JIFS-233295

[3] Y. S. Huang, *BCI-algebra*, Science Press, Beijing, 2006.

[4] K. Iséki, *On BCI-algebras*, Mathematics Seminar Notes, **8** (1980), 125–130. https://api.semanticscholar.org/CorpusID:119048727

[5] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Mathematica Japonica, **23** (1978), 1–26.

[6] Y. B. Jun, *Łukasiewicz fuzzy subalgebras in BCK-algebras and BCI-algebras*, Annals of Fuzzy Mathematics and Informatics, **23** (2) (2022), 213–223. https://doi.org/10.30948/afmi.2022.23.2.213

[7] Y. B. Jun, *Łukasiewicz fuzzy ideals in BCK-algebras and BCI-algebras*, Journal of Algebra and Related Topics, **11** (1) (2023), 1–14.

[8] Y. B. Jun, *Positive implicative BE-filters of BE-algebras based on Łukasiewicz fuzzy sets*, Journal of Algebraic Hyperstructures and Logical Algebras, **4** (1) (2023), 1–11.

[9] Y. B. Jun and S. S. Ahn, *Łukasiewicz fuzzy BE-algebras and BE-filters*, European Journal of Pure and Applied Mathematics, **15** (3) (2022), 924–937. https://doi.org/10.29020/nybg.ejpam.v15i3.4446

[10] J. Meng and Y. B. Jun, *BCK-algebras*, Kyungmoonsa Co., Seoul, 1994.

[11] M. Mohseni Takallo, M. Aaly Kologani, Y. B. Jun and R. A. Borzooei, *Łukasiewicz fuzzy filters in hoops*, Journal of Algebraic Systems, **12** (1) (2024), 1–20. https://doi.org/10.22044/JAS.2022.12139.1632

[12] P. M. Pu and Y. M. Liu, *Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence*, Journal of Mathematical Analysis and Applications, **76** (1980), 571–599. https://doi.org/10.1016/0022-247X(80)90048-7

[13] G. R. Rezaei and Y. B. Jun, *Commutative ideals of BCI-algebras based on Łukasiewicz fuzzy sets*, Journal of Algebraic Hyperstructures and Logical Algebras, **3** (4) (2022), 25–36. https://doi.org/10.52547/HATEF.JAHLA.3.4.2

[14] S. Z. Song and Y. B. Jun, *Łukasiewicz fuzzy positive implicative ideals in BCK-algebras*, Journal of Algebraic Hyperstructures and Logical Algebras, **3** (2) (2022), 47–58. https://doi.org/10.52547/HATEF.JAHLA.3.2.4

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