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HYERS-ULAM STABILITY OF FUZZY HILBERT C^*-MODULE HOMOMORPHISMS AND FUZZY HILBERT C^*-MODULE DERIVATIONS

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Abstract. In the present paper, we introduce the notion of a fuzzy Hilbert C^*-module and study the Hyers-Ulam stability of fuzzy Hilbert C^*-module homomorphisms and fuzzy Hilbert C^*-module derivations in fuzzy Hilbert C^*-modules using the fixed point method.

1. Introduction and preliminaries

The field of fuzzy theory holds substantial importance in mathematics and applied sciences, where Zadeh's concept of fuzzy sets [42] has been widely adopted across various mathematical disciplines (see [6, 22, 24, 26, 35, 40]). George and Viramani [15] contributed new insights by modifying existing definitions, and Bag and Samantha [7] introduced a novel approach to fuzzy norms. Further advancements in fuzzy norm concepts were later achieved by Saadati and Vaezpour [38] and Ameri [5].

Definition 1.1. [7, 11] Let Ξ be a linear space. A fuzzy set $\mathcal{N}$ is considered a fuzzy norm on $\Xi \times (0, \infty)$ such that the following conditions hold: For all $x, y \in \Xi$ and $\alpha, \beta > 0$,

1) $\mathcal{N}(x, \alpha) > 0$;
2) $\mathcal{N}(x, \alpha) = 1,\ iff\ x = 0$;
3) $\mathcal{N}(ax, \alpha) = \mathcal{N}\left(x, \frac{\alpha}{|a|}\right),\ \forall a \neq 0$;
4) $\mathcal{N}(x + y, \alpha + \beta) \geq min\{\mathcal{N}(x, \alpha), \mathcal{N}(y, \beta)\}$;
5) $\mathcal{N}(x, \cdot)\ is\ continuous\ for\ each\ x \in \Xi$;
6) $\lim_{\alpha \to \infty} \mathcal{N}(x, \alpha) = 1$.

The pair $(\Xi, \mathcal{N})$ is called a fuzzy normed space.

Recently, Chaharpashlou *et al.* [10] introduced the notion of fuzzy inner product $\mathcal{A}$-modules, which can be viewed as an enhancement of the fuzzy inner products and fuzzy Hilbert spaces presented in [34]. For more information on fuzzy inner products and fuzzy Hilbert spaces, see [13, 16]. The concept of a Hilbert C^*-module is a generalization of the concept of Hilbert space, was introduced by Kaplansky [19].

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Before further definitions, it is important to introduce some notations here. Let $\mathcal{A}$ be a C^*-algebras. An element $\iota \in \mathcal{A}$ is said to positive if it is self adjoint and has a non-negative spectrum. We will denote the set of all positive elements of $\mathcal{A}$ by $\mathcal{A}^+$. $\mathcal{A}$ can be considered with the partial order: For any $\iota, \kappa \in \mathcal{A}$, $\iota \geq \kappa$ if and only if $\iota - \kappa \in \mathcal{A}^+$. The absolute value for each $\iota \in \mathcal{A}$ can be introduced as $|\iota| = (\iota\iota^*)^{\frac{1}{2}}$ [25, 30].

Definition 1.2. [25] Let $\mathcal{A}$ be a C^*-algebra and Ξ be a complex linear space equipped with a compatible left $\mathcal{A}$-module action (i.e., $\lambda(\iota x) = (\lambda\iota)x = \iota(\lambda x)$ for $x \in \Xi$, $\iota \in \mathcal{A}$, $\lambda \in \mathbb{C}$). The mapping $\langle\cdot,\cdot\rangle : \Xi \times \Xi \longrightarrow \mathcal{A}$ is called an (left) inner product if for all $x, y, z \in \Xi, \alpha, \beta \in \mathbb{C}, \iota \in \mathcal{A}$,

(i) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$;
(ii) $\langle \iota x, y\rangle = \iota\langle x, y\rangle$;
(iii) $\langle x, y\rangle^* = \langle y, x\rangle$;
(iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

If on Ξ the mapping $\langle\cdot,\cdot\rangle$ is an inner product, then Ξ is called a (left) pre-Hilbert $\mathcal{A}$-module or (left) inner product $\mathcal{A}$-module. Ξ is called a Hilbert C^*-module over $\mathcal{A}$ if it is complete with the induced norm introduced by $\|x\| := \|\langle x, x\rangle\|^{\frac{1}{2}}$ for $x \in \Xi$.

As an example, it is easy to see that every complex Hilbert space is a Hilbert C^*-module over $\mathbb{C}$, with its inner product. Moreover, every C^*-algebra $\mathcal{A}$ can be regarded as a Hilbert C^*-module over $\mathcal{A}$, where

$$\langle \iota, \kappa\rangle := \iota\kappa^*, \qquad (\iota, \kappa \in \mathcal{A}).$$

Definition 1.3. [10] Let $\mathcal{A}$ be an arbitrary C^*-algebra. A mapping $f : \mathcal{A}^+ \to ball(\mathcal{A}^+) = \{\iota \in \mathcal{A}^+ : \|\iota\| \leq 1\}$ is considered vanish at infinity if for every $\varepsilon > 0$ the set of $\{\iota \in \mathcal{A}^+ : \|f(\iota)\| \geq \varepsilon\}$ is compact. Also, $F_0(\mathcal{A}^+)$ is the notation used to represent the set of all mappings which are vanish at infinity.

Notice that if $\mathcal{A} = \mathbb{C}$, then we obtain

$$F_0(\mathcal{A}^+) = \{f : \mathbb{R}^+ \to [0, 1] : \lim_{\alpha\to\infty} f(\alpha) = 0\}.$$

Definition 1.4. [10] Let Ξ be a complex linear space, $\mathcal{A}$ be a unital C^*-algebra and μ represent a fuzzy set from $\Xi \times \Xi \times \mathcal{A}$ to ball$(\mathcal{A}^+)$. Then the pair (Ξ, μ) is called a fuzzy inner product $\mathcal{A}$-module if the following hold:

(FIPA1) $\mu(x, x, \iota) = 0, \quad \forall x \in \Xi, \quad \forall \iota \in \mathcal{A} \setminus \mathcal{A}^+$;

(FIPA2) $\mu(x, x, \iota) = 1_{\mathcal{A}}, \quad \forall \iota \in \mathcal{A}^+ \quad iff \quad x = 0$;

(FIPA3) $\mu(\alpha x, y, \iota) = \mu(x, y, \frac{\iota}{|\alpha|}), \quad \forall x, y \in \Xi, \quad \forall \iota \in \mathcal{A},\ \forall \alpha \in \mathbb{C} \setminus \{0\}$;

(FIPA4) $\mu(x, y, \iota^*) = \mu(y, x, \iota), \quad \forall x, y \in \Xi, \quad \forall \iota \in \mathcal{A}$;

(FIPA5) $\mu(x + y, z, |\iota| + |\kappa|) \geq \min\{\mu(x, z, |\iota|), \mu(y, z, |\kappa|)\}$,
$\forall x, y, z \in \Xi,\ \forall \iota, \kappa \in \mathcal{A}$;

(FIPA6) $\mu(x, x, \cdot) : \mathcal{A}^+ \to ball(\mathcal{A}^+), x \in \Xi,$ *is left continuous*
and $\mu(x, x, \cdot) - 1_{\mathcal{A}} \in F_0(\mathcal{A}^+)$;

(FIPA7) $\mu(x, y, |\iota\kappa|) \geq \min\{\mu(x, x, |\iota|^2), \mu(y, y, |\kappa|^2)\},\ \forall x, y \in \Xi,\ \forall \iota, \kappa \in \mathcal{A}$.

Definition 1.5. [10, Theorem 2.9] Let (Ξ, μ) be a fuzzy inner product $\mathcal{A}$-module such that $(\mathcal{A}^+, \leq)$ is totally ordered. Then a mapping $\mathcal{N} : \Xi \times \mathcal{A}^+ \longrightarrow ball(\mathcal{A}^+)$, given by

$$\mathcal{N}(x, \iota) = \mu(x, x, |\iota|^2) \qquad (x \in \Xi, \iota \in \mathcal{A}^+),$$

is a fuzzy norm on Ξ.

The stability problem of functional equations was arose from a question posed by Ulam [41] during his talk before a Mathematical Colloquium at the University of Wisconsin in 1940. Specifically, the question was: Let $(G_1, *)$ be a group and $(G_2, \diamond, d)$ be a metric group with the metric d. For a given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(x * y), f(x) \diamond f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $d(f(x), h(x)) < \epsilon$ for all $x \in G_1$? In 1941, Hyers [18] provided the first affirmative answer for approximately additive mappings in the context of Banach spaces. In 1978, Rassias [36] generalized the Hyers' result for linear mappings. Găvruţa [14] introduced a general control function and further extended the Rassias' results. The stability problems of various functional equations, functional inequalities and differential equations have been widely explored by numerous researchers (see [8, 17, 20, 21, 29, 33]).

In 2008, Mirmostafaee and Moslehian [28] introduced the idea of generalized Hyers-Ulam-Rassias stability in the fuzzy sense by introducing three different versions of fuzzy approximate additive function in fuzzy normed spaces. In 2009, Park [32] studied the fuzzy stability of a functional equation associated with inner product spaces. Since then, the stability of functional equations in fuzzy normed spaces has attracted the attention of scholars (see [23, 37]). The main objective of this paper is to investigate the fuzzy Hyers-Ulam stability of homomorphisms and derivations in the framework of fuzzy Hilbert C^*-modules.

Following Definition 1.5 we introduce the idea of limit in the framework of fuzzy inner product $\mathcal{A}$-modules. Throughout the paper we will denote the set of strictly positive elements of $\mathcal{A}$ by $\mathcal{A}^+_{>0}$.

Definition 1.6. Let (Ξ, μ) be a fuzzy inner product $\mathcal{A}$-module.
(1) A sequence $\{x_n\}$ in Ξ is said to be convergent to $x \in \Xi$ if for all $\iota \in \mathcal{A}^+_{>0}$

$$\lim_{n\to\infty} \mathcal{N}(x_n - x, \iota) = 1_{\mathcal{A}}.$$

(2) A sequence $\{x_n\}$ in Ξ is said to be Cauchy if for every $\epsilon > 0$ there exists p in $\mathbb{N}$ such that $\mathcal{N}(x_{n+p} - x_n, \iota) > 1_{\mathcal{A}} - \epsilon$, for all $\iota \in \mathcal{A}^+_{>0}$.
(3) (Ξ, μ) is called a complete fuzzy inner product $\mathcal{A}$-module or fuzzy Hilbert $\mathcal{A}$-module or simply a fuzzy Hilbert C^*-module if every Cauchy sequence converges in (Ξ, μ).

Definition 1.7. Let $\mathcal{A}$ be a unital C^*-algebra with unit $1_{\mathcal{A}}$ and $\mathcal{B}$ be a unital C^*-algebra with unit $1_{\mathcal{B}}$, and Ξ be a fuzzy Hilbert $\mathcal{A}$-module and ∇ be a fuzzy Hilbert $\mathcal{B}$-module.
(1) $H : \nabla \to \Xi$ is called a fuzzy Hilbert C^*-module homomorphism if

$$H(\langle x, y\rangle\, z) = \langle H(x), H(y)\rangle\, H(z)$$

for all $x, y, z \in \nabla$.
(2) $D : \Xi \to \Xi$ is called a fuzzy Hilbert C^*-module derivation if

$$H(\langle x, y\rangle\, z) = \langle H(x), y\rangle\, z + \langle x, H(y)\rangle\, z + \langle x, y\rangle\, H(z)$$

for all $x, y, z \in \Xi$.

We recall a fundamental result in the fixed point theory.

THEOREM 1.8. [9] Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

Using the fixed point method, many authors have studied the Hyers-Ulam stability of functional equations and differential equations (see [1–4]).

The paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of fuzzy Hilbert C^*-module homomorphisms (for Hilbert C^*-module homomorphisms, see [39]) in fuzzy Hilbert C^*-modules using the fixed point method. In Section 3, we establish the Hyers-Ulam stability of fuzzy Hilbert C^*-module derivations (for Hilbert C^*-module derivations, see [12]) on fuzzy Hilbert C^*-modules by employing the fixed point method.

Throughout the paper, Ξ denotes a fuzzy Hilbert C^*-module, $\mathcal{A}$ denotes a unital C^*-algebra and $\mathcal{A}^+_{>0}$ denotes the set of strictly positive elements of $\mathcal{A}$.

2. Hyers-Ulam stability of fuzzy Hilbert C^*-module homomorphisms in fuzzy Hilbert C^*-modules

We need the following lemma for our main results.

LEMMA 2.1. [31] Let X and Y be complex linear spaces and $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Then the mapping f is $\mathbb{C}$-linear.

THEOREM 2.2. *Let $f : \nabla \to \Xi$ be a mapping and assume that there exists a function $\varphi : \nabla^3 \to [0, \infty)$ such that*

$$\mathcal{N}\left(f(\mu x + y) - \mu f(x) - f(y), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x, y, 0)}\right) 1_{\mathcal{A}}, \tag{1}$$

$$\mathcal{N}\left(f(\langle x, y\rangle z) - \langle f(x), f(y)\rangle f(z), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x, y, z)}\right) 1_{\mathcal{A}} \tag{2}$$

for all $x, y, z \in \nabla$, $\iota \in \mathcal{A}^+_{>0}$ and all $\mu \in \mathbb{T}$. If there exists $0 \leq L < 1$ such that $\varphi(x, y, z) \leq 2L\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$ for all $x, y, z \in \nabla$, then there exists a unique fuzzy Hilbert C^*-module homomorphism $H : \nabla \to \Xi$ such that

$$\mathcal{N}\left(f(x) - H(x), \iota\right) \geq \left(\frac{(2 - 2L)\|\iota\|}{(2 - 2L)\|\iota\| + \varphi(x, x, 0)}\right) 1_{\mathcal{A}} \tag{3}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$.

Proof. Consider the set $\mathcal{S} = \{g : \nabla \to \Xi\}$, and define the generalized metric on $\mathcal{S}$:

$$\begin{aligned} d(g,h) : &= \inf\{c \in (0,\infty) : \mathcal{N}\left(g(x) - h(x), c\iota\right) \\ &\geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x,x,0)}\right) 1_{\mathcal{A}}, \forall x \in \nabla, \iota \in \mathcal{A}^+_{>0}\}. \end{aligned}$$

It can easily be shown that $(\mathcal{S}, d)$ is a complete metric space (see [27]).

Now, consider the linear mapping $J : \mathcal{S} \to \mathcal{S}$ such that

$$(Jg)(x) := \frac{1}{2} g(2x).$$

for all $x \in \nabla$.

Let $d(g,h) = \epsilon$ for any $g, h \in \mathcal{S}$. Then

$$\mathcal{N}\left(g(x) - h(x), \epsilon\iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x,x,0)}\right) 1_{\mathcal{A}}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$. Then

$$\begin{aligned} \mathcal{N}\left(Jg(x) - Jh(x), L\epsilon\iota\right) =& \mathcal{N}\left(\frac{1}{2}g(2x) - \frac{1}{2}h(2x), L\epsilon\iota\right) \\ =& \mathcal{N}\left(g(2x) - h(2x), 2L\epsilon\iota\right) \text{ (by (FIPA3) and Definition 1.5)} \\ \geq& \left(\frac{2L\|\iota\|}{2L\|\iota\| + \varphi(2x,2x,0)}\right) 1_{\mathcal{A}} \geq \left(\frac{2L\|\iota\|}{2L\|\iota\| + 2L\varphi(x,x,0)}\right) 1_{\mathcal{A}} \\ \geq& \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x,x,0)}\right) 1_{\mathcal{A}} \end{aligned}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$. So $d(g,h) = \epsilon$ implies that $d(Jg, Jh) = L\epsilon$. This shows that J is strictly contractive with the Lipschitz constant $L < 1$.

Letting $x = y$, $\mu = 1$ in (1), we obtain

$$\mathcal{N}\left(f(2x) - 2f(x), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x,x,0)}\right) 1_{\mathcal{A}} \tag{4}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$.

By (FIPA3), the above inequality implies that

$$\mathcal{N}\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}\iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x,x,0)}\right) 1_{\mathcal{A}}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$. Hence $d(f, Jf) \leq \frac{1}{2}$.

By Theorem 1.8, there exists a mapping $H : \Xi \to \Xi$ which is a fixed point of J such that

$$H(2x) = 2H(x) \tag{5}$$

for all $x \in \nabla$. Also $\lim_{n\to\infty} d(J^n f, H) = 0$. This implies that

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = H(x) \tag{6}$$

for all $x \in \nabla$. Notice that H is the unique fixed point of J in the set

$$U = \{g \in \mathcal{S} : d(g,f) < \infty\}.$$

This implies that H is the unique fixed point satisfying (5) such that there exists $c \in (0, \infty)$ satisfying

$$\mathcal{N}\left(f(x) - H(x), c\iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x, x, 0)}\right) 1_{\mathcal{A}}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^{+}_{>0}$.
Applying Theorem 1.8 once again, we obtain $d(f, H) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{2-2L}$. This implies that

$$\mathcal{N}\left(f(x) - H(x), \iota\right) \geq \left(\frac{(2-2L)\|\iota\|}{(2-2L)\|\iota\| + \varphi(x, x, 0)}\right) 1_{\mathcal{A}}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^{+}_{>0}$. Thus (3) holds.

Now we show that H is $\mathbb{C}$-linear. From the assumption $\varphi(x, y, z) \leq 2L\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$ for all $x, y, z \in \nabla$, it follows that

$$0 \leq \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) \leq \lim_{n\to\infty} \frac{2^n L^n}{2^n}\varphi(2^n x, 2^n y, 2^n z) = 0. \tag{7}$$

In (1), replacing x by $2^n x$, y by $2^n y$ respectively, using (FIPA3) and dividing both sides by 2^n, letting $n \to \infty$, using (7), by Definition 1.6 and applying (6), we obtain

$$H(\mu x + y) = \mu H(x) + H(y) \tag{8}$$

for all $\mu \in \mathbb{T}$ and all $x, y \in \nabla$. For $\mu = 1$, (8) implies that H is additive and so $H(0) = 0$, together with these, Lemma 2.1 implies that H is $\mathbb{C}$-linear.

Replacing x, y, z by $2^n x, 2^n y, 2^n z$ respectively in the left hand side of (2), we obtain

$$\mathcal{N}\left(f(\langle 2^n x, 2^n y\rangle\, 2^n z) - \langle f(2^n x), f(2^n y)\rangle\, f(2^n z), \iota\right).$$

Applying (2), we obtain

$$\mathcal{N}\left(f(2^{3n}\langle x, y\rangle\, z) - \langle f(2^n x), f(2^n y)\rangle\, f(2^n z), 2^{3n}\iota\right) \geq \left(\frac{2^{3n}\|\iota\|}{2^{3n}\|\iota\| + \varphi(2^n x, 2^n y, 2^n z)}\right) 1_{\mathcal{A}}.$$

This further implies that

$$\begin{aligned}\mathcal{N}\left(2^{-3n} f(2^{3n}\langle x, y\rangle\, z) - \left\langle \frac{f(2^n x)}{2^n}, \frac{f(2^n y)}{2^n}\right\rangle \frac{f(2^n z)}{2^n}, \iota\right) &\geq \left(\frac{2^{3n}\|\iota\|}{2^{3n}\|\iota\| + \varphi(2^n x, 2^n y, 2^n z)}\right) 1_{\mathcal{A}} \\ &= \left(\frac{\|\iota\|}{\|\iota\| + 2^{-3n}\varphi(2^n x, 2^n y, 2^n z)}\|\right) 1_{\mathcal{A}} \\ &\geq \left(\frac{\|\iota\|}{\|\iota\| + 2^{-3n}8^n L^n \varphi(x, y, z)}\|\right) 1_{\mathcal{A}}\end{aligned}$$

for all $x, y, z \in \nabla$ and all $\iota \in \mathcal{A}^{+}_{>0}$, since $\varphi(x, y, z) \leq 2L\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) \leq 8L\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$ for all $x, y, z \in \nabla$.

Taking the limit as $n \to \infty$, for all $\iota \in \mathcal{A}^{+}_{>0}$, using (7), by Definition 1.6 and applying (6), we obtain

$$\mathcal{N}\left(H(\langle x, y\rangle\, z) - \langle H(x), H(y)\rangle\, H(z), \iota\right) \geq 1_{\mathcal{A}}$$

for all $x, y, z \in \nabla$ and $\iota \in \mathcal{A}^{+}_{>0}$. By (FIPA2), we obtain

$$H(\langle x, y\rangle\, z) = \langle H(x), H(y)\rangle\, H(z)$$

for all $x, y, z \in \nabla$. Thus H is a fuzzy Hilbert C^*-module homomorphism. □

The following corollary gives us the Hyers-Ulam-Rassias stability of fuzzy Hilbert C^*-module homomorphisms.

Corollary 2.3. *Let $p \in (0,1)$, $\theta \geq 0$ and $f : \nabla \to \Xi$ be a mapping such that*

$$\mathcal{N}\left(f(\mu x + y) - \mu f(x) - f(y), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \theta(\|x\|^p + \|y\|^p)}\right) 1_{\mathcal{A}},$$

$$\mathcal{N}\left(f(\langle x, y\rangle z) - \langle f(x), f(y)\rangle f(z), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}\right) 1_{\mathcal{A}}$$

for all $x, y, z \in \nabla$, $\iota \in \mathcal{A}^+_{>0}$ and all $\mu \in \mathbb{T}$. Then there exists a unique fuzzy Hilbert C^*-module homomorphism $H : \nabla \to \Xi$ such that

$$\mathcal{N}\left(f(x) - H(x), \iota\right) \geq \left(\frac{(2 - 2^p)\|\iota\|}{(2 - 2^p)\|\iota\| + 2\theta\|x\|^p}\right) 1_{\mathcal{A}}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and $L = 2^{p-1}$. □

Theorem 2.4. *Let $f : \nabla \to \Xi$ be a mapping and assume that there exists a function $\varphi : \nabla^3 \to [0, \infty)$ such that*

$$\mathcal{N}\left(f(\mu x + y) - \mu f(x) - f(y), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x, y, 0)}\right) 1_{\mathcal{A}}, \tag{9}$$

$$\mathcal{N}\left(f(\langle x, y\rangle z) - \langle f(x), f(y)\rangle f(z), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x, y, z)}\right) 1_{\mathcal{A}} \tag{10}$$

for all $x, y, z \in \nabla$, $\iota \in \mathcal{A}^+_{>0}$ and all $\mu \in \mathbb{T}$. If there exists $0 \leq L < 1$ such that $\varphi(x, y, z) \leq \frac{L}{8}\varphi(2x, 2y, 2z)$ for all $x, y, z \in \nabla$, then there exists a unique fuzzy Hilbert C^*-module homomorphism $H : \nabla \to \Xi$ such that

$$\mathcal{N}\left(f(x) - H(x), \iota\right) \geq \left(\frac{(2 - 2L)\|\iota\|}{(2 - 2L)\|\iota\| + L\varphi(x, x, 0)}\right) 1_{\mathcal{A}} \tag{11}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$.

Proof. Consider the linear mapping $J : \mathcal{S} \to \mathcal{S}$ such that

$$(Jg)(x) := 2g(\frac{x}{2})$$

for all $x \in \nabla$.
Let $d(g, h) = \epsilon$ for any $g, h \in \mathcal{S}$. Then by (4) we have

$$\mathcal{N}\left(g(x) - h(x), \epsilon\iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x, x, 0)}\right) 1_{\mathcal{A}}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$. Thus

$$\begin{aligned}\mathcal{N}\left(Jg(x)-Jh(x),L\epsilon\iota\right)=&\mathcal{N}\left(2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right),L\epsilon\iota\right)\\=&\mathcal{N}\left(g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right),\frac{L}{2}\epsilon\iota\right)\\\geq&\left(\frac{\frac{L}{2}\|\iota\|}{\frac{L}{2}\|\iota\|+\varphi(\frac{x}{2},\frac{x}{2},0)}\right)1_A\geq\left(\frac{\frac{L}{2}\|\iota\|}{\frac{L}{2}\|\iota\|+\frac{L}{2}\varphi(x,x,0)}\right)1_{\mathcal{A}}\\\geq&\left(\frac{\|\iota\|}{\|\iota\|+\varphi(x,x,0)}\right)1_{\mathcal{A}}\end{aligned}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$, since $\varphi(x,y,z) \leq \frac{L}{8}\varphi(2x,2y,2z) \leq \frac{L}{2}\varphi(2x,2y,2z)$ for all $x,y,z \in \nabla$. So $d(g,h)=\epsilon$ implies that $d(Jg,Jh)=L\epsilon$. This shows that J is strictly contractive with the Lipschitz constant $L<1$.

From (4) and (FIPA3), it follows that

$$\mathcal{N}\left(f(x)-2f\left(\frac{x}{2}\right),\frac{L}{2}\iota\right)\geq\left(\frac{\|\iota\|}{\|\iota\|+\varphi(x,x,0)}\right)1_{\mathcal{A}}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$. Thus $d(f,Jf) \leq \frac{L}{2}$.

By Theorem 1.8, there exists a mapping $H : \nabla \to \Xi$ which is a fixed point of J such that

$$H\left(\frac{x}{2}\right)=\frac{1}{2}H\left(x\right) \tag{12}$$

for all $x \in \nabla$. Also $\lim_{n\to\infty} d(J^n f,H)=0$. This implies that

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)=H(x) \tag{13}$$

for all $x \in \nabla$. Notice that H is the unique fixed point of J in the set

$$U=\{g\in\mathcal{S}:d(g,f)<\infty\}.$$

This implies that H is the unique fixed point satisfying (12) such that there exists $c \in (0,\infty)$ satisfying

$$\mathcal{N}\left(f(x)-H(x),c\iota\right)\geq\left(\frac{\|\iota\|}{\|\iota\|+\varphi(x,x,0)}\right)1_{\mathcal{A}}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$.

Applying Theorem 1.8 once again, we obtain $d(f,H) \leq \frac{1}{1-L}d(f,Jf) \leq \frac{L}{2-2L}$. This implies that

$$\mathcal{N}\left(f(x)-H(x),\iota\right)\geq\left(\frac{(2-2L)\|\iota\|}{(2-2L)\|\iota\|+L\varphi(x,x,0)}\right)1_{\mathcal{A}}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$. Thus (11) holds.

Now to show that H is $\mathbb{C}$-linear. From the assumption that $\varphi(x,y,z) \leq \frac{L}{8}\varphi(2x,2y,2z) \leq \frac{L}{2}\varphi(2x,2y,2z)$ for all $x,y,z \in \nabla$, it follows that

$$\lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)=0. \tag{14}$$

In (9), replacing x by $\frac{x}{2^n}$, y by $\frac{y}{2^n}$ respectively, using (FIPA3) and multiplying both sides by 2^n, letting $n \to \infty$, using (14), by Definition 1.6 and applying (13), we obtain

$$H(\mu x + y) = \mu H(x) + H(y) \tag{15}$$

for all $x, y \in \nabla$ and $\mu \in \mathbb{T}$. For $\mu = 1$, (15) implies that H is additive and so $H(0) = 0$, together with these, Lemma 2.1 implies that H is $\mathbb{C}$-linear.

Replacing x, y, z by $\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}$ in the left side of (10), we obtain

$$\mathcal{N}\left(f\left(\left\langle \frac{x}{2^n}, \frac{y}{2^n}\right\rangle \frac{z}{2^n}\right) - \left\langle f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right)\right\rangle f\left(\frac{z}{2^n}\right), \iota\right).$$

Applying (10), we obtain

$$\begin{aligned}\mathcal{N}&\left(f\left(\left\langle \frac{x}{2^n}, \frac{y}{2^n}\right\rangle \frac{z}{2^n}\right) - \left\langle f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right)\right\rangle f\left(\frac{z}{2^n}\right), 2^{-3n}\iota\right)\\ &\geq \left(\frac{2^{-3n}\|\iota\|}{2^{-3n}\|\iota\| + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)}\right)1_{\mathcal{A}}.\end{aligned}$$

This further implies that

$$\begin{aligned}\mathcal{N}&\left(2^{3n}f\left(\left\langle \frac{x}{2^n}, \frac{y}{2^n}\right\rangle \frac{z}{2^n}\right) - \left\langle 2^n f\left(\frac{x}{2^n}\right), 2^n f\left(\frac{y}{2^n}\right)\right\rangle 2^n f\left(\frac{z}{2^n}\right), \iota\right)\\ &\geq \left(\frac{2^{-3n}\|\iota\|}{2^{-3n}\|\iota\| + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)}\right)1_{\mathcal{A}}\\ &\geq \left(\frac{\|\iota\|}{\|\iota\| + 2^{3n}\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)}\right)1_{\mathcal{A}}\\ &\geq \left(\frac{\|\iota\|}{\|\iota\| + 2^{3n}2^{-3n}L^n\varphi(x, y, z)}\right)1_{\mathcal{A}}.\end{aligned}$$

Taking the limit as $n \to \infty$, for all $\iota \in \mathcal{A}^+_{>0}$, using (14), by Definition 1.6 and applying (13), we obtain

$$\mathcal{N}\left(H(\langle x, y\rangle z) - \langle H(x), H(y)\rangle H(z), \iota\right) \geq 1_{\mathcal{A}}$$

for all $x, y, z \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$. By (FIPA2), we obtain

$$H(\langle x, y\rangle z) = \langle H(x), H(y)\rangle H(z)$$

for all $x, y, z \in \nabla$. Thus H is a fuzzy Hilbert C^*-module homomorphism. □

Corollary 2.5. *Let $p > 3$, $\theta \geq 0$ and $f : \nabla \to \Xi$ be a mapping such that*

$$\mathcal{N}\left(f(\mu x + y) - \mu f(x) - f(y), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \theta(\|x\|^p + \|y\|^p)}\right)1_{\mathcal{A}},$$

$$\mathcal{N}\left(f(\langle x, y\rangle z) - \langle f(x), f(y)\rangle f(z), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}\right)1_{\mathcal{A}}$$

for all $x, y, z \in \nabla$, $\iota \in \mathcal{A}^+_{>0}$ and all $\mu \in \mathbb{T}$. Then there exists a unique fuzzy Hilbert C^*-module homomorphism $H : \nabla \to \Xi$ such that

$$\mathcal{N}\left(f(x) - H(x), \iota\right) \geq \left(\frac{(2^p - 2)\|\iota\|}{(2^p - 2)\|\iota\| + 2\theta\|x\|^p}\right)1_{\mathcal{A}}$$

for all $x \in \nabla$ and $\iota \in \mathcal{A}^+_{>0}$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and $L = 2^{1-p}$. □

3. Hyers-Ulam stability of fuzzy Hilbert C^*-module derivations on fuzzy Hilbert C^*-modules

In this section, we prove the Hyers-Ulam stability of fuzzy Hilbert C^*-module derivations on fuzzy Hilbert C^*-modules.

THEOREM 3.1. *Let $f : \Xi \to \Xi$ be a mapping and assume that there exists a function $\varphi : \Xi^3 \to [0, \infty)$ such that*

$$\mathcal{N}\left(f(\mu x + y) - \mu f(x) - f(y), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x, y, 0)}\right) 1_{\mathcal{A}},$$

$$\mathcal{N}\left(f(\langle x, y\rangle z) - \langle f(x), y\rangle z - \langle x, f(y)\rangle z - \langle x, y\rangle f(z), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x, y, z)}\right) 1_{\mathcal{A}} \tag{16}$$

for all $x, y, z \in \Xi$, $\iota \in \mathcal{A}^+_{>0}$ and all $\mu \in \mathbb{T}$. If there exists $0 \leq L < 1$ such that $\varphi(x, y, z) \leq 2L\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$ for all $x, y, z \in \Xi$, then there exists a unique fuzzy Hilbert C^*-module derivation $D : \Xi \to \Xi$ such that

$$\mathcal{N}\left(f(x) - D(x), \iota\right) \geq \left(\frac{(2 - 2L)\|\iota\|}{(2 - 2L)\|\iota\| + \varphi(x, x, 0)}\right) 1_{\mathcal{A}} \tag{17}$$

for all $x \in \Xi$ and $\iota \in \mathcal{A}^+_{>0}$.

Proof. From Theorem 2.2, there exists a unique $\mathbb{C}$-linear mapping $D : \Xi \to \Xi$ satisfying (17) such that

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = D(x) \tag{18}$$

for all $x \in \Xi$.

We show that D is a Hilbert C^*-module derivation. Replacing x, y, z by $2^n x, 2^n y$ and $2^n z$ in the left side of (16), we obtain

$$\mathcal{N}\left(f(\langle 2^n x, 2^n y\rangle\, 2^n z) - \langle f(2^n x), 2^n y\rangle\, 2^n z - \langle 2^n x, f(2^n y)\rangle\, 2^n z - \langle 2^n x, 2^n y\rangle\, f(2^n z), \iota\right).$$

Applying (16), we have

$$\mathcal{N}\left(f(2^{3n} \langle x, y\rangle\, z) - \langle f(2^n x), y\rangle\, z - \langle x, f(2^n y)\rangle\, z - \langle x, y\rangle\, f(2^n z), 2^{3n}\iota\right) \geq \left(\frac{2^{3n}\|\iota\|}{2^{3n}\|\iota\| + \varphi(2^n x, 2^n y, 2^n z)}\right) 1_{\mathcal{A}}$$

for all $x, y, z \in \Xi$ and all $\iota \in \mathcal{A}^+_{>0}$. So we have

$$\begin{aligned}\mathcal{N}\left(2^{-3n} f(2^{3n} \langle x, y\rangle\, z) - \left\langle \frac{f(2^n x)}{2^n}, y\right\rangle z - \left\langle x, \frac{f(2^n y)}{2^n}\right\rangle z - \langle x, y\rangle\, \frac{f(2^n z)}{2^n}, \iota\right) &\geq \left(\frac{\|\iota\|}{\|\iota\| + 2^{-3n}\varphi(2^n x, 2^n y, 2^n z)}\right) 1_{\mathcal{A}} \\ &\geq \left(\frac{\|\iota\|}{\|\iota\| + 2^{-3n} 8^n L^n \varphi(x, y, z)}\right) 1_{\mathcal{A}}\end{aligned}$$

for all $x, y, z \in \Xi$ and all $\iota \in \mathcal{A}^+_{>0}$, since $\varphi(x, y, z) \leq 2L\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) \leq 8L\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$ for all $x, y, z \in \Xi$.

Taking the limit as $n \to \infty$, for all $\iota \in \mathcal{A}^+_{>0}$, using (7), by Definition 1.6 and applying (18), we obtain

$$\mathcal{N}\left(D(\langle x, y\rangle z) - \langle D(x), y\rangle z - \langle x, D(y)\rangle z - \langle x, y\rangle D(z), \iota\right) \geq 1_{\mathcal{A}}$$

for all $x, y, z \in \Xi$ and $\iota \in \mathcal{A}_{>0}$. By (FIPA2), we obtain

$$D(\langle x, y\rangle z) = \langle D(x), y\rangle z + \langle x, D(y)\rangle z + \langle x, y\rangle D(z)$$

for all $x, y, z \in \Xi$. Thus D is a fuzzy Hilbert C^*-module derivation. □

The following corollary gives us the Hyers-Ulam-Rassias stability of fuzzy Hilbert C^*-module derivations.

Corollary 3.2. *Let $p \in (0,1)$, $\theta \geq 0$ and $f : \Xi \to \Xi$ be a mapping such that*

$$\mathcal{N}\left(f(\mu x + y) - \mu f(x) - f(y), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \theta(\|x\|^p + \|y\|^p)}\right) 1_{\mathcal{A}},$$

$$\mathcal{N}\left(f(\langle x, y\rangle z) - \langle f(x), y\rangle z - \langle x, f(y)\rangle z - \langle x, y\rangle f(z)\iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}\right) 1_{\mathcal{A}}$$

for all $x, y, z \in \Xi$, $\iota \in \mathcal{A}^+_{>0}$ and all $\mu \in \mathbb{T}$. Then there exists a unique fuzzy Hilbert C^*-module derivation $D : \Xi \to \Xi$ such that

$$\mathcal{N}\left(f(x) - H(x), \iota\right) \geq \left(\frac{(2 - 2^p)\|\iota\|}{(2 - 2^p)\|\iota\| + 2\theta\|x\|^p}\right) 1_{\mathcal{A}}$$

for all $x \in \Xi$ and $\iota \in \mathcal{A}^+_{>0}$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and $L = 2^{p-1}$. □

Theorem 3.3. *Let $f : \Xi \to \Xi$ be a mapping and assume that there exists a function $\varphi : \Xi^3 \to [0, \infty)$ such that*

$$\mathcal{N}\left(f(\mu x + y) - \mu f(x) - f(y), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x, y, z)}\right) 1_{\mathcal{A}},$$

$$\mathcal{N}\left(f(\langle x, y\rangle z) - \langle f(x), y\rangle z - \langle x, f(y)\rangle z - \langle x, y\rangle f(z), \iota\right) \geq \left(\frac{\|\iota\|}{\|\iota\| + \varphi(x, y, z)}\right) 1_{\mathcal{A}} \tag{19}$$

for all $x, y, z \in \Xi$, $\iota \in \mathcal{A}^+_{>0}$ and all $\mu \in \mathbb{T}$. If there exists $0 \leq L < 1$ such that $\varphi(x, y, z) \leq \frac{L}{8}\varphi(2x, 2y, 2z)$ for all $x, y, z \in \Xi$, then there exists a unique fuzzy Hilbert C^*-module derivation $D : \Xi \to \Xi$ such that

$$\mathcal{N}\left(f(x) - D(x), \iota\right) \geq \left(\frac{(2 - 2L)\|\iota\|}{(2 - 2L)\|\iota\| + L\varphi(x, x, 0)}\right) 1_{\mathcal{A}} \tag{20}$$

for all $x \in \Xi$ and $\iota \in \mathcal{A}^+_{>0}$.

Proof. By Theorem 2.4, there exists a unique $\mathbb{C}$-linear mapping $D : \Xi \to \Xi$ satisfying (20) such that

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = D(x) \tag{21}$$

for all $x \in \Xi$.

To show D is a derivation, replacing x, y, z by $\frac{x}{2^n}$, $\frac{y}{2^n}$ and $\frac{z}{2^n}$ in the left side of (19), we have

$$\mathcal{N}\left(f\left(\left\langle \frac{x}{2^n}, \frac{y}{2^n}\right\rangle \frac{z}{2^n}\right) - \left\langle f\left(\frac{x}{2^n}\right), y\right\rangle z - \left\langle x, f\left(\frac{x}{2^n}\right)\right\rangle z - \langle x, y\rangle f\left(\frac{z}{2^n}\right), \iota\right).$$

Applying (19), we obtain

$$\mathcal{N}\left(f\left(\left\langle \frac{x}{2^n}, \frac{y}{2^n}\right\rangle \frac{z}{2^n}\right) - \left\langle f\left(\frac{x}{2^n}\right), y\right\rangle z - \left\langle x, f\left(\frac{x}{2^n}\right)\right\rangle z - \langle x, y\rangle f\left(\frac{z}{2^n}\right), 2^{-3n}\iota\right)$$
$$\geq \left(\frac{2^{-3n}\|\iota\|}{2^{-3n}\|\iota\| + \varphi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n})}\right) 1_{\mathcal{A}}$$

for all $x, y, z \in \Xi$ and all $\iota \in \mathcal{A}^+_{>0}$. So we have we obtain

$$\mathcal{N}\left(2^{3n} f\left(\left\langle \frac{x}{2^n}, \frac{y}{2^n}\right\rangle \frac{z}{2^n}\right) - \left\langle 2^n f\left(\frac{x}{2^n}\right), y\right\rangle z - \left\langle x, 2^n f\left(\frac{x}{2^n}\right)\right\rangle z - \langle x, y\rangle 2^n f\left(\frac{z}{2^n}\right), \iota\right)$$
$$\geq \left(\frac{\|\iota\|}{\|\iota\| + 2^{3n}\varphi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n})}\right) 1_{\mathcal{A}}$$
$$\geq \left(\frac{\|\iota\|}{\|\iota\| + 2^{3n}8^{-n}L^n\varphi(x, y, z)}\right) 1_{\mathcal{A}}$$

for all $x, y, z \in \Xi$ and all $\iota \in \mathcal{A}^+_{>0}$.

Taking the limit as $n \to \infty$, for all $\iota \in \mathcal{A}^+_{>0}$, using (14), by Definition 1.6 and applying (21), we obtain

$$D(\langle x, y\rangle z) = \langle D(x), y\rangle z + \langle x, D(y)\rangle z + \langle x, y\rangle D(z)$$

for all $x, y, z \in \Xi$. Thus D is a fuzzy Hilbert C^*-module derivation. □

Corollary 3.4. *Let $p > 3$, $\theta \geq 0$ and $f : \Xi \to \Xi$ be a mapping such that*

$$\mathcal{N}(f(\mu x + y) - \mu f(x) - f(y), \iota) \geq \left(\frac{\|\iota\|}{\|\iota\| + \theta(\|x\|^p + \|y\|^p)}\right) 1_{\mathcal{A}},$$

$$\mathcal{N}(f(\langle x, y\rangle z) - \langle f(x), y\rangle z - \langle x, f(y)\rangle z - \langle x, y\rangle f(z)\iota) \geq \left(\frac{\|\iota\|}{\|\iota\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}\right) 1_{\mathcal{A}}$$

for all $x, y, z \in \Xi$, $\iota \in \mathcal{A}^+_{>0}$. Then there exists a unique fuzzy Hilbert C^*-module derivation $D : \Xi \to \Xi$ such that

$$\mathcal{N}(f(x) - H(x), \iota) \geq \left(\frac{(2^p - 2)\|\iota\|}{(2^p - 2)\|\iota\| + 2\theta\|x\|^p}\right) 1_{\mathcal{A}}$$

for all $x \in \Xi$ and $\iota \in \mathcal{A}^+_{>0}$.

Proof. The proof follows from Theorem 3.3 by taking $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and $L = 2^{1-p}$. □

4. Conclusion

In this paper, we introduced the idea of fuzzy Hilbert C^*-modules. Furthermore, we proved the Hyers-Ulam stability of fuzzy Hilbert C^*-module homomorphisms and fuzzy Hilbert C^*-module derivations on fuzzy Hilbert C^*-modules, using the fixed point method.

Declarations

Availability of data and materials

No datasets were generated or analyzed during the current study.

Human and animal rights

We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest

The authors declare that they have no competing interests.

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