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# SOME EXAMPLES OF WEAKLY FACTORIAL RINGS

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Abstract. Let $D$ be a principal ideal domain, $X$ be an indeterminate over $D$, $D[X]$ be the polynomial ring over $D$, and $R_n = D[X]/(X^n)$ for an integer $n \geq 1$. Clearly, $R_n$ is a commutative Noetherian ring with identity, and hence each nonzero nonunit of $R_n$ can be written as a finite product of irreducible elements. In this paper, we show that every irreducible element of $R_n$ is a primary element, and thus every nonunit element of $R_n$ can be written as a finite product of primary elements.

## 1. Introduction

Let $D$ be an integral domain, $X$ be an indeterminate over $D$, $D[X]$ be the polynomial ring over $D$, and $R_n = D[X]/(X^n)$ for an integer $n \geq 1$. Clearly, $R_n$ is a commutative ring with identity $1 + (X^n)$, and since $(X^n) \cap D = (0)$, $D$ can be considered as a subring of $R_n$. Note that if $\alpha \in R_n$, then $\alpha = a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + (X^n)$ for some unique $a_i \in D$; so if we let $x = X + (X^n)$, then $x$ is a prime element of $R_n$, $\alpha = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$, and $\alpha = 0$ if and only if $a_0 = a_1 = \cdots = a_{n-1} = 0$. Also, if $\beta = b_0 + b_1x + \cdots b_{n-1}x^{n-1} \in R_n$, then

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$\alpha+\beta = \sum_{k_0}^{n-1}(a_k+b_k)x^k$ and $\alpha\cdot\beta = a_0b_0+(a_1b_0+a_0b_1)x+\cdots+(a_{n-1}b_0+a_{n-2}b_1+\cdots+a_1b_{n-2}+a_0b_{n-1})x^{n-1} = \sum_{k=0}^{n-1}(\sum_{i+j=k}a_ib_j)x^k$.

Let $R$ be a commutative ring with identity and $U(R)$ be the set of units of $R$. An $a \in R$ is said to be *primary* if $aR$ is a primary ideal. An integral domain is called a *weakly factorial domain* if its nonzero nonunit can be written as a finite product of primary elements [1]. For convenience, in this paper, we will say that $R$ is a *weakly factorial ring* if every nonzero nonunit of $R$ can be written as a finite product of primary elements. Hence, weakly factorial domains are weakly factorial rings. Two elements $a, b \in R$ are said to be *associates* if $aR = bR$, i.e., $a = bc_1$ and $b = ac_2$ for some $c_1, c_2 \in R$. An $a \in R$ is said to be *irreducible* if $a = bc$ implies that either $b$ or $c$ is associated with $a$.

Let $D$ be a principal ideal domain (PID). Clearly, $R_1 = D$, and thus $R_1$ is a weakly factorial ring. Moreover, in [2, Corollary 11], it was proved that $R_2$ is a weakly factorial ring. In this short paper, we show that $R_n$ is a weakly factorial ring for all integers $n \geq 1$. Note that $R_n$ is a commutative Noetherian ring with identity, and hence each nonzero nonunit of $R_n$ can be written as a finite product of irreducible elements. Thus, to prove that $R_n$ is a weakly factorial ring, it suffices to show that every irreducible element of $R_n$ is primary. This will be proved by a series of lemmas.

## 2. Main Results

Let $D$ be an integral domain, $D[X]$ be the polynomial ring over $D$, and $R_n = D[X]/(X^n)$ for an integer $n \geq 1$. In this section, we show that $R_n$ is a weakly factorial ring by a series of lemmas.

Lemma 1. (cf. [2, Lemma 1]) *Let $\alpha = a_0+a_1x+\cdots+a_{n-1}x^{n-1} \in R_n$. Then $\alpha$ is a unit of $R_n$ if and only if $a_0$ is a unit of $D$.*

*Proof.* If $\alpha$ is a unit, then there is a $\beta = b_0+b_1x+\cdots+b_{n-1}x^{n-1} \in R_n$ such that $\alpha\cdot\beta = 1$. Thus, $a_0b_0 = 1$. Conversely, assume that $a_0$ is a unit of $D$, and let $c \in D$ with $a_0c = 1$. Note that $\alpha R_n = c\alpha R_n$; so replacing $\alpha$ with $\alpha\cdot c$ if necessary, we may assume that $a_0 = 1$. Let $c_0, c_1, \ldots, c_{n-1} \in D$ be such that

$$\begin{cases} c_0 = 1 \\ c_1 + a_1 c_0 = 0 \\ c_2 + c_1 a_1 + c_0 a_2 = 0 \\ \cdots \\ c_{n-1} + c_{n-2} a_1 + \cdots + c_0 a_{n-1} \quad = 0 \end{cases}$$

and let $\gamma = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}$. Clearly, such $c_i$'s exist and $\alpha\gamma = 1$. □

LEMMA 2. (cf. [2, Lemma 2]) *Let $\alpha, \beta \in R_n$. Then $\alpha$ and $\beta$ are associates if and only if there is a $\theta \in U(R_n)$ such that $\alpha = \theta\beta$. Hence $\alpha \in R_n$ is irreducible if and only if $\alpha = \beta\gamma$ for $\beta, \gamma \in R_n$ implies that either $\beta$ or $\gamma$ is a unit.*

*Proof.* Let $\alpha = a_i x^i + a_{i+1} x^{i+1} + \cdots + a_{n-1} x^{n-1}$ and $\beta = b_j x^j + b_{j+1} x^{j+1} + \cdots + b_{n-1} x^{n-1}$ such that $a_i \neq 0$, $b_j \neq 0$, and $0 \leq i \leq j$. If $\alpha$ and $\beta$ are associates, then $\alpha = \beta \cdot \theta$ for some $\theta \in R_n$. Note that $i \leq j$; so if we let $\gamma = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}$, then $j = i$ and $a_i = c_0 b_j$. Similarly, we can find an element $d \in D$ such that $b_j = a_i d$. Hence $a_i = c_0 d a_i$, and since $a_i \neq 0$, we have $c_0 d = 1$ or $c_0 \in U(D)$. Thus, $\theta$ is a unit of $R_n$ by Lemma 1. The converse is clear. □

LEMMA 3. (cf. [2, Theorem 5]) *Let $D$ be a PID and $\alpha = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \in R_n$. If $\alpha$ is irreducible, then either $(i)$ $a_0 = 0$ and $a_1 \in U(D)$ or $(ii)$ $a_0 = up^k$ for some prime $p \in D$, $u \in U(D)$, and integer $k \geq 1$.*

*Proof.* Assume that $a_0 = 0$. Then $a_1 \neq 0$, because $a_2 x^2 + \cdots + a_{n-1} x^{n-1} = x(a_2 x + \cdots + a_{n-1} x^{n-2})$ and both $x$ and $a_2 x + \cdots + a_{n-1} x^{n-2}$ are not units by Lemma 1. Moreover, if $a_1$ is a nonzero nonunit, then $a_1 x + \cdots + a_{n-1} x^{n-1} = x(a_1 + a_2 x \cdots + a_{n-1} x^{n-2})$, and since both $x$ and $a_1 + a_2 x \cdots + a_{n-1} x^{n-2}$ are not units by Lemma 1, $\alpha$ is not irreducible by Lemma 2, a contradiction. Thus, $a_1$ is a unit of $D$.

Next, assume that $a_0 \neq 0$. If $a_0$ is not of the form $up^k$, then there are nonzero $b_0, c_0 \in D$ such that $a_0 = b_0 c_0$ and $\gcd(b_0, c_0) = 1$. Since $D$ is a PID, there exist $b_1, c_1 \in D$ so that $b_0 c_1 + b_1 c_0 = a_1$. Again, $D$ being a PID guarantees that there are $b_2, c_2 \in D$ such that $b_0 c_2 + b_2 c_0 = a_2 - b_1 c_1$. Repeating this process, we can choose $b_2, \ldots, b_{n-1}, c_2, \ldots, c_{n-1} \in D$ so that

$$(b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}) \cdot (c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}) = \alpha,$$

hence $\alpha$ is not irreducible by Lemmas 1 and 2. Thus, $a_0$ must be of the form $up^k$ for some prime $p \in D$, $u \in U(D)$, and integer $k \geq 1$. □

We are now ready to prove the main result of this paper.

Theorem 4. (cf. [2, Corollary 11]) *If $D$ is a PID, then the ring $R_n = D[X]/(X^n)$ is a weakly factorial ring for all integers $n \geq 1$.*

*Proof.* Note that $R_n$ is a Noetherian ring; hence each element of $R_n$ can be written as a finite product of irreducible elements. Hence if we show that each irreducible element of $R_n$ is primary, then $R_n$ is a weakly factorial ring.

Let $\alpha = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \in R_n$ be irreducible. By Lemma 3, there are only two cases we have to consider. First, assume $a_0 = 0$ and $a_1 \in U(D)$. Then $\alpha R_n = xR_n$ by Lemma 1, and hence $\alpha$ is prime (so primary). Next, assume $a_0 = up^k$ for some $u \in U(D)$, prime $p \in D$ and integer $k \geq 1$. It is known that if $\sqrt{\alpha R_n}$ is a maximal ideal, then $\alpha R_n$ is primary [2, Lemma 10]; so it suffices to show that $\sqrt{\alpha R_n}$ is maximal. Let $\beta \in R_n \setminus \sqrt{\alpha R_n}$, and put $\beta = b_0 + b_1x + \cdots + b_{n-1}x^{n-1}$. Note that if $\delta = c_1x + \cdots + c_{n-1}x^{n-1} \in R_n$, then $\delta^n = 0$, and hence $\delta \in \sqrt{\alpha R_n}$. Hence $b_0 \notin \sqrt{\alpha R_n}$ and $p \in \sqrt{\alpha R_n}$. Note also that if $b_0 \in pD$, then $b_0 = pz$ for some $z \in D$, and so $b_0 = pz \in \sqrt{\alpha R_n}$, a contradiction. So $b_0 \notin pD$, and since $D$ is a PID, we have $b_0z_1 + pz_2 = 1$ for some $z_1, z_2 \in D$. Thus, $1 = \beta z_1 + pz_2 - z_1(b_1x + \cdots + b_{n-1}x^{n-1}) \in \beta R_n + \sqrt{\alpha R_n}$. Therefore, $\sqrt{\alpha R_n}$ is maximal. □

Corollary 5. *If $\mathbb{Z}$ is the ring of integers, then $\mathbb{Z}[X]/(X^n)$ is a weakly factorial ring for all integers $n \geq 1$.*

*Proof.* This follows directly from Theorem 4 because $\mathbb{Z}$ is a PID. □

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