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ON THE WEAK GLOBAL DIMENSION OF A SUBCLASS OF PRÜFER NON-COHERENT RINGS

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Abstract. It is known that if R is a coherent Prüfer ring, which is necessarily a Gaussian ring, then its weak global dimension $\mathrm{w.gl.dim}(R)$ must be 0, 1, or ∞. In this paper, we investigate the possible values of the weak global dimension for a broader class of Prüfer rings that are not necessarily coherent. Our analysis employs four conceptually distinct proofs, each relying on different homological techniques, including localization at the nilradical, finitistic projective dimension, and flatness properties. The results extend the classical framework to a non-coherent setting by incorporating the effective $\mathcal{H}^D$ framework, which serves as a surrogate for coherence in controlling homological dimensions. This work aims to deepen the understanding of the weak global dimension in the context of non-coherent Prüfer rings and provide a unified perspective on its behavior.

1. Introduction

In this introductory section, we outline certain conventions and review standard background material. The set of nilpotent elements of a ring R is denoted by $\mathrm{Nil}(R)$, while $Z(R)$ denotes the set of zero-divisors of R. A ring is called a *ϕ-ring* if its nilradical $\mathrm{Nil}(R)$ is a *divided prime*, meaning that $\mathrm{Nil}(R) \subset xR$ for every $x \in R \setminus \mathrm{Nil}(R)$. An ideal I of R is said to be *nonnil* if $I \nsubseteq \mathrm{Nil}(R)$. The notation $\mathcal{H}$ (resp., $\overline{\mathcal{H}}$) denotes the class of all rings with a divided prime nilradical (resp., those with a divided prime but non-maximal nilradical). A ring R is called a *strongly ϕ-ring* if $R \in \mathcal{H}$ and $Z(R) = \mathrm{Nil}(R)$.

For a ring R and an R-module M, we define

$$\phi\text{-}\mathrm{tor}(M) = \{x \in M \mid sx = 0 \text{ for some } s \in R \setminus \mathrm{Nil}(R)\}.$$

An R-module M is called a *ϕ-torsion module* (resp., a *ϕ-torsion-free module*) if $\phi\text{-}\mathrm{tor}(M) = M$ (resp., $\phi\text{-}\mathrm{tor}(M) = 0$). The module M is said to be *uniformly ϕ-torsion* (or *u-ϕ-torsion*) if $sM = 0$ for some $s \in R \setminus \mathrm{Nil}(R)$, and it is called *$\phi$-divisible* if $M = sM$ for every $s \in R \setminus \mathrm{Nil}(R)$. The classical projective and flat dimensions of an R-module M are denoted by $\mathrm{pd}_R(M)$ and $\mathrm{fd}_R(M)$, respectively.

A submodule N of an R-module M is called a *ϕ-submodule* if M/N is a ϕ-torsion module. Similarly, N is called a *uniformly ϕ-submodule* (or *u-ϕ-submodule*) if M/N

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is a u-ϕ-torsion R-module. A submodule N of M is said to be *pure* if the sequence

$$0 \longrightarrow F \otimes_R N \longrightarrow F \otimes_R M$$

is exact for every R-module F. Moreover, N is called *ϕ-pure* if this sequence is exact for every finitely presented ϕ-torsion R-module F.

Two new classes of ϕ-rings that generalize the notion of coherence were introduced by K. Bacem and A. Benhissi. A ϕ-ring R is said to be *ϕ-coherent* (resp., *nonnil-coherent*) if $R/\operatorname{Nil}(R)$ is a coherent domain (resp., every finitely generated nonnil ideal of R is finitely presented).

D. F. Anderson and A. Badawi introduced the notion of *ϕ-Dedekind rings*. A ϕ-ring R is called *ϕ-Dedekind* if $R/\operatorname{Nil}(R)$ is a Dedekind domain. Later, they introduced the concept of *ϕ-Prüfer rings*: a ϕ-ring R is *ϕ-Prüfer* if $R/\operatorname{Nil}(R)$ is a Prüfer domain. It is known that every ϕ-Prüfer ring is a Prüfer ring, and if $Z(R) = \operatorname{Nil}(R)$, then every Prüfer ring is also a ϕ-Prüfer ring.

In [22], G. H. Tang, F. G. Wang, and W. Zhao introduced the concept of *ϕ-von Neumann regular rings*. An R-module M is said to be *ϕ-flat* if, for every R-monomorphism $f : A \to B$ with ϕ-torsion cokernel, the induced map

$$f \otimes 1 : A \otimes_R M \longrightarrow B \otimes_R M$$

is an R-monomorphism [22, Definition 3.1]. An R-module M is ϕ-flat *if and only if* $M_{\mathfrak{p}}$ is ϕ-flat for every prime ideal $\mathfrak{p}$ of R, or equivalently, if $M_{\mathfrak{m}}$ is ϕ-flat for every maximal ideal $\mathfrak{m}$ of R [22, Theorem 3.5]. A ϕ-ring R is called a *ϕ-von Neumann regular ring* if every R-module is ϕ-flat, which is equivalent to the condition that $R/\operatorname{Nil}(R)$ is a von Neumann regular ring [22, Theorem 4.1].

In [10], the authors introduced the concept of *ϕ-(weak) global dimension* for rings with prime nilradicals. An R-module P is called *ϕ-u-projective* if $\operatorname{Ext}^1_R(P, N) = 0$ for every u-ϕ-torsion R-module N. The *ϕ-projective dimension* of an R-module M, denoted ϕ-$\operatorname{pd}_R M$, is said to be at most $n \geq 1$ (where $n \in \mathbb{N}$) if either $M = 0$, or M is a nonzero module that is not ϕ-u-projective but satisfies $\operatorname{Ext}^{n+1}_R(M, N) = 0$ for all u-ϕ-torsion modules N. If n is the least non-negative integer such that $\operatorname{Ext}^{n+1}_R(M, N) = 0$ for every u-ϕ-torsion module N, then we define ϕ-$\operatorname{pd}_R M = n$. If no such n exists, we define ϕ-$\operatorname{pd}_R M = \infty$.

For a ring R, the *ϕ-global dimension* is either 0 or the supremum of all values of ϕ-$\operatorname{pd}_R(R/I)$, where I is a nonnil ideal of R such that R/I is not ϕ-u-projective. In particular, if R is a ring with $Z(R) = \operatorname{Nil}(R)$, then the ϕ-global dimension of R is the supremum of ϕ-$\operatorname{pd}_R(R/I)$ taken over all nonnil ideals I of R.

Similarly, the *ϕ-flat dimension* of an R-module M, denoted ϕ-$\operatorname{fd}_R M$, is at most $n \geq 1$ (where $n \in \mathbb{N}$) if either $M = 0$, or M is a nonzero module that is not ϕ-flat but satisfies $\operatorname{Tor}^R_{n+1}(M, N) = 0$ for every u-ϕ-torsion R-module N. If n is the least non-negative integer such that $\operatorname{Tor}^R_{n+1}(M, N) = 0$ for every u-ϕ-torsion module N, then ϕ-$\operatorname{fd}_R M = n$. If no such n exists, we set ϕ-$\operatorname{fd}_R M = \infty$.

For rings R with $Z(R) = \operatorname{Nil}(R)$, the ϕ-weak global dimension of R is defined as follows:

$$\begin{aligned}
\phi\text{-}\operatorname{w.gl.dim}(R) &= \sup \{\phi\text{-}\operatorname{fd}_R M \mid M \text{ is a } \phi\text{-torsion module}\} \\
&= \sup \{\phi\text{-}\operatorname{fd}_R(R/I) \mid I \text{ is a nonnil ideal of } R\} \\
&= \sup \{\phi\text{-}\operatorname{fd}_R(R/I) \mid I \text{ is a finitely generated nonnil ideal of } R\}.
\end{aligned}$$

Thus, the ϕ-weak global dimension of a ring R is either 0 or the supremum of all values ϕ-$\mathrm{fd}_R(R/I)$, where I is a nonnil ideal of R such that R/I is not ϕ-flat.

Section 2 is devoted to introducing a new subclass within $\mathcal{H}$, denoted by $\mathcal{H}^D$. A ring R belongs to $\mathcal{H}^D$ if it is a strongly ϕ-ring and its total ring of quotients $Q(R)$ is Noetherian. Such a ring is called a *ϕ-D-ring*. Within the framework of ϕ-rings, all rings R in $\mathcal{H}$ satisfying $Z(R) = \mathrm{Nil}(R)$ play a role analogous to integral domains in classical ring theory. In particular, since the quotient fields of integral domains are Noetherian, every integral domain lies in $\mathcal{H}^D$. To more effectively address the notion of domains in the context of ϕ-rings R with $Z(R) = \mathrm{Nil}(R)$, the study of ϕ-D-rings becomes essential.

In Theorem 2.3, we extend a classical result from [23, Theorem 1.6.15], adapting it to our setting. We then examine, through four distinct proofs based on different homological techniques, the possible values of the weak global dimension of ϕ-Prüfer rings within $\mathcal{H}^D$.

Section 3 focuses on demonstrating that, in ϕ-D-rings with a non-maximal nilradical, any finitely generated ϕ-u-projective module with finite flat dimension must be projective. This result is established using the framework of locally ϕ-(n,d)-properties and provides further insights into the structural behavior of ϕ-Prüfer rings.

To illustrate our results with explicit examples, we employ the construction of the *trivial ring extension*. Let R be a ring and E an R-module. The trivial ring extension of R by E, denoted by $R \propto E$, is defined as the ring whose additive structure is the external direct sum $R \oplus E$, and whose multiplication is given by

$$(a,e)(b,f) := (ab, af + be)$$

for all $a, b \in R$ and $e, f \in E$. This construction, also known as the *idealization* and denoted by $R(+)E$, has been widely discussed in the literature (see [6, 13, 15, 17]).

2. On ϕ-D-rings

We now introduce a new subclass within the category of strongly ϕ-rings, defined as follows:

Definition 2.1. A strongly ϕ-ring R is called a *ϕ-D-ring* if the localization of R at its nilradical, $R_{\mathrm{Nil}(R)}$, is a Noetherian ring. We denote by $\mathcal{H}^D$ the class of all ϕ-D-rings.

Remark 2.2. According to [5, Lemma 2.3], for any ring R in the class $\mathcal{H}^D$, the localized ring $R_{\mathrm{Nil}(R)}$ is either an integral domain or a local Artinian ring whose nonzero maximal ideal coincides with its nilradical.

As established in [23, Theorem 1.6.15], every finitely generated torsion-free module over an integral domain can be embedded in a finitely generated free module. This result can be extended to the class $\mathcal{H}^D$.

Recall that the *small finitistic projective dimension* of a ring R, denoted $\mathrm{fPD}(R)$, is defined as the supremum of the projective dimensions of all R-modules M having finite projective dimension and admitting a finite resolution by finitely generated projective modules. In contrast, the *finitistic projective dimension* of R, denoted $\mathrm{FPD}(R)$, is the supremum of the projective dimensions of all R-modules M with finite projective dimension.

Theorem 2.3. *Let R be a ring in the class $\mathcal{H}^D$, and let M be a finitely generated torsion-free R-module. Set $K := R_{\mathrm{Nil}(R)}$. Then the following hold:*

1. *If $\mathrm{Nil}(K) = 0$, then M can be embedded in a finitely generated free R-module as a u-ϕ-submodule.*
2. *If $\mathrm{Nil}(K) \neq 0$ and $\mathrm{fd}_R(M) < \infty$, then M can also be embedded in a finitely generated free R-module as a u-ϕ-submodule.*

Proof. (1) Since K is a ϕ-von Neumann regular ring with a finitely generated nilradical, it follows from [5, Lemma 2.3] that K is a field. Therefore, by [23, Theorem 1.6.15], the module M can be embedded in a finitely generated free R-module as a u-ϕ-submodule.

(2) By [5, Lemma 2.3], K is a local Artinian ring whose nonzero nilradical coincides with its maximal ideal. Combining results from [21, Corollary 5], [23, Theorem 3.10.25], and the inequality $\mathrm{fPD}(R) \leq \mathrm{FPD}(R)$, we deduce that K is a coherent ring with $\mathrm{fPD}(K) = 0$. Hence, by [23, Theorem 2.5.24], every finitely presented K-module of finite flat dimension is flat.

Since $\mathrm{fd}_R(M) < \infty$, it follows that $\mathrm{fd}_K(M_{\mathrm{Nil}(R)}) < \infty$. Because M is finitely generated over R, the module $M_{\mathrm{Nil}(R)}$ is finitely presented over K, and thus is a finitely generated free K-module.

Let $x_1, \ldots, x_n \in M$ be such that $\{\frac{x_1}{1}, \ldots, \frac{x_n}{1}\}$ forms a K-basis for $M_{\mathrm{Nil}(R)}$. Then the R-submodule $F := Rx_1 + \cdots + Rx_n$ is a finitely generated free R-submodule of M.

Let $\{z_1, \ldots, z_m\}$ be a generating set of M. For each $i = 1, \ldots, m$, there exists $s_i \in R \setminus \mathrm{Nil}(R)$ such that $s_i z_i \in F$. Set $s := s_1 \cdots s_m$. Then $sM \subseteq F$, and we can define a monomorphism $f : M \to F$ by $f(x) = sx$. Since $sF \subseteq M$, it follows that M is a u-ϕ-submodule of F. □

Our main objective is to show that the possible values of the weak global dimension of ϕ-Prüfer rings in the class $\mathcal{H}^D$ are precisely 0, 1, or ∞.

Theorem 2.4. *Let $R \in \mathcal{H}^D$. If R is a ϕ-Prüfer ring, then* $\mathrm{w.gl.dim}(R) = 0$, 1, *or* ∞.

First Proof. Let $R \in \mathcal{H}^D$ be a ϕ-Prüfer ring.

If $\mathrm{Nil}(R) = 0$, then R is a Prüfer domain, and hence $\mathrm{w.gl.dim}(R) \leq 1$.

Now assume that $\mathrm{Nil}(R) \neq 0$. Then there exists a maximal ideal $\mathfrak{m}$ of R such that $\mathrm{Nil}(R)_\mathfrak{m} \neq 0$. Set $D := R_\mathfrak{m}$. We will show that $\mathrm{fd}_D(\mathrm{Nil}(D)) = \infty$. Note that $D \in \mathcal{H}^D$.

Assume, for contradiction, that $\mathrm{fd}_D(\mathrm{Nil}(D)) < \infty$. Then, by Theorem 2.3, there exists a short exact sequence of D-modules:

$$0 \longrightarrow \mathrm{Nil}(D) \longrightarrow F \longrightarrow F/\mathrm{Nil}(D) \longrightarrow 0,$$

where F is a finitely generated free D-module and $F/\mathrm{Nil}(D)$ is a u-ϕ-torsion D-module. This implies the existence of an element $s \in D \setminus \mathrm{Nil}(D)$ such that

$$sD \subseteq sF \subseteq \mathrm{Nil}(D),$$

which is a contradiction. Therefore, $\mathrm{fd}_D(\mathrm{Nil}(D)) = \infty$, and hence $\mathrm{w.gl.dim}(D) = \infty$.

Since $\mathrm{w.gl.dim}(D) \leq \mathrm{w.gl.dim}(R)$, it follows that $\mathrm{w.gl.dim}(R) = \infty$. Thus, the possible values of the weak global dimension of R are 0, 1, or ∞. □

Second Proof. We begin by recalling that any coherent ring with small finitistic projective dimension zero has weak global dimension either 0 or ∞. Let R be a coherent ring with $\mathrm{fPD}(R) = 0$. Consider a finitely generated submodule N of a finitely generated free module F such that $\mathrm{pd}_R(N) < \infty$. Since R is coherent, N is finitely presented, and hence F/N has a finite projective resolution by finitely generated projective modules. Therefore, F/N is flat by [23, Theorem 3.10.6].

It follows that N is a pure submodule of F, and hence a direct summand of F by [12, Theorems 8 and 9]. Thus, for any such submodule N, we have $\mathrm{pd}_R(N) = 0$, implying that $\mathrm{w.gl.dim}(R)$ is either 0 or ∞.

Next, recall that every Artinian ring is coherent with small finitistic projective dimension zero, by [21, Corollary 5] and [23, Theorem 3.10.25].

Now, consider the case where R is a coherent ϕ-Prüfer ring. Then, by [7, Proposition 6.1], we have $\mathrm{w.gl.dim}(R) = 0$, 1, or ∞.

If R is a ϕ-Prüfer ring that is not coherent, then the only possible value for $\mathrm{w.gl.dim}(R)$ is ∞. Since R is not coherent, we must have $\mathrm{Nil}(R) \neq 0$. By [5, Lemma 2.3], the localization $R_{\mathrm{Nil}(R)}$ is a local Artinian ring with nonzero nilradical as its maximal ideal. From the initial part of this proof, we then obtain $\mathrm{w.gl.dim}(R_{\mathrm{Nil}(R)}) = \infty$. Therefore, by [23, Corollary 3.8.6(2)], we conclude that $\mathrm{w.gl.dim}(R) = \infty$.

Thus, in all cases, for a ϕ-Prüfer ring $R \in \mathcal{H}^D$, the weak global dimension of R is 0, 1, or ∞. □

Third Proof. Let $R \in \mathcal{H}^D$ be a ϕ-Prüfer ring. Note that $R_{\mathrm{Nil}(R)}$ is a Noetherian ϕ-Prüfer ring. This follows from the isomorphism

$$\frac{R_{\mathrm{Nil}(R)}}{\mathrm{Nil}(R)_{\mathrm{Nil}(R)}} \cong \left(\frac{R}{\mathrm{Nil}(R)}\right)_{\mathrm{Nil}(R)},$$

which implies, by [23, Theorem 3.7.13 and Corollary 3.8.6(2)], that

$$\mathrm{w.gl.dim}\left(\frac{R_{\mathrm{Nil}(R)}}{\mathrm{Nil}(R)_{\mathrm{Nil}(R)}}\right) \leq \mathrm{w.gl.dim}\left(\frac{R}{\mathrm{Nil}(R)}\right) \leq 1.$$

Moreover, from [2, Theorem 2.6], we conclude that $R_{\mathrm{Nil}(R)}$ is indeed a ϕ-Prüfer ring. In view of [2, Theorem 2.14] and [7, Proposition 6.1], it follows that

$$\mathrm{w.gl.dim}(R_{\mathrm{Nil}(R)}) = 0,\ 1,\ \text{or } \infty.$$

If $\mathrm{w.gl.dim}(R_{\mathrm{Nil}(R)}) = \infty$, then by [23, Corollary 3.8.6(2)], we also have $\mathrm{w.gl.dim}(R) = \infty$.

Now, assume instead that $\mathrm{w.gl.dim}(R_{\mathrm{Nil}(R)}) \leq 1$. According to [16, Theorem], this implies $\mathrm{Nil}(R)_{\mathrm{Nil}(R)} = 0$, and hence $\mathrm{Nil}(R) = 0$. Therefore, R is a Prüfer domain, and thus $\mathrm{w.gl.dim}(R) \leq 1$.

In conclusion, for any ϕ-Prüfer ring $R \in \mathcal{H}^D$, the weak global dimension of R can take only three possible values: 0, 1, or ∞. □

Fourth Proof. Assume $R \in \mathcal{H}^D$ is a ϕ-Prüfer ring. Then $R_{\mathrm{Nil}(R)}$ is a Noetherian Prüfer ring. If $R_{\mathrm{Nil}(R)}$ is an integral domain, then R itself is a Prüfer domain, and so $\mathrm{w.gl.dim}(R) \leq 1$.

Suppose instead that $R_{\mathrm{Nil}(R)}$ is not an integral domain. Then its nilradical $\mathrm{Nil}(R_{\mathrm{Nil}(R)})$ is a finitely generated ideal and hence nilpotent. Importantly, we have $(0 : \mathrm{Nil}(R_{\mathrm{Nil}(R)})) \neq$

0 in $R_{\mathrm{Nil}(R)}$. By [18, Theorem 9], any finitely generated R-module with finite projective dimension must be free. However, from [23, Proposition 6.7.12], it is known that $\mathrm{Nil}(R_{\mathrm{Nil}(R)})$ is not a projective $R_{\mathrm{Nil}(R)}$-module. Therefore, we conclude that

$$\mathrm{pd}_{R_{\mathrm{Nil}(R)}}\big(\mathrm{Nil}(R_{\mathrm{Nil}(R)})\big) = \infty,$$

which implies

$$\mathrm{w.gl.dim}(R_{\mathrm{Nil}(R)}) = \mathrm{gl.dim}(R_{\mathrm{Nil}(R)}) = \infty.$$

Consequently, by [23, Corollary 3.8.6(2)], we obtain $\mathrm{w.gl.dim}(R) = \infty$.

Thus, in all cases, for a ϕ-Prüfer ring $R \in \mathcal{H}^D$, the weak global dimension of R can only be 0, 1, or ∞. □

The following example exhibits a ϕ-Prüfer ring in $\mathcal{H}^D$ that is not coherent.

EXAMPLE 2.5. Let V be a (nontrivial) Prüfer domain with quotient field K. Consider the trivial extension (idealization)

$$R := V \propto K = V \oplus K, \qquad (a,u)\cdot(b,v) = (ab,\ av + bu).$$

For a concrete instance, take $V = \mathbb{Z}$ and $K = \mathbb{Q}$; then $R = \mathbb{Z} \propto \mathbb{Q}$.

1. *R is a ϕ–ring and even strongly ϕ.* The nilradical is $N := 0 \propto K$ with $N^2 = 0$. For any $(a,u) \notin N$ we have $a \neq 0$, and since $K = aK$, it follows that $N \subseteq (a,u)R$. Thus N is divided prime, and R is a ϕ–ring. Moreover, because K is torsion-free over V, the only zero-divisors are the nilpotents in N; hence $Z(R) = \mathrm{Nil}(R) = N$, and thus R is a *strongly* ϕ–ring.
2. *R is ϕ–Prüfer (equivalently, Prüfer in the strongly ϕ case).* We have $R/\mathrm{Nil}(R) \cong V$, and since V is a Prüfer domain, it follows by definition that R is ϕ–Prüfer. Furthermore, because $Z(R) = \mathrm{Nil}(R)$, the ϕ–Prüfer property is equivalent to the classical Prüfer property for R.
3. *$R \in \mathcal{H}^D$.* Localizing at the nilradical inverts all pairs with a nonzero first component; hence

   $$R_{\mathrm{Nil}(R)} \cong K \propto K,$$

   which is a local Artinian (hence Noetherian) ring with square-zero maximal ideal. Therefore, R is a ϕ–D–ring, i.e., $R \in \mathcal{H}^D$.
4. *R is **not** coherent.* Consider $x := (0,1) \in R$. Then

   $$\mathrm{Ann}_R(x) = \{(a,u) \in R \mid (a,u)\cdot(0,1) = (0,a) = 0\} = 0 \propto K = \mathrm{Nil}(R).$$

   As an R–ideal, $\mathrm{Nil}(R)$ is *not* finitely generated: if $\mathrm{Nil}(R) = \sum_{i=1}^{n} R\cdot(0,u_i)$, then

   $$\mathrm{Nil}(R) = 0 \propto \sum_{i=1}^{n} V u_i = 0 \propto W,$$

   where W is a finitely generated V–submodule of K. However, $W \neq K$ since K is not finitely generated as a V–module. Therefore, $\mathrm{Ann}_R(x)$ is not finitely generated. Since in a coherent ring $\mathrm{Ann}_R(x) = \ker(R \xrightarrow{\cdot x} R)$ must be finitely generated, it follows that R is not coherent.

In summary, $R = V \propto K$ is a ϕ–Prüfer ring in $\mathcal{H}^D$, but it is **not** coherent.

3. On Locally ϕ-(n,d) Property and ϕ-Prüfer Rings

This section presents new insights into ϕ-Prüfer rings. We begin with the following definitions:

DEFINITION 3.1. Let R be a ring. An R-module M is said to be *n-presented* if it admits an n-finite presentation, that is, there exists an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where each F_i is a finitely generated free R-module. Moreover, if M is a ϕ-torsion R-module, it is called *ϕ-n-presented*, and the above sequence is referred to as a *ϕ-n-presentation* of M.

DEFINITION 3.2. Let R be a ϕ-ring, and let $n \geq -1$ and $d \geq 0$ be integers. We define:

(1) R is said to be ϕ-(n,d) if every ϕ-n-presented R-module has ϕ-projective dimension at most d.
(2) R is said to be *locally* ϕ-(n,d) if, for every maximal ideal $\mathfrak{m}$ of R, the localization $R_{\mathfrak{m}}$ is a ϕ-(n,d) ring.

The following result provides a condition under which a locally ϕ-(n,d) ring is also a ϕ-(n,d) ring.

THEOREM 3.3. *If R is a locally ϕ-(n,d) ring and $n \geq d+1$, then R is a ϕ-(n,d) ring.*

Before proving Theorem 3.3, we present an essential lemma, adapted from [8, Lemma 3.1].

LEMMA 3.4. *Let M be an R-module and S a multiplicative subset of R. If M admits a finite n-presentation, then for all $0 \leq i < n$, we have:*

$$S^{-1}\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^i_{S^{-1}R}\big(S^{-1}M,\, S^{-1}N\big)\,,$$

and $S^{-1}\operatorname{Ext}^n_R(M,N)$ is isomorphic to a submodule of $\operatorname{Ext}^n_{S^{-1}R}(S^{-1}M,\, S^{-1}N)$.

Proof of Theorem 3.3. If R does not belong to the class $\overline{\mathcal{H}}$, the conclusion follows trivially. Suppose instead that $R \in \overline{\mathcal{H}}$ and that R is locally a ϕ-(n,d) ring with $n \geq d+1$.

Let N be a u-ϕ-torsion R-module, and let M be a ϕ-n-presented R-module. For any maximal ideal $\mathfrak{m}$ of R, if M is ϕ-u-projective, then it is projective by [10, Corollary 5.36].

If M is not ϕ-u-projective, Lemma 3.4 ensures that

$$\operatorname{Ext}^{d+1}_R(M,N) = 0,$$

both in the case $d+1 = n$ and when $d+1 < n$. Therefore, R is a ϕ-(n,d) ring. □

PROPOSITION 3.5. *Let R be a ϕ-(n,d) ring with $n \leq 1$. Then every localization $S^{-1}R$ is also a ϕ-(n,d) ring.*

Proof. We note the following:

1. Every finitely generated $S^{-1}R$-module is the localization of a finitely generated R-module.

2. Every finitely presented $S^{-1}R$-module is the localization of a finitely presented R-module.
3. Every u-ϕ-torsion $S^{-1}R$-module is the localization of a u-ϕ-torsion R-module.

These observations, together with Lemma 3.4, imply that $S^{-1}R$ satisfies the ϕ-(n, d) condition. □

Remark 3.6. It is worth noting that the properties of being a ϕ-$(1, 0)$ ring or a ϕ-$(0, 0)$ ring (i.e., a ϕ-von Neumann regular ring) are preserved under localization. Likewise, the property of being a ϕ-$(1, 1)$ ring—which, under the condition $Z(R) = \mathrm{Nil}(R)$, characterizes ϕ-Prüfer rings (see Theorem 3.7)—is also stable under localization.

The following theorem characterizes ϕ-Prüfer rings when $Z(R) = \mathrm{Nil}(R)$, and some of its results are related to [20, 24].

Theorem 3.7. *Let R be a ϕ-ring. The following statements are equivalent:*

1. *R is a ϕ-Prüfer strongly ϕ-ring.*
2. *ϕ-$\mathrm{w.gl.dim}(R) \leq 1$.*
3. *Every finitely generated ϕ-submodule of a free R-module is projective.*
4. *Every finitely generated nonnil ideal of R is projective.*
5. *Every finitely generated nonnil ideal of R is flat.*
6. *Every nonnil ideal of R is flat.*
7. *Either $\mathrm{Nil}(R)$ is a maximal ideal of R, or R is a nonnil-coherent ring in which every maximal ideal is ϕ-flat.*

Before proving Theorem 3.7, we first establish the following lemma:

Lemma 3.8. *A ring in which every maximal ideal is ϕ-flat is a ϕ-$(2, 1)$ ring.*

To prove Lemma 3.8, we begin with the following auxiliary result:

Lemma 3.9. *Let $(R, \mathfrak{m})$ be a local ring. Suppose M is a finitely generated R-module with a minimal generating set $\{m_1, m_2, \ldots, m_n\}$, and let F be a finitely generated free R-module with basis $\{x_1, x_2, \ldots, x_n\}$. Define $K := \ker(f)$, where $f : F \to M$ is the R-module homomorphism given by $f(x_i) = m_i$ for $1 \leq i \leq n$. Then $K \subseteq \mathfrak{m}F$.*

Proof. Assume, for contradiction, that $K \not\subseteq \mathfrak{m}F$. Then there exists $z := \sum_{i=1}^{n} r_i x_i \in K \setminus \mathfrak{m}F$, implying that $r_i \notin \mathfrak{m}$ for some $1 \leq i \leq n$, say r_1. Thus, r_1 is a unit in R. Since $z \in \ker(f)$, we have $\sum_{i=1}^{n} r_i m_i = 0$, which yields

$$m_1 = -r_1^{-1} \sum_{i=2}^{n} r_i m_i,$$

contradicting the minimality of the generating set. Hence, $K \subseteq \mathfrak{m}F$. □

Proof of Lemma 3.8. Assume all maximal ideals of R are ϕ-flat, and suppose $(R, \mathfrak{m})$ is local. Consider the short exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow R/\mathfrak{m} \longrightarrow 0.$$

Then, for any ϕ-torsion R-module M, we have

$$\mathrm{Tor}_2^R(R/\mathfrak{m}, M) \cong \mathrm{Tor}_1^R(\mathfrak{m}, M) = 0.$$

Let M be a ϕ-2-presented R-module with a minimal ϕ-2-presentation

$$F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

By Lemma 3.9, we have $\ker(F_i \to F_{i-1}) \subseteq \mathfrak{m}F_i$ for $i = 1, 2$. Consider the exact sequence

$$0 \longrightarrow K_1 \longrightarrow F_1 \longrightarrow M \longrightarrow 0.$$

Tensoring with $R/\mathfrak{m}$ gives the exact sequence

$$0 \longrightarrow K_1 \otimes_R R/\mathfrak{m} \longrightarrow F_1 \otimes_R R/\mathfrak{m} \longrightarrow M \otimes_R R/\mathfrak{m} \longrightarrow 0.$$

The minimality assumption implies that $f_1 \otimes 1$ is an isomorphism, so $K_1 \otimes_R R/\mathfrak{m} = 0$. By Nakayama's Lemma, $K_1 = 0$. Thus, $\phi\text{-pd}_R(M) \leq 1$, and R is a locally ϕ-$(2,1)$ ring. By Theorem 3.3, R is a ϕ-$(2,1)$ ring. □

Proof of Theorem 3.7. (1) ⇔ (4), (4) ⇔ (5), and (5) ⇔ (6): These equivalences follow directly from [19, Theorem 2.13].

(1) and (6) ⇒ (7): If $\mathrm{Nil}(R)$ is not maximal, then all maximal ideals are nonnil and hence flat, implying they are ϕ-flat. Moreover, (1) implies that R is nonnil-coherent.

(1) ⇒ (2): If R is a ϕ-Prüfer ring with $Z(R) = \mathrm{Nil}(R)$, then by [10, Corollary 5.27], we have $\phi\text{-w.gl.dim}(R) \leq 1$.

(2) ⇒ (1): Suppose $\phi\text{-w.gl.dim}(R) \leq 1$. If $\mathrm{Nil}(R)$ is a maximal ideal, then R is a ϕ-von Neumann regular ring, and hence $Z(R) = \mathrm{Nil}(R)$ by [10, Corollary 5.15].

Assume instead that $\mathrm{Nil}(R)$ is not maximal and that $\mathrm{Nil}(R) \subsetneq Z(R)$. Then there exists $s \in Z(R) \setminus \mathrm{Nil}(R)$. Since R is a ϕ-ring and hence connected, R/sR is not ϕ-flat by [10, Theorem 5.13 and Corollary 5.36]. This implies that $\langle s \rangle$ is a ϕ-flat ideal. Consider the short exact sequence

$$0 \longrightarrow (0 : s) \longrightarrow R \longrightarrow \langle s \rangle \longrightarrow 0.$$

This sequence is ϕ-pure by [10, Theorem 5.4], so the induced homomorphism

$$\varphi : (0 : s) \otimes_R R/\langle s \rangle \longrightarrow R/\langle s \rangle$$

is a monomorphism. However, the kernel of φ is $\langle s \rangle / s(0 : s)$, implying $\langle s \rangle = s(0 : s)$ and thus $s = 0$, a contradiction. Therefore, $Z(R) = \mathrm{Nil}(R)$ and R is a ϕ-Prüfer ring.

(1) and (2) ⇒ (3): Suppose R is a ϕ-Prüfer ring with $Z(R) = \mathrm{Nil}(R)$, and let N be a finitely generated ϕ-submodule of a free R-module F. Then F/N is a ϕ-torsion R-module.

If $\mathrm{Nil}(R)$ is maximal, then $F = N$ and N is projective. Otherwise, if $R \in \overline{\mathcal{H}}$, then by [23, Theorem 1.6.15], F is finitely generated, so F/N is finitely presented. Since R is nonnil-coherent, [9, Theorem 2.6] implies F/N is ϕ-2-presented. Thus, N is finitely presented. Since $\phi\text{-w.gl.dim}(R) \leq 1$, N is ϕ-flat, and by [10, Theorem 5.13 and Corollary 5.36], N is projective.

(3) ⇒ (2): If $\mathrm{Nil}(R)$ is a maximal ideal, then R is a ϕ-von Neumann regular ring, and $\phi\text{-w.gl.dim}(R) = 0$ by [10, Theorem 5.29]. If $\mathrm{Nil}(R)$ is not maximal, consider a finitely generated nonnil ideal I of R. Since R/I is ϕ-torsion and I is a ϕ-submodule of R, it follows from (3) that I is projective, and hence $\phi\text{-w.gl.dim}(R) \leq 1$.

(7) ⇒ (2): Assume condition (7). If $\mathrm{Nil}(R)$ is maximal, then R is a ϕ-von Neumann regular ring by [10, Theorem 5.14], and hence $\phi\text{-w.gl.dim}(R) = 0$. If $\mathrm{Nil}(R)$ is not maximal, then by Lemma 3.8, R is a ϕ-$(2,1)$ ring. In this case, every finitely generated nonnil ideal J of R is finitely presented due to the nonnil-coherence of R. Thus, $\phi\text{-pd}_R(R/J) \leq 1$, and hence $\phi\text{-fd}_R(R/J) \leq 1$. Since R is connected, R/J cannot be ϕ-flat by [10, Theorem 5.13 and Corollary 5.36]. Therefore, $\phi\text{-w.gl.dim}(R) \leq 1$. □

Corollary 3.10. *In a ϕ-D-ring R with a nonzero and non-maximal nilradical, any finitely generated ϕ-u-projective R-module M with $\mathrm{fd}_R(M) < \infty$ is a projective R-module.*

Proof. This result follows directly from Theorem 2.3, Theorem 3.7, and [1, Corollary 2.26], which collectively establish that in a ϕ-D-ring with a non-maximal nilradical, a finitely generated ϕ-u-projective module of finite flat dimension must necessarily be projective. $\square$

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