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A HOPF BIFURCATION OF MULTIDIMENSIONAL ATTRACTION-REPULSION CHEMOTAXIS SYSTEM WITH NONLINEAR SENSITIVE FUNCTIONS

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Abstract. This paper is concerned with a multi-dimensional attraction-repulsion chemotaxis system with nonlinear sensitive functions. A corresponding free boundary problem is derived, and proved the existence of stationary solutions and Hopf bifurcation which are essentially determined by the competition of attraction and repulsion.

1. Introduction

The following an attraction-repulsion chemotaxis system have been extensively studied in [6, 8, 15, 19]:

$$(1)\quad \begin{cases} \sigma\varepsilon\, U_t = \varepsilon^2\, \nabla^2 U - \varepsilon\nabla\cdot(\kappa_1 U\nabla\chi(V)) + \varepsilon\nabla\cdot(\kappa_2 U\nabla\xi(W)) + F(U, a_0), \\ DV_t = \nabla^2 V + \mu U - V, \\ DW_t = \nabla^2 W + U + V - W - s_0, \quad t > 0,\ \mathbf{x}\in\mathbb{R}^n. \end{cases}$$

where $U(\mathbf{x}, t)$ is a cell density, $V(\mathbf{x}, t)$ and $W(\mathbf{x}, t)$ represent the concentration of an attractive cue and of repulsive signal, respectively. The parameters $\varepsilon, \sigma, \kappa_1, \kappa_2, \mu, s_0$ are positive constants, and $\chi(V)$ and $\xi(W)$ are the chemical sensitivity functions of the chemical attractant and repulsive. The parameters κ_1 and κ_2 measure the strength of the attraction and repulsion, respectively. The nonlinear term $F(U, a_0)$ models characterizing the chemical growth and degradation.

In this paper, we study the system (1) with the case where a_0 is a function of W (see [9])

$$(2)\quad a(W) = \frac{1}{2}\big(1 + \tanh(kW + a_0)\big),$$

where k, a_0 are positive constant and k is the intensity of W.

The linear system characterized by $\chi(V) = V$, $\xi(W) = W$, with $F = 0$ and $\varepsilon = 1$, was introduced in [9, 13] to model the aggregation behavior of microglial cells observed in Alzheimer's disease. A similar formulation was also employed in [15] to investigate quorum-sensing mechanisms in chemotactic processes. In the one-dimensional setting, the system is globally well-posed provided that the condition $\kappa_2 - \mu\kappa_1 > 0$ holds, as demonstrated in [7, 11]. In higher dimensions, the well-posedness of the system is fundamentally determined by the interplay between attractive and repulsive effects,

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encapsulated by the sign of $\kappa_2 - \mu\kappa_1$, as discussed in [10, 16]. Moreover, it has been established in [10, 17] that the system remains globally well-posed under the same condition, even when $F = 0$ and $D = 0$ (or $D = 1$). Further developments include the work in [1], where it was shown that if $F = 0$, $\varepsilon = 1$, $\chi(V) = \ln V$, and $\xi(W)$ is constant, then the model (1) is capable of reproducing propagating wave patterns observed in experimental settings.

We consider a free boundary problem of the attraction-repulsion nonlinear system of (1) in the multi-dimensional case, specifically under the conditions $\chi'(V) > 0$ and $\xi'(W) > 0$.

Suppose that there exists a unique $(n-1)$ -dimensional hypersurface $\eta(t)$, which is simply single closed curve within the domain in such a way that $\mathbb{R}^n = \Omega_1(t) \cup \eta(t) \cup \Omega_0(t)$, where $\Omega_1(t) = \{\mathbf{x} \in \mathbb{R}^n \ : U(\mathbf{x}, t) > a(W)\}$ and $\Omega_0(t) = \{\mathbf{x} \in \mathbb{R}^n \ : U(\mathbf{x}, t) < a(W)\}$. The equation of $\eta(t)$ is given by (see [12, 14, 18]):

$$\frac{d\,\eta(t)}{dt} \cdot \nu = \frac{1}{\sigma}\Big(C(v_i) + \kappa_1\chi'(V) \cdot \frac{\partial V}{\partial \nu} - \kappa_2\xi'(W) \cdot \frac{\partial W}{\partial \nu}\Big), \ \mathbf{x} \in \eta(t),$$

where v_i is the value of V on the interface $\eta(t)$ and ν is the outward normal vector on $\eta(t)$. The velocity of the interface $C(\cdot)$ is a continuously differentiable function defined on an interval $\mathbf{I} := (-a(W), 1 - a(W))$ and thus it can be normalized by

$$C(V; a(W)) = \frac{1 - 2V - 2a(W)}{\sqrt{(V + a(W))(1 - a(W) - V)}}.$$

An analysis of the dynamics of this process has been shown (see example [3, 9]) to lead a free boundary problem consisting of the initial-boundary value problem

$$(3) \qquad \begin{cases} V_t = \nabla^2 V - (\mu+1)V + \mu, \quad t > 0, \ \mathbf{x} \in \Omega_1(t) \\ V_t = \nabla^2 V - (\mu+1)V, \quad t > 0, \ \mathbf{x} \in \Omega_0(t) \\ V(\eta(t) - 0, t) = V(\eta(t) + 0, t), \\ \frac{d}{d\nu}V(\eta(t) - 0, t) = \frac{d}{d\nu}V(\eta(t) + 0, t), \\ \lim_{|\mathbf{x}|\to\infty} V(\mathbf{x}, t) = 0 \end{cases}$$

and

$$(4) \qquad \begin{cases} W_t = \nabla^2 W - W + 1 - s_0, \quad t > 0, \ \mathbf{x} \in \Omega_1(t) \\ W_t = \nabla^2 W - W - s_0, \quad t > 0, \ \mathbf{x} \in \Omega_0(t) \\ W(\eta(t) - 0, t) = W(\eta(t) + 0, t), \\ \frac{d}{d\nu}W(\eta(t) - 0, t) = \frac{d}{d\nu}W(\eta(t) + 0, t), \\ \lim_{|\mathbf{x}|\to\infty} W(\mathbf{x}, t) = -s_0. \end{cases}$$

The organization of the paper is as follows: In section 2, we introduces a change of variables that regularizes problem (3) and (4), allowing results from the theory of nonlinear evolution equations to be applied. This transformation ensures that the solution has sufficient regularity for a bifurcation analysis. In section 3, we establishes the existence of equilibrium solutions for (3) and (4), and presents the linearization of these problems under the condition $\kappa_2\xi'(W) - \mu\kappa_1\chi'(V) > 0$. The last section 4 examines the conditions on $a(W)$ and $\xi(W)$ that guarantee the existence of periodic solutions. Additionally, it investigates the bifurcation of the interface problem as the parameter σ varies in both two and three dimensions.

2. Regularization of the interface equation

We now search for an existence problem of radially symmetric equilibrium solutions of (3) and (4) with $|\mathbf{x}| = r$, $(n = 1, 2, 3)$, where the center and the interface are located at the origin and $r = \eta$, respectively. The problem is given by :

$$(5)\quad \begin{cases} V_t = \frac{\partial^2 V}{\partial r^2} + \frac{n-1}{r}\frac{\partial V}{\partial r} - (\mu+1)V + \mu, & t>0,\ r\in\Omega_1(t),\\ V_t = \frac{\partial^2 V}{\partial r^2} + \frac{n-1}{r}\frac{\partial V}{\partial r} - (\mu+1)V, & t>0,\ r\in\Omega_0(t),\\ \frac{\partial V}{\partial r}(0,t)=0,\ \lim\limits_{r\to\infty} V(r,t)=0\,, & t>0,\\ W_t = \frac{\partial^2 W}{\partial r^2} + \frac{n-1}{r}\frac{\partial W}{\partial r} - W + 1 - s_0, & t>0,\ r\in\Omega_1(t),\\ W_t = \frac{\partial^2 W}{\partial r^2} + \frac{n-1}{r}\frac{\partial W}{\partial r} - W - s_0, & t>0,\ r\in\Omega_0(t),\\ \frac{\partial W}{\partial r}(0,t)=0,\ \lim\limits_{r\to\infty} W(r,t) = -s_0\,, & t>0,\\ \sigma\eta'(t) = C(V(\eta); a(W(\eta))) + \kappa_1\chi'(V)\,V_r(\eta,t) - \kappa_2\xi'(W)W_r(\eta,t), \eta(0)=\eta_0, \end{cases}$$

where $\Omega_1(t) = \{r\,:0<r<\eta(t)\}$ and $\Omega_0(t) = \{r\,:\eta(t)<r<\infty\}$.

As a first step we obtain more regularity for the solution by semigroup methods, considering $\hat{W}(r,t) = W(r,t) + s_0$. Let A be an operator defined by $A := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \mu + 1$ with domain $D(A) = \{V \in H^{2,2}((0,\infty))\,:\,\frac{\partial V}{\partial r}(0,t) = 0, \lim_{r\to\infty} V(r,t) = 0\}$. Let $A_0 := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + 1$ with domain $D(A_0) = \{\hat{W} \in H^{2,2}((0,\infty))\,:\,\frac{\partial \hat{W}}{\partial r}(0,t) = 0, \lim_{r\to\infty}\hat{W}(r,t) = 0\}$ with $\hat{W}(r,t) = W(r,t) + s_0$. In order to apply semigroup theory to (5), we choose the space $X := L_2(0,\infty)$ with norm $\|\cdot\|_2$.

To get differential dependence on initial conditions, we decompose V in (5) into two parts: u, which is a solution to a more regular problem and g, which is less regular but explicitly known in terms of the Green's function G of the operator A. Namely, we define $g : [0,\infty)\times[0,\infty) \to \mathbb{R}$, by

$$g(r,\eta) := A^{-1}(\mu H(\cdot - \eta)(x)) = \mu \int_0^\infty G(r,y)\,H(\eta - y)\,dy,$$

where $G : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is a Green's function of A satisfying the Neumann boundary conditions;

$$G(r,z) = \begin{cases} \frac{1}{\sqrt{1+\mu}}\cosh(r\sqrt{1+\mu})\,e^{-\sqrt{1+\mu}z}\,, & 0<r<z\\ \frac{1}{\sqrt{1+\mu}}\,e^{-\sqrt{1+\mu}r}\cosh(z\sqrt{1+\mu})\,, & z<r \qquad (n=1), \end{cases}$$

$$G(r,z) = \begin{cases} zK_0(z\sqrt{1+\mu})\,I_0(r\sqrt{1+\mu})\,, & 0<r<z\\ zI_0(z\sqrt{1+\mu})\,K_0(r\sqrt{1+\mu})\,, & z<r \qquad (n=2), \end{cases}$$

where I_0 and K_0 are modified Bessel functions and

$$G(r,z) = \begin{cases} z\,e^{-z\sqrt{1+\mu}}\,\frac{\sinh(r\sqrt{1+\mu})}{r\sqrt{1+\mu}}\,, & 0<r<z\\ z\sinh(z\sqrt{1+\mu})\,\frac{e^{-r\sqrt{1+\mu}}}{r\sqrt{1+\mu}}\,, & z<r \qquad (n=3), \end{cases}$$

and $\gamma : [0,\infty) \to \mathbb{R}$,

$$\gamma(\eta) := g(\eta,\eta).$$

If we take a transformation $u(t)(r) = V(r,t) - g(r,\eta(t))$, we have $(u_r)(t)(r) = V_r(r,t) - g_r(r,\eta(t))$. Since $G_r(r,\eta)$ is discontinuous, we cannot obtain one step more regular than that of (5).

To overcome this difficulty, let $p(r,t) = V_r(r,t)$ and define $\hat{g} : [0,\infty)\times[0,\infty) \to \mathbb{R}$,

$$\hat{g}(r,\eta) := A^{-1}(\mu\delta(\cdot-\eta)(x)) = \mu\int_0^\infty \hat{G}(r,y)\,\delta(\eta-y)\,dy,$$

where $\hat{G} : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is a Green's function of A satisfying the Dirichlet boundary conditions, and $\hat{\gamma} : [0,\infty) \to \mathbb{R}$,

$$\hat{\gamma}(\eta) := \hat{g}(\eta,\eta).$$

We define $j : [0,\infty)\times[0,\infty) \to \mathbb{R}$,

$$j(r,\eta) := A_0^{-1}(H(\cdot-\eta)(r)) = \int_0^\infty J(x,y)H(\eta-y)\,dy$$

and $\alpha : [0,\infty) \to \mathbb{R}$,

$$\alpha(\eta) := j(\eta,\eta).$$

Here $J : [0,\infty)^2 \to \mathbb{R}$ is a Green's function of A_0 satisfying the boundary conditions. Define $w(t)(r) = \hat{W}(r,t) - j(r,\eta(t))$, $q(r,t) = \hat{W}_r(r,t)$ and define $\hat{j} : [0,\infty)\times[0,\infty) \to \mathbb{R}$,

$$\hat{j}(r,\eta) := A_0^{-1}(\delta(\cdot-\eta)(r)) = \int_0^\infty \hat{J}(r,y)\,\delta(\eta-y)\,dy,$$

where $\hat{J} : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is a Green's function of A_0 satisfying the Dirichlet boundary conditions and $\hat{\alpha} : [0,\infty) \to \mathbb{R}$,

$$\hat{\alpha}(\eta) := \hat{j}(\eta,\eta).$$

Applying the transformations $u(t)(r) = V(r,t) - g(r,\eta(t))$, $v(t)(r) = p(r,t) - \hat{g}(r,\eta(t))$ and $w(t)(r) = \hat{W}(r,t) - j(r,\eta(t))$, $s(t)(r) = q(r,t) - \hat{j}(r,\eta(t))$, then (5) becomes

$$(6)\qquad \begin{cases} u_t + Au = -\frac{1}{\sigma}\mu G(r,\eta)R(u,w,\eta) \\ v_t + Av = \frac{1}{\sigma}\frac{\mu}{\eta}\hat{G}(r,\eta)R(u,w,\eta) \\ w_t + A_0 w = -\frac{1}{\sigma}J(r,\eta)R(u,w,\eta) \\ s_t + A_0 s = \frac{1}{\sigma\eta}\hat{J}(r,\eta)R(u,w,\eta) \\ \eta'(t) = \frac{1}{\sigma}R(u,w,\eta), \quad t>0 \end{cases}$$

where $R(u,w,\eta) = C(u(\eta)+\gamma(\eta); a(w(\eta)+\alpha(\eta)-s_0)) + \kappa_1(v(\eta)+\hat{\gamma}(\eta)) - \kappa_2(s(\eta)+\hat{\alpha}(\eta))$.

Thus, we obtain an abstract evolution equation equivalent to (6) :

$$(7)\qquad \begin{cases} \frac{d}{dt}(u,v,w,s,\eta) + \widetilde{A}(u,v,w,s,\eta) = \frac{1}{\sigma}f(u,v,w,s,\eta), \\ (u,v,w,s,\eta)(0) = (u_0(x),v_0(x),w_0(x),s_0(x),\eta_0), \end{cases}$$

where $\widetilde{A}$ is a 5×5 matrix where (1,1) and (2,2)-entries are an operator A, (3,3) and (4,4)-entries are an operator A_0 and all the others are zero. The nonlinear forcing term f is

$$f(u,v,w,s,\eta) = \begin{pmatrix} f_1(\eta) \cdot (f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,s,\eta) - f_{23}(u,v,w,s,\eta)) \\ f_2(\eta) \cdot (f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,s,\eta) - f_{23}(u,v,w,s,\eta)) \\ f_3(\eta) \cdot (f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,s,\eta) - f_{23}(u,v,w,s,\eta)) \\ f_4(\eta) \cdot (f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,s,\eta) - f_{23}(u,v,w,s,\eta)) \\ f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,s,\eta) - f_{23}(u,v,w,s,\eta) \end{pmatrix},$$

where $f_1 : (0,\infty) \to X$, $f_1(\eta)(r) := -\mu G(r,\eta)$, $f_2 : (0,\infty) \to X$, $f_2(\eta)(x) := \frac{\mu}{\eta}\hat{G}(r,\eta)$, $f_3 : (0,\infty) \to X$, $f_3(\eta)(r) := -J(r,\eta)$, $f_4 : (0,\infty) \to X$, $f_4(\eta)(r) := \frac{1}{\eta}\hat{J}(r,\eta)$, $f_{21} : Y \to \mathbb{C}$, $f_{21}(u,v,w,s,\eta) := C(u(\eta)+\gamma(\eta); a(w(\eta)+\alpha(\eta)-s_0))$, $f_{22} : Y \to \mathbb{C}$, $f_{22}(u,v,w,s,\eta) := \kappa_1\chi'(u(\eta)+\gamma(\eta))(v(\eta)+\hat{\gamma}(\eta))$, $f_{23}(u,v,w,s,\eta) := \kappa_2\xi'(w(\eta)+\alpha(\eta))(s(\eta)+\hat{\alpha}(\eta))$ and $Y := \{(u,v,w,s,\eta) \in C^1(0,\infty) \times C^1(0,\infty) \times C^1(0,\infty) \times C^1(0,\infty) \times (0,\infty) \ : \ u(\eta)+\gamma(\eta) \in I,\ v(\eta)+\hat{\gamma}(\eta) \in I,\ w(\eta)+\alpha(\eta) \in I,\ s(\eta)+\hat{\alpha}(\eta) \in I\} \subset_{\text{open}} C^1(\mathbb{R}) \times C^1(\mathbb{R}) \times C^1(\mathbb{R}) \times C^1(\mathbb{R}) \times \mathbb{R}$.

The well-posedness of solutions of (7) is shown in [5, 8, 18] with the help of the semigroup theory using domains of fractional powers $\theta \in (3/4, 1]$ of A, A_0 and $\widetilde{A}$. Moreover, the nonlinear term f is a continuously differentiable function from $Y \cap \widetilde{X}^\theta$ to $\widetilde{X}$, where $\widetilde{X} := D(\widetilde{A}) = D(A) \times D(A) \times D(A_0) \times D(A_0) \times \mathbb{R}$, $X^\theta := D(A^\theta)$; $X_0^\theta := D(A_0^\theta)$ and $\widetilde{X}^\theta := D(\widetilde{A}^\theta) = X^\theta \times X^\theta \times X_0^\theta \times X_0^\theta \times \mathbb{R}$.

The velocity of η is written by

(8)
$$\begin{aligned} &C(u(\eta)+\gamma(\eta); a(w(\eta)+\alpha(\eta)-s_0)) \\ &= \frac{1 - 2(u(\eta)+\gamma(\eta)+a(w(\eta)+\alpha(\eta)-s_0))}{\sqrt{(u(\eta)+\gamma(\eta)+a(w(\eta)+\alpha(\eta)-s_0))(1-(u(\eta)+\gamma(\eta)+a(w(\eta)+\alpha(\eta)-s_0))}}, \end{aligned}$$

where $a(w(\eta)+\alpha(\eta)-s_0) = \frac{1}{2}\big(1+\tanh(k(w(\eta)+\alpha(\eta)-s_0)+a_0)\big)$.

The derivative of f can be obtained from the following in [4]:

Lemma 2.1. *The functions $G(\cdot,\eta) : (0,\infty) \to X$, $\hat{G}(\cdot,\eta) : (0,\infty) \to X$, $J(\cdot,\eta) : (0,\infty) \to X$, $\hat{J}(\cdot,\eta) : (0,\infty) \to X$, $C(\cdot) : Y \to \mathbb{C}$ and $f : Y \to X \times \mathbb{R}$ are continuously differentiable with derivatives given by*

$$\begin{aligned} &Df_{21}(u,v,w,s,\eta)(\tilde{u},\tilde{v},\tilde{w},\tilde{s},\tilde{\eta}) = C'(u(\eta)+\gamma(\eta); a(w(\eta)+\alpha(\eta)-s_0)) \\ &\qquad \cdot\big(u'(\eta)\tilde{\eta} + \tilde{u}(\eta) + \gamma'(\eta)\tilde{\eta} + a'(w(\eta)+\alpha(\eta)-s_0))(w'(\eta)\tilde{\eta} + \tilde{w}(\eta) + \alpha'(\eta)\tilde{\eta})\big) \end{aligned}$$

$$\begin{aligned} &Df_{22}(u,v,w,s,\eta)(\tilde{u},\tilde{v},\tilde{w},\tilde{s},\tilde{\eta}) = \kappa_1\chi'(u(\eta)+\gamma(\eta))\big(v'(\eta)\tilde{\eta} + \tilde{v}(\eta) + \hat{\gamma}'(\eta)\tilde{\eta}\big) \\ &\qquad +\kappa_1\chi''(u(\eta)+\gamma(\eta))(u'(\eta)\tilde{\eta} + \tilde{u}(\eta) + \gamma'(\eta)\tilde{\eta})(v(\eta)+\hat{\gamma}(\eta)) \end{aligned}$$

$$\begin{aligned} &Df_{23}(u,v,w,s,\eta)(\tilde{u},\tilde{v},\tilde{w},\tilde{s},\tilde{\eta}) = \kappa_2\xi'(w(\eta)+\alpha(\eta))\big(s'(\eta)\tilde{\eta} + \tilde{s}(\eta) + \hat{\alpha}'(\eta)\tilde{\eta}\big) \\ &\qquad +\kappa_2\xi''(w(\eta)+\alpha(\eta))(w'(\eta)\tilde{\eta} + \tilde{w}(\eta) + \alpha'(\eta)\tilde{\eta})(s(\eta)+\hat{\alpha}(\eta)) \end{aligned}$$

$$\begin{aligned} &Df(u,v,w,s,\eta)(\tilde{u},\tilde{v},\tilde{w},\tilde{s},\tilde{\eta}) \\ &= (f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,s,\eta)) \cdot (f_1'(\eta), f_2'(\eta), f_3'(\eta), f_4'(\eta), 0)\,\tilde{\eta} \\ &\quad +(Df_{21}(u,v,w,s,\eta) + Df_{22}(u,v,w,s,\eta) - Df_{23}(u,v,w,s,\eta))(\tilde{u},\tilde{v},\tilde{w},\tilde{s},\tilde{\eta}) \\ &\quad \cdot(f_1(\eta), f_2(\eta), f_3(\eta), f_4(\eta), 1). \end{aligned}$$

3. Equilibrium solutions and Linearization of the interface equation

In this section, we shall examine the existence of equilibrium solutions of (7). We assume that $\chi'(v) > 0$ for all v and $\xi'(w) > 0$ for all w.

We look for $(u^*, v^*, w^*, s^*, \eta^*) \in D(\widetilde{A}) \cap Y$ satisfying the following equations:

$$(9) \qquad \begin{cases} Au^* = -\frac{1}{\sigma}\,\mu G(\cdot,\eta^*)P(\eta^*) \\ Av^* = \frac{1}{\sigma\eta^*}\,\mu \hat{G}(\cdot,\eta^*)P(\eta^*) \\ A_0 w^* = -\frac{1}{\sigma}\,J(\cdot,\eta^*)P(\eta^*) \\ A_0 s^* = \frac{1}{\sigma\eta^*}\,\hat{J}(\cdot,\eta^*)P(\eta^*) \\ 0 = P(\eta^*) \end{cases}$$

with $u^{*\prime}(0) = 0 = u(\infty)$, $v^*(0) = 0 = v^*(\infty)$, $w^{*\prime}(0) = 0 = w^*(\infty)$, $s^*(0) = 0 = s^*(\infty)$ and $P(\eta) = C(u^*(\eta^*) + \gamma(\eta^*); a(w^*(\eta^*) + \alpha(\eta^*) - s_0))) + \kappa_1\chi'(u^*(\eta) + \gamma(\eta^*))(v^*(\eta^*) + \hat{\gamma}(\eta^*)) - \kappa_2\xi'(w(\eta^*) + \alpha(\eta^*))(s^*(\eta^*) + \hat{\alpha}(\eta^*))$.

Theorem 3.1. *Suppose that* $C'(\gamma(\eta); a(\alpha(\eta) - s_0)) + \kappa_1\chi''(\gamma(\eta))\,\hat{\gamma}(\eta) < 0$ *and* $C'(\gamma(\eta);$
$a(\alpha(\eta) - s_0))a'(\alpha(\eta) - s_0) - \kappa_2\xi''(\alpha(\eta)) < 0$ *for all* $\eta > 0$. *Furthermore, assume* $ks_0 > a_0$, $\frac{\mu}{1+\mu} + \tanh(k(\frac{1}{2} - s_0) + a_0) > 0$ *and* $\kappa_2\,\xi'(\alpha(\eta)) > \mu\kappa_1\chi'(\gamma(\eta))$ *for all* $\eta > 0$. *Then equation (7) has at least one equilibrium solution* $(0,0,0,0,\eta^*)$, $\eta^* \in (0,\infty)$. *The linearization of* f *at the stationary solution* $(0,0,0,0,\eta^*)$ *is*

$$Df(0,0,0,0,\eta^*)(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) = \begin{pmatrix} -\mu G(\cdot,\eta^*)\,Q(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) \\ \frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)\,Q(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) \\ -J(\cdot,\eta^*)\,Q(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) \\ \frac{1}{\eta^*}\,\hat{J}(\cdot,\eta^*)\,Q(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) \\ Q(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) \end{pmatrix},$$

where $Q(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) = \left(-4+\kappa_1\chi''(\gamma(\eta^*)\hat{\gamma}(\eta^*)\right)(\hat{u}(\eta^*)+\gamma'(\eta^*)\hat{\eta})+\kappa_1\chi'(\gamma(\eta^*))\,(\hat{v}(\eta^*)+\hat{\gamma}'(\eta^*)\hat{\eta}) - \left(\kappa_2\xi''(\alpha(\eta^*))\hat{\alpha}'(\eta^*)+4a'(\alpha(\eta^*)-s_0)\right)(\hat{w}(\eta^*)+\alpha'(\eta^*)\hat{\eta}) - \kappa_2\xi'(\alpha(\eta^*))(\hat{s}(\eta^*)+\hat{\alpha}'(\eta^*)\hat{\eta})$. *The pair* $(0,0,0,0,\eta^*)$ *corresponds to a unique steady state* $(V^*, V_r^*, \hat{W}^*, \hat{W}_r^*, \eta^*)$ *of (5) for* $\sigma \neq 0$ *with* $V^*(r) = g(r,\eta^*)$, $V_r^*(r) = \hat{g}(r,\eta^*)$, $W^*(r) = j(r,\eta^*) - s_0$ *and* ${W_r}^*(r) = \hat{j}(r,\eta^*)$.

Proof. From the system of equations (9), we have $u^* = 0, v^* = 0, w^* = 0$ and $s^* = 0$. In order to show existence of η^*, we define

$$\Gamma(\eta) := C(S(0,0,\eta)) + \kappa_1\,\chi'(\gamma(\eta))\hat{\gamma}(\eta) - \kappa_2\xi'(\alpha(\eta))\hat{\alpha}(\eta)$$

where $S(u,w,\eta) = u(\eta) + \gamma(\eta) + a(w(\eta) + \alpha(\eta) - s_0)$. We shall show that $\Gamma'(\eta) < 0$ for all $\eta > 0$ and then $\Gamma(\eta) = 0$ is solvable with η^* if $\Gamma(0) > 0$ and $\Gamma(\infty) < 0$ for all $\eta > 0$.

$$\begin{aligned} \Gamma'(\eta) &= C'(S(0,0,\eta))\big(\gamma'(\eta) + a'(\alpha(\eta) - s_0))\alpha'(\eta) + \kappa_1\chi''(\gamma(\eta))\,\hat{\gamma}(\eta)\big)\,\gamma'(\eta) \\ &\quad +\kappa_1\chi'(\gamma(\eta))\hat{\gamma}'(\eta) - \kappa_2\xi'(\alpha(\eta))\hat{\alpha}'(\eta) - \kappa_2\xi''(\alpha(\eta))\alpha'(\eta)\,\hat{\alpha}(\eta)) \\ &= \big(C'(S(0,0,\eta)) + \kappa_1\chi''(\gamma(\eta))\,\hat{\gamma}(\eta)\big)\,\gamma'(\eta) + \kappa_1\chi'(\gamma(\eta))\hat{\gamma}'(\eta) - \kappa_2\xi'(\alpha(\eta))\hat{\alpha}'(\eta) \\ &\quad +\big(C'(S(0,0,\eta))\,a'(\alpha(\eta) - s_0)) - \kappa_2\xi''(\alpha(\eta))\,\hat{\alpha}(\eta)\big)\,\alpha'(\eta). \end{aligned}$$

Assuming that $\kappa_2\xi'(\alpha(\eta)) > \kappa_1\mu\chi'(\gamma(\eta))$ holds for all $\eta > 0$, we obtain the inequality $\kappa_1\chi'(\gamma(\eta))\hat{\gamma}'(\eta) - \kappa_2\xi'(\alpha(\eta))\hat{\alpha}'(\eta) < \big(\kappa_1\mu\chi'(\gamma(\eta)) - \kappa_2\xi'(\alpha(\eta))\big)\,\hat{\alpha}'(\eta) < 0$ since $0 < \hat{\gamma}'(\eta) < \mu\hat{\alpha}'(\eta)$ for all $\eta > 0$. This implies that $\Gamma'(\eta) < 0$ for all $\eta > 0$ under the assumptions. Moreover, $\Gamma(0) = C(\gamma(0)) > 0$ is satisfied if $1 - 2\gamma(0) - 2a(\alpha(0) - s_0) = -\tanh(k(\alpha(0) - s_0) + a_0) > 0$ which is equivalent to the condition $ks_0 - a_0 > 0$. Also,

$$\begin{aligned}\lim_{\eta\to\infty}\Gamma(\eta) &= \lim_{\eta\to\infty} C(S(0,0,\eta)) + \kappa_1\,\chi'(\gamma(\eta))\hat{\gamma}(\eta) - \kappa_2\xi'(\alpha(\eta))\hat{\alpha}(\eta)\\ &< \lim_{\eta\to\infty}\Big(C(S(0,0,\eta)) + \tfrac{1}{2}\big(\tfrac{1}{\sqrt{1+\mu}} - 1\big)\kappa_2\xi'(\alpha(\eta))\Big).\end{aligned}$$

Thus, $\lim_{\eta\to\infty}\Gamma(\eta) < 0$ if the following inequality holds:

$$\tfrac{1}{2} - \lim_{\eta\to\infty}(\gamma(\eta) - a(\alpha(\eta) - s_0)) < 0,$$

which is equivalent to the condition if $\frac{\mu}{1+\mu} + \tanh(k(\frac{1}{2} - s_0) + a_0) > 0$.

Furthermore, the formula for $Df(0,0,0,0,\eta^*)$ follows from the relation $C'(S(0,0,\eta^*)) = -4$, and the corresponding steady state $(V^*, p^*, W^*, q^*, \eta^*)$ for (5) is derived via the transformation and Theorem 2.1 in [4]. □

4. A Hopf bifurcation

In this section, we establish that a Hopf bifurcation occurs along the curve of equilibrium points parameterized by $\sigma \mapsto (0,0,0,0,\eta^*,\tau^*)$. First, let us introduce the following relevant definition.

Definition 4.1. Under the assumptions of Theorem 3.1, define (for $1 \geq \theta > 3/4$) the linear operator B from $\widetilde{X}^\theta$ to $\widetilde{X}$ by

$$B := Df(0,0,0,0,\eta^*)\,.$$

We then define $(0,0,0,0,\eta^*)$ to be a Hopf point for (7) if and only if there exists an $\epsilon_0 > 0$ and a C^1-curve

$$(-\epsilon_0 + \tau^*, \tau^* + \epsilon_0) \mapsto (\lambda(\tau), \phi(\tau)) \in \mathbb{C} \times \widetilde{X}_{\mathbf{C}}$$

($Y_{\mathbf{C}}$ denotes the complexification of the real space Y) of eigendata for $-\widetilde{A} + \tau B$ with

(i) $(-\widetilde{A} + \tau B)(\phi(\tau)) = \lambda(\tau)\phi(\tau), \quad (-\widetilde{A} + \tau B)(\overline{\phi(\tau)}) = \overline{\lambda(\tau)}\,\overline{\phi(\tau)}$;
(ii) $\lambda(\tau^*) = i\beta$ with $\beta > 0$;
(iii) $\mathrm{Re}\,(\lambda) \neq 0$ for all λ in the spectrum of $(-\widetilde{A} + \tau^* B) \setminus \{\pm i\beta\}$;
(iv) $\mathrm{Re}\,\lambda'(\tau^*) \neq 0$ (transversality);

where $\tau = 1/\sigma$.

Next, we check (7) for Hopf points. For this, we solve the eigenvalue problem:

$$-\widetilde{A}(u,v,w,s,\eta) + \tau B(u,v,w,s,\eta) = \lambda I_5(u,v,w,s,\eta),$$

where I_5 is an 5×5 identity matrix. This is equivalent to:

$$(10)\qquad \begin{cases} (A+\lambda)u = \tau\mu\, G(\cdot,\eta^*)R(u,v,w,s,\eta), \\ (A+\lambda)v = -\tau\,\frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)\,R(u,v,w,s,\eta), \\ (A_0+\lambda)w = \tau\, J(\cdot,\eta^*)R(u,v,w,s,\eta), \\ (A_0+\lambda)s = -\tau\frac{1}{\eta^*}\,\hat{J}(\cdot,\eta^*)R(u,v,w,s,\eta), \\ \lambda\,\eta = \tau R(u,v,w,s,\eta), \end{cases}$$

where $R(u,v,w,z,\eta) = -4(u(\eta^*)+\gamma'(\eta^*)\eta)+\kappa_1 d_2(u(\eta^*)+\gamma'(\eta^*)\eta)+\kappa_1 d_1(v(\eta^*)+\hat{\gamma}'(\eta^*)\eta)-(\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0))(w(\eta^*)+\alpha'(\eta^*)\eta)-\kappa_2 d_4(s(\eta^*)+\hat{\alpha}'(\eta^*)\eta)$, $d_1 = \chi'(\gamma(\eta^*)), d_2 = \chi''(\gamma(\eta^*))\hat{\gamma}(\eta^*)$, $d_3 = \xi''(\alpha(\eta))\hat{\alpha}(\eta^*)$ and $d_4 = \xi'(\alpha(\eta^*))$.

Henceforth, we set $d_1 = \chi'(\gamma(\eta^*)), d_2 = \chi''(\gamma(\eta^*))\hat{\gamma}(\eta^*)$, $d_3 = \xi''(\alpha(\eta))\hat{\alpha}(\eta^*)$, $d_4 = \xi'(\alpha(\eta^*))$, and assume $d_1 > 0$ and $d_4 > 0$.

We shall show that an equilibrium solution is a Hopf point.

Theorem 4.2. *Suppose that $ks_0 > a_0$, $\frac{\mu}{1+\mu}+\tanh(k(\frac{1}{2}-s_0)+a_0) > 0$, $\kappa_2\, d_4 > \mu\kappa_1 d_1$, $4-\kappa_1 d_2 > 0$ and $4a'(\alpha(\eta)-s_0)+\kappa_2 d_3 > \kappa_2\frac{d_4}{\eta^*}$. Additionally, suppose the operator $-\widetilde{A}+\tau^* B$ has a unique pair $\{\pm i\beta\}$, $\beta > 0$ of purely imaginary eigenvalues for some $\tau^* > 0$. Then, $(0,0,0,0,\eta^*,\tau^*)$ is a Hopf point for (7).*

Proof. We assume without loss of generality that $\beta > 0$, and Φ^* is the (normalized) eigenfunction of $-\widetilde{A}+\tau^* B$ with eigenvalue $i\beta$. We have to show that $(\Phi^*, i\beta)$ can be extended to a C^1-curve $\tau\mapsto(\Phi(\tau),\lambda(\tau))$ of eigendata for $-\widetilde{A}+\tau B$ with $\mathrm{Re}(\lambda'(\tau^*))\neq 0$.

For this, let $\Phi^* = (\psi_0, v_0, w_0, s_0, \eta_0)\in D(A)\times D(A)\times D(A_0)\times D(A_0)\times\mathbb{R}$. First, we note that $\eta_0\neq 0$. Otherwise, by (10), $(A+i\beta)\psi_0 = \mu\, i\beta\,\eta_0\, G(\cdot,\eta^*) = 0$ and $(A+i\beta)v_0 = -\frac{\mu}{\eta^*}\, i\beta\,\eta_0\,\hat{G}(\cdot,\eta^*) = 0$, which is not possible given A is symmetric. So, without loss of generality, let $\eta_0 = 1$. Then $E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*) = 0$ by (10), where

$$E : D(A)_{\mathbf{C}}\times D(A)_{\mathbf{C}}\times D(A_0)_{\mathbf{C}}\times D(A_0)_{\mathbf{C}}\times\mathbb{C}\times\mathbb{R}\longrightarrow X_{\mathbf{C}}\times X_{\mathbf{C}}\times X_{\mathbf{C}}\times X_{\mathbf{C}}\times\mathbb{C},$$

$$E(u,v,w,s,\lambda,\tau) := \begin{pmatrix} (A+\lambda)u-\tau\mu\, G(\cdot,\eta^*)Q^*(\eta^*) \\ (A+\lambda)v+\tau\frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)Q^*(\eta^*) \\ (A_0+\lambda)w-\tau J(\cdot,\eta^*)Q^*(\eta^*) \\ (A_0+\lambda)s+\tau\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)Q^*(\eta^*) \\ \lambda-\tau Q^*(\eta^*) \end{pmatrix}$$

where $Q^*(\eta^*) = (-4+\kappa_1 d_2)(u(\eta^*)+\gamma'(\eta^*))+\kappa_1 d_1(v(\eta^*)+\hat{\gamma}'(\eta^*))-\kappa_2 d_4(s(\eta^*)+\hat{\alpha}'(\eta^*))-(\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0))(w(\eta^*)+\alpha'(\eta^*))$. The equation $E(u,v,w,s,\lambda,\tau) = 0$ is equivalent to λ being an eigenvalue of $-\widetilde{A}+\tau B$ with eigenfunction $(u,v,w,s,1)$. We shall apply the implicit function theorem to E to check that E is of C^1- class and that

$$(11)\quad D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*)\in L((D(A)_{\mathbf{C}})^2\times(D(A_0)_{\mathbf{C}})^2\times\mathbb{C}\times\mathbb{R}, X_{\mathbf{C}}^4\times\mathbb{C})$$

is an isomorphism. In addition, the mapping

$$D_{(u,v,w,z,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*)(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\lambda}) = \begin{pmatrix} (A+i\beta)\hat{u} + \hat{\lambda}\psi_0 + \tau^*\mu G(\cdot,\eta^*)\hat{Q}^*(\eta^*) \\ (A+i\beta)\hat{v} + \hat{\lambda}v_0 - \tau^*\frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)\hat{Q}^*(\eta^*) \\ (A_0+i\beta)\hat{w} + \hat{\lambda}w_0 + \tau^* J(\cdot,\eta^*)\hat{Q}^*(\eta^*) \\ (A_0+i\beta)\hat{z} + \hat{\lambda}s_0 - \tau^*\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)\hat{Q}^*(\eta^*) \\ \hat{\lambda} - \tau^*\,\hat{Q}^*(\eta^*), \end{pmatrix}$$

where $\hat{Q}^*(\eta^*) = (-4+\kappa_1 d_2)(\hat{u}(\eta^*) + \gamma'(\eta^*)) + \kappa_1 d_1(\hat{v}(\eta^*) + \hat{\gamma}'(\eta^*)) - \kappa_2 d_4(\hat{s}(\eta^*) + \hat{\alpha}'(\eta^*)) - (\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0))(\hat{w}(\eta^*) + \alpha'(\eta^*))$ is a compact perturbation of the mapping

$$(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\lambda}) \longmapsto \big((A+i\beta)\hat{u}, (A+i\beta)\hat{v}\,, (A_0+i\beta)\hat{w}, (A_0+i\beta)\hat{s}, \hat{\lambda}\big)$$

which is invertible. Thus, $D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*)$ is a Fredholm operator of index 0. Therefore, in order to verify (11), it suffices to show that the system of equations

$$D_{(u,v,w,z,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*)(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\lambda}) = 0$$

which is equivalent to

$$(12)\qquad \begin{cases} (A+i\beta)\hat{u} + \hat{\lambda}\psi_0 = \tau^*\mu\, G(\cdot,\eta^*)\hat{Q}^*(\eta^*) \\ (A+i\beta)\hat{v} + \hat{\lambda}v_0 = -\tau^*\frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)\hat{Q}^*(\eta^*) \\ (A_0+i\beta)\hat{w} + \hat{\lambda}w_0 = \tau^*\, J(\cdot,\eta^*)\hat{Q}^*(\eta^*) \\ (A_0+i\beta)\hat{z} + \hat{\lambda}s_0 = -\tau^*\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)\hat{Q}^*(\eta^*) \\ \hat{\lambda} = \tau^*\,\hat{Q}^*(\eta^*) \end{cases}$$

necessarily implies that $\hat{u} = 0$, $\hat{v} = 0$, $\hat{w} = 0$, $\hat{s} = 0$ and $\hat{\lambda} = 0$. If we define $\phi := \psi_0 - \mu G(\cdot,\eta^*)$, $\xi := v_0 + \frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)$, $\rho := w_0 - J(\cdot,\eta^*)$ and $\zeta := s_0 + \frac{1}{\eta^*}\,\hat{J}(\cdot,\eta^*)$, then (12) becomes

(13) $(A+i\beta)\hat{u} + \hat{\lambda}\phi = 0,$

(14) $(A+i\beta)\hat{v} + \hat{\lambda}\xi = 0,$

(15) $(A_0+i\beta)\hat{w} + \hat{\lambda}\rho = 0,$

(16) $(A_0+i\beta)\hat{s} + \hat{\lambda}\zeta = 0,$

(17) $\dfrac{\hat{\lambda}}{\tau^*} = (-4+\kappa_1 d_2)\hat{u}(\eta^*) + \kappa_1 d_1\hat{v}(\eta^*) - (\kappa_2 d_3 + 4a'(\alpha(\eta) - s_0)\hat{w}(\eta^*) - \kappa_2 d_4\hat{s}(\eta^*)$

On the other hand, since $E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*) = 0$, ϕ, ξ, ρ and ζ are solutions to the equations, we have:

$$(A + i\,\beta)\phi = -\mu\,\delta(\eta - r), \tag{18}$$

$$(A + i\beta)\xi = \frac{\mu}{\eta^*}\delta(\eta - r), \tag{19}$$

$$(A_0 + i\,\beta)\rho = -\delta(\eta - r), \tag{20}$$

$$(A_0 + i\,\beta)\zeta = \frac{1}{\eta^*}\delta(\eta - r), \tag{21}$$

$$\begin{aligned} \frac{i\beta}{\tau^*} &= (-4 + \kappa_1 d_2)(\phi(\eta^*) - \mu G(\eta^*, \eta^*) + \gamma'(\eta^*)) \\ &+\kappa_1 d_1(\xi(\eta^*) + \frac{\mu}{\eta^*}\hat{G}(\eta^*, \eta^*) + \hat{\gamma}'(\eta^*)) \\ &-\big(\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0)\big)\big(\rho(\eta^*) - J(\eta^*, \eta^*) + \alpha'(\eta^*))\big) \\ &-\kappa_2 d_4(\zeta(\eta^*) + \frac{1}{\eta^*}\hat{J}(\eta^*, \eta^*) + \hat{\alpha}'(\eta^*)). \end{aligned} \tag{22}$$

Multiplying (14) and (19) by $r^{n-1}\phi$, and (13) and (18) by $r^{n-1}\xi$ and subtracting one from the other, we now obtain

$$\hat{u}(\eta^*) = -\eta^*\,\hat{v}(\eta^*),\ \hat{w}(\eta^*) = -\eta^*\,\hat{s}(\eta^*), \tag{23}$$

$$\phi(\eta^*) = -\eta^*\,\xi(\eta^*),\ \rho(\eta^*) = -\eta^*\,\zeta(\eta^*). \tag{24}$$

Multiplying (18) by $r^{n-1}\overline{\phi}$ we now obtain

$$\beta||r^{\frac{n-1}{2}}\phi||^2 = -\mu(\eta^*)^{n-1}\mathrm{Im}(\phi(\eta^*)). \tag{25}$$

The imaginary part of (22) is given by

$$\begin{aligned} \tfrac{\mu}{\tau^*}(\eta^*)^{n-1} &= (4 - \kappa_1 d_2)||r^{\frac{n-1}{2}}\phi||^2 + \eta^*\kappa_1 d_1||r^{\frac{n-1}{2}}\xi||^2 - \mu\eta^*\kappa_2 d_4||r^{\frac{n-1}{2}}\zeta||^2 \\ &\quad +\mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0))||r^{\frac{n-1}{2}}\rho||^2. \end{aligned} \tag{26}$$

Multiplying (13) by $(-4 + \kappa_1 d_2)r^{n-1}\overline{\phi}$, (14) by $-\eta^*\kappa_1 d_1 r^{n-1}\overline{\xi}$, (15) by $-\mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0))r^{n-1}\overline{\rho}$ and (16) by $\mu\eta^*\kappa_2 d_4 r^{n-1}\overline{\zeta}$ and adding the resultants to each, we now obtain

(27)

$$\int r^{n-1}\big((-4 + \kappa_1 d_2)\hat{u}\overline{\phi} - \eta^*\kappa_1 d_1\hat{v}\overline{\xi} - \mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0))\hat{w}\overline{\rho} + \mu\eta^*\kappa_2 d_4\hat{s}\overline{\zeta}\big) = 0$$

by (26).

Now, multiplying (13) by $(-4+\kappa_1 d_2)\,r^{n-1}\overline{\hat{u}}$, (18) by $-\eta^*\kappa_1 d_1\,r^{n-1}\overline{\hat{v}}$, (15) by $-\mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0))\,r^{n-1}\overline{\hat{w}}$ and (16) by $\mu\eta^*\kappa_2 d_4\,r^{n-1}\overline{\hat{s}}$ and adding the resultants to each, we now obtain the imaginary part

(28)

$$\begin{aligned} 0 &= (-4 + \kappa_1 d_2)||r^{\frac{n-1}{2}}\hat{u}||^2 - \eta^*\kappa_1 d_1||r^{\frac{n-1}{2}}\hat{v}||^2 - \mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0))||r^{\frac{n-1}{2}}\hat{w}||^2 \\ &\quad +\mu\eta^*\kappa_2 d_4||r^{\frac{n-1}{2}}\hat{s}||^2 \\ &= \big(4 - \kappa_1 d_2 + \tfrac{\kappa_1 d_1}{\eta^*}\big)||r^{\frac{n-1}{2}}\hat{u}||^2 + \mu\big(\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0) - \tfrac{\kappa_2 d_4}{\eta^*}\big)||r^{\frac{n-1}{2}}\hat{w}||^2 \end{aligned}$$

since $||r^{\frac{n-1}{2}}\hat{u}||^2 = (\eta^*)^2||r^{\frac{n-1}{2}}\hat{v}||^2$. By assumptions, $4-\kappa_1 d_2 > 0$ and $4a'(\alpha(\eta)-s_0)+\kappa_2 d_3 > \kappa_2 \frac{d_4}{\eta^*}$ we have $\hat{u}=0$ and $\hat{w}=0$ and so, $\hat{v}=0$ and $\hat{s}=0$. By (17), we have $\hat{\lambda}=0$. □

THEOREM 4.3. *Under the same condition as in Theorem 4.2,* $(0,0,0,0,\eta^*,\tau^*)$ *satisfies the transversality condition. Hence, it is a Hopf point for (7).*

Proof. By implicit differentiation of $E(\psi_0(\tau), v_0(\tau), w_0(\tau), s_0(\tau), \lambda(\tau), \tau) = 0$, we find that

$$D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*)(\psi_0'(\tau^*), v_0'(\tau^*), w_0'(\tau^*), s_0'(\tau^*), \lambda'(\tau^*))$$
$$= \begin{pmatrix} -\mu\, G(\cdot,\eta^*)P(\eta^*) \\ \frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)P(\eta^*) \\ -J(\cdot,\eta^*)P(\eta^*) \\ \frac{1}{\eta^*}\,\hat{J}(\cdot,\eta^*)P(\eta^*) \\ P(\eta^*) \end{pmatrix},$$

where $P(\eta^*) = (-4+\kappa_1 d_2)(\psi_0(\eta^*)+\gamma'(\eta^*)) + \kappa_1\, d_1(v_0(\eta^*)+\hat{\gamma}'(\eta^*)) - \kappa_2 d_4(s_0(\eta^*)+\hat{\alpha}'(\eta^*)) - (\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0))(w_0(\eta^*)+\alpha'(\eta^*))$. This means that the functions $\tilde{u} := \psi_0'(\tau^*)$, $\tilde{v} := v_0'(\tau^*)$, $\tilde{w} := w_0'(\tau^*)$, $\tilde{s} := s_0'(\tau^*)$ and $\tilde{\lambda} := \lambda'(\tau^*)$ satisfy the equations

$$\begin{cases} (A+i\beta)\tilde{u} + \tilde{\lambda}\psi_0 + \tau^*\mu G(\cdot,\eta^*)\widetilde{P}(\eta^*) = -\mu G(\cdot,\eta^*)P(\eta^*), \\ (A+i\beta)\tilde{v} + \tilde{\lambda}\xi_0 - \tau^*\frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)\widetilde{P}(\eta^*) = \frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)P(\eta^*) \\ (A_0+i\beta)\tilde{w} + \tilde{\lambda}\rho_0 + \tau^* J(\cdot,\eta^*)\widetilde{P}(\eta^*) = -J(\cdot,\eta^*)P(\eta^*), \\ (A_0+i\beta)\tilde{s} + \tilde{\lambda}\zeta_0 - \tau^*\frac{1}{\eta^*}\,\hat{J}(\cdot,\eta^*)\widetilde{P}(\eta^*) = \frac{1}{\eta^*}\,\hat{J}(\cdot,\eta^*)P(\eta^*) \\ \tilde{\lambda} - \tau^*\widetilde{P}(\eta^*) = P(\eta^*), \end{cases} \tag{29}$$

where $\widetilde{P}(\eta^*) = (-4+\kappa_1 d_2)\tilde{u}(\eta^*) + \kappa_1 d_1 \tilde{v}(\eta^*) - (\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0))\tilde{w}(\eta^*) - \kappa_2 d_4 \tilde{s}(\eta^*)$. By letting $\phi := \psi_0 + \mu G(\cdot,\eta^*)$, $\xi = v_0 - \frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)$, $\rho = w_0 + J(\cdot,\eta^*)$ and $\zeta = s_0 - \hat{J}(\cdot,\eta^*)$ as before, we obtain

$$(A+i\beta)\tilde{u} + \tilde{\lambda}\phi = 0, \tag{30}$$
$$(A+i\beta)\tilde{v} + \tilde{\lambda}\xi = 0, \tag{31}$$
$$(A_0+i\beta)\tilde{w} + \tilde{\lambda}\rho = 0, \tag{32}$$
$$(A_0+i\beta)\tilde{s} + \tilde{\lambda}\zeta = 0, \tag{33}$$
$$\tilde{\lambda} - \tau^*\big((-4+\kappa_1 d_2)\tilde{u}(\eta^*) + \kappa_1 d_1 \tilde{v}(\eta^*) - \kappa_2 d_4 \tilde{s}(\eta^*)\big) \\ -(\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0))\tilde{w}(\eta^*)\big) = \frac{i\beta}{\tau^*}. \tag{34}$$

Multiplying (30) by $(-4+\kappa_1 d_2)r^{n-1}\overline{\phi}$, (31) by $-\eta^*\kappa_1 d_1 r^{n-1}\overline{\xi}$, (32) by $-\mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0))r^{n-1}\overline{\rho}$ and (33) by $\mu\eta^*\kappa_2 d_4 r^{n-1}\overline{\zeta}$ and adding the resultants to each, we now obtain

$$(-4+\kappa_1 d_2)\mu(\eta^*)^{n-1}\tilde{u}(\eta^*) - \kappa_1 d_1 \mu\,(\eta^*)^{n-1}\tilde{v}(\eta^*) - \mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0))(\eta^*)^{n-1}$$
$$\tilde{w}(\eta^*) + \mu\eta^*\kappa_2 d_4(\eta^*)^{n-1}\tilde{s}(\eta^*) + \tilde{\lambda}((-4+\kappa_1 d_2)||r^{\frac{n-1}{2}}\phi||^2 - \eta^*\kappa_1 d_1|r^{\frac{n-1}{2}}|\xi||^2$$
$$-\mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0))||r^{\frac{n-1}{2}}\rho||^2 + \mu\eta^*\kappa_2 d_4||r^{\frac{n-1}{2}}\zeta||^2)$$
$$+2i\beta\int r^{n-1}\big((-4+\kappa_1 d_2)\tilde{u}\overline{\phi} - \eta^*\kappa_1 d_1\tilde{v}\overline{\xi} - \mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0))\tilde{w}\,\overline{\rho}$$
$$+\mu\eta^*\kappa_2 d_4\tilde{s}\,\overline{\zeta}\big) = 0.$$

From (26) and (34), the above equation implies that

(35)
$$\frac{\mu(\eta^*)^{n-1}}{(\tau^*)^2} = 2\int r^{n-1}\Big((-4+\kappa_1 d_2)\tilde{u}\overline{\phi} - \eta^*\kappa_1 d_1\tilde{v}\overline{\xi} - \mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0))\tilde{w}\,\overline{\rho} + \mu\eta^*\kappa_2 d_4\tilde{s}\,\overline{\zeta}\Big)$$

Multiplying (30) by $(-4+\kappa_1 d_2)\overline{\tilde{u}}$, (31) by $-\eta^*\kappa_1 d_1\overline{\tilde{v}}$, (32) by $-\mu(\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0)\overline{\tilde{w}}$ and (33) by $\mu\eta^*\kappa_2 d_4\overline{\tilde{s}}$ and adding the resultants to each, and applying (35), we now obtain

$$\begin{aligned}\tilde{\lambda}\,\tfrac{\mu}{2(\tau^*)^2} &= (4-\kappa_1 d_2)||A^{1/2}r^{\frac{n-1}{2}}\tilde{u}||^2 + \eta^*\kappa_1 d_1||A^{1/2}r^{\frac{n-1}{2}}\tilde{v}||^2\\ &+\mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0))||A_0^{1/2}r^{\frac{n-1}{2}}\tilde{w}||^2 - \mu\eta^*\kappa_2 d_4||A_0^{1/2}r^{\frac{n-1}{2}}\tilde{s}||^2\\ &+i\beta\Big((4-\kappa_1 d_2)||r^{\frac{n-1}{2}}\tilde{u}||^2 + \eta^*\kappa_1 d_1||r^{\frac{n-1}{2}}\tilde{v}||^2\\ &+\mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0))||r^{\frac{n-1}{2}}\tilde{w}||^2 - \mu\eta^*\kappa_2 d_4||r^{\frac{n-1}{2}}\tilde{s}||^2\Big).\end{aligned}$$

The real part of the above is given by

(36)
$$\begin{aligned}\tfrac{\mu}{2(\tau^*)^2}\mathrm{Re}\tilde{\lambda} &= (4-\kappa_1 d_2)||A^{1/2}r^{\frac{n-1}{2}}\tilde{u}||^2 + \eta^*\kappa_1 d_1||A^{1/2}r^{\frac{n-1}{2}}\tilde{v}||^2\\ &+\mu(\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0))||A_0^{1/2}r^{\frac{n-1}{2}}\tilde{w}||^2 - \mu\eta^*\kappa_2 d_4||A_0^{1/2}r^{\frac{n-1}{2}}\tilde{s}||^2.\end{aligned}$$

Now, multiplying (30) by $2i\beta r^{n-1}\overline{\tilde{u}}$ and (31) by $\tilde{r}^{n-1}\lambda\overline{\tilde{u}}$ and subtracting resultants from each other, we now obtain

$$||A^{1/2}r^{\frac{n-1}{2}}\tilde{u}||^2 = (\eta^*)^2||A^{1/2}r^{\frac{n-1}{2}}\tilde{v}||^2 \quad\text{and}\quad ||r^{\frac{n-1}{2}}\tilde{u}||^2 = (\eta^*)^2||r^{\frac{n-1}{2}}\tilde{v}||^2.$$

Thus (36) implies that

$$\tfrac{\mu(\eta^*)^{n-1}}{2(\tau^*)^2}\mathrm{Re}\tilde{\lambda} = \big(4-\kappa_1 d_2+\tfrac{\kappa_1 d_1}{\eta^*}\big)||A^{1/2}r^{\frac{n-1}{2}}\tilde{u}||^2+\mu\big(\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0)-\kappa_2\tfrac{d_4}{\eta^*}\big)||A_0^{1/2}r^{\frac{n-1}{2}}\tilde{w}||^2$$

which is positive since $4-\kappa_1 d_2 > 0$ and $\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0) > \kappa\frac{d_4}{\eta^*}$. We have $\mathrm{Re}\lambda'(\tau^*) > 0$ for $\beta > 0$, and thus, by the Hopf-bifurcation theorem in [2], there exists a family of periodic solutions which bifurcates from the stationary solution as τ passes τ^*. □

We now establish the existence of a unique $\tau^* > 0$ for which the point $(0,0,0,0,\eta^*,\tau^*)$ corresponds to a Hopf bifurcation point; thus τ^* is the origin of a branch of nontrivial periodic orbits.

LEMMA 4.4. *Suppose that* $4-\frac{\kappa_1 d_1}{\eta^*} > 0$ *and* $\kappa_2 d_3 + 4a'(\alpha(\eta^*)-s_0) > \kappa\frac{d_4}{\eta^*}$. *Let* G_β *and* $\hat{G}_\beta$ *be Green functions of the differential operator* $A+i\beta$ *satisfying (15) and (19), respectively. Then, the expression* $(-4+\kappa_1 d_2)\,\mathrm{Re}\,(G_\beta(\eta^*,\eta^*)) - \frac{\kappa_1 d_1}{\eta^*}\mathrm{Re}\,(\hat{G}_\beta(\eta^*,\eta^*))$

and $-(\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0))\mathrm{Re}(J_\beta(\eta^*,\eta^*))+\frac{d_4}{\eta^*}\mathrm{Re}(\hat{J}_\beta(\eta^*,\eta^*))$ *are strictly increasing in* $\beta>0$ *with*

$$\mathrm{Re}\,G_0(\eta^*,\eta^*)=G(\eta^*,\eta^*),\quad \lim_{\beta\to\infty}\mathrm{Re}\,G_\beta(\eta^*,\eta^*)=0\,.$$

Moreover,

$$\begin{aligned}&(-4+\kappa_1 d_2)\,\mathrm{Im}G_\beta(\eta^*,\eta^*)-\tfrac{\kappa_1 d_1}{\eta^*}\,\mathrm{Im}\,\hat{G}_\beta(\eta^*,\eta^*)\\&-(\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0))\mathrm{Im}(J_\beta(\eta^*,\eta^*))+\kappa_2 d_4\mathrm{Im}(\tfrac{1}{\eta^*}\hat{J}_\beta(\eta^*,\eta^*))\end{aligned}$$

is positive for $\beta>0$.

Proof. First, we have $(A+i\beta)^{-1}=(A-i\beta)(A^2+\beta^2)^{-1}$, so if $L(\beta):=\mathrm{Re}\,(A+i\beta)^{-1}$, then $L(\beta)=A(A^2+\beta^2)^{-1}$. Moreover, $L(\beta)\longrightarrow A^{-1}$ as $\beta\to 0$ and $L(\beta)\longrightarrow 0$ as $\beta\to\infty$, which results in the corresponding limiting behavior for $\mathrm{Re}\,\big(G_\beta(\eta^*,\eta^*)\big)$.

Now to show that $\beta\mapsto((-4+\kappa_1 d_2)\,\mathrm{Re}\,(G_\beta(\eta^*,\eta^*))-\frac{\kappa_1 d_1}{\eta^*}\mathrm{Re}\,(\hat{G}_\beta(\eta^*,\eta^*))$ is strictly decreasing, define $h(\beta)(r):=(-4+\kappa_1 d_2)\,G_\beta(r,\eta^*)-\frac{\kappa_1 d_1}{\eta^*}\,\hat{G}_\beta(r,\eta^*)-(-4+\kappa_1 d_2)\,G(r,\eta^*)+\frac{\kappa_1 d_1}{\eta^*}\,\hat{G}(r,\eta^*)$. Then (in the weak sense initially)

$$(A+i\beta)h(\beta)=i\beta(4-\kappa_1 d_2)\,G(\cdot,\eta^*)+i\beta\tfrac{\kappa_1 d_1}{\eta^*}\hat{G}(\eta^*,\eta^*).\tag{37}$$

As a result $h(\beta)\in D(A)_{\mathbf{C}}$ and $h:\mathbb{R}^+\to D(A)_{\mathbf{C}}$ is differentiable with $ih(\beta)+(A+i\beta)h'(\beta)=-i\,G(\cdot,\eta^*)$, therefore

$$(A+i\beta)h'(\beta)=i\big((4-\kappa_1 d_2)\,G_\beta(\cdot,\eta^*)-\tfrac{\kappa_1 d_1}{\eta^*}\hat{G}_\beta(\cdot,\eta^*)\big).$$

Thus, we get

$$\begin{aligned}&|Ar^{\frac{n-1}{2}}h'(\beta)||^2-\beta^2||r^{\frac{n-1}{2}}h'(\beta)||^2\,dr+2i\beta||A^{1/2}r^{\frac{n-1}{2}}h'(\beta)||^2\\&=i\,((4-\kappa_1 d_2\,+\tfrac{\kappa_1 d_1}{\eta^*})\overline{h'(\beta)(\eta^*)}.\end{aligned}\tag{38}$$

From (38) it follows that

$$(4-\kappa_1 d_2+\tfrac{\kappa_1 d_1}{\eta^*})\mathrm{Re}\,(h'(\beta)(\eta^*))=2\beta||A^{1/2}r^{\frac{n-1}{2}}h'(\beta)||^2$$

and thus $\mathrm{Re}\,(h'(\beta)(\eta^*))>0$ if $4-\kappa_1 d_2>0$. In order to show $(-4+\kappa_1 d_2)\,\mathrm{Im}(G_\beta(\eta^*,\eta^*))-\frac{\kappa_1 d_1}{\eta^*}\,\mathrm{Im}(\,\hat{G}_\beta(\eta^*,\eta^*))>0$ for $\beta>0$, we multiply (37) by $r^{n-1}\overline{h(\beta)(r)}$ and integrate the resulting equation, then we have:

$$-i\,\beta((-4+\kappa_1 d_2)-\tfrac{\kappa_1 d_1}{\eta^*})\overline{h(\beta)(\eta^*)}=||Ah(\beta)|^2+i\beta||A^{1/2}h(\beta)||^2,$$

which implies that $\beta((4-\kappa_1 d_2)+\frac{\kappa_1 d_1}{\eta^*})\mathrm{Im}h(\beta)(\eta^*)=\int|Ah(\beta)|^2>0$. Since $(4-\kappa_1 d_2)>0$, we have $\mathrm{Im}h(\beta)(\eta^*)>0$ for $\beta>0$.

We define $k(\beta)(r):=-(\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0))\,J_\beta(r,\eta^*)+\frac{\kappa_2 d_4}{\eta^*}\,\hat{J}_\beta(r,\eta^*)+(\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0)\,J(r,\eta^*)-\frac{\kappa_2 d_4}{\eta^*}\,\hat{J}(r,\eta^*)$. Then

$$\begin{cases}((\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0)-\kappa_2\frac{d_4}{\eta^*})\mathrm{Re}\,(k'(\beta)(\eta^*))=2\beta||A_0^{1/2}k'(\beta)||^2\\ \beta((\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0)-\kappa_2\frac{d_4}{\eta^*})\mathrm{Im}k(\beta)(\eta^*)=||A_0k(\beta)||^2.\end{cases}$$

If $\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0)>\kappa_2\frac{d_4}{\eta^*}$ we have $\mathrm{Re}k'(\beta)(\eta^*)>0$ and $\mathrm{Im}k(\beta)(\eta^*)>0$ for $\beta>0$. Thus, $(-4+\kappa_1 d_2)\,\mathrm{Im}G_\beta(\eta^*,\eta^*)-\frac{\kappa_1 d_1}{\eta^*}\,\mathrm{Im}\,\hat{G}_\beta(\eta^*,\eta^*)-(\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0))\mathrm{Im}(J_\beta(\eta^*,\eta^*))+\kappa_2\frac{d_4}{\eta^*}\mathrm{Im}(\hat{J}_\beta(\eta^*,\eta^*))>0$ for $\beta>0$ if $4-\kappa_1 d_2>0$ and $\kappa_2 d_3+$

$4a'(\alpha(\eta^*) - s_0) > \frac{\kappa_2 d_4}{\eta^*}$. Similarly, $(-4+\kappa_1 d_2)\,\mathrm{Re}\,(G_\beta(\eta^*,\eta^*)) - \frac{\kappa_1 d_1}{\eta^*}\mathrm{Re}\,(\hat{G}_\beta(\eta^*,\eta^*)) - \kappa_2 d_3 \mathrm{Re}(J_\beta(\eta^*,\eta^*)) + \frac{\kappa_2 d_4}{\eta^*}\mathrm{Re}(\hat{J}_\beta(\eta^*,\eta^*))$ is a strictly increasing function of $\beta > 0$ if $4 - \kappa_1 d_2 > 0$ and $\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0) > \frac{\kappa_2 d_4}{\eta^*}$.

□

THEOREM 4.5. *Under the same condition as in Theorem 4.2, for a unique critical point $\tau^* > 0$, there exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (10) with $\beta > 0$.*

Proof. To complete the argument, it suffices to show that the function $(u,v,w,s,\beta,\tau) \mapsto E(u,v,w,s,i\beta,\tau)$ possesses a unique root for which $\beta > 0$ and $\tau > 0$. This is equivalent to solving the system of equations given in (10) with $\lambda = i\beta$, $u = V + \mu G(\cdot,\eta^*)$, $v = p - \frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*), w = W + J(\cdot,\eta^*)$ and $s = q - \frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)$,

$$\begin{cases} (A+i\beta)V = -\mu\,\delta(\eta^*-r), \\ (A+i\beta)p = \frac{\mu}{\eta^*}\,\delta(\eta^*-r), \\ (A_0+i\beta)W = -\delta(\eta^*-r), \\ (A_0+i\beta)q = \frac{1}{\eta^*}\delta(\eta^*-r), \\ \frac{i\beta}{\tau^*} = (-4+\kappa_1 d_2)(V(\eta^*) - \mu G(\eta^*,\eta^*) + \gamma'(\eta^*)) + \kappa_1 d_1(p(\eta^*) + \frac{\mu}{\eta^*}\hat{G}(\eta^*,\eta^*) + \hat{\gamma}'(\eta^*)) \\ \quad -(\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0))(W(\eta^*) - J(\eta^*,\eta^*) + \alpha'(\eta^*)) \\ \quad +\kappa_2 d_4(q(\eta^*) + \frac{1}{\eta^*}\hat{J}(\eta^*,\eta^*) + \hat{\alpha}'(\eta^*)). \end{cases}$$

The real and imaginary parts of the above equation are given by

$$\begin{aligned} \frac{\beta}{\tau^*} &= (-4+\kappa_1 d_2)\mathrm{Im}(\mu G_\beta(\eta^*,\eta^*)) - \frac{\mu\kappa_1 d_1}{\eta^*}\mathrm{Im}(\hat{G}_\beta(\eta^*,\eta^*)) \\ &\quad -(\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0))\mathrm{Im}(J_\beta(\eta^*,\eta^*)) + \frac{\kappa_2 d_4}{\eta^*}\mathrm{Im}(-\hat{J}_\beta(\eta^*,\eta^*)) \end{aligned} \tag{39}$$

and

$$\begin{aligned} 0 = \;& (-4+\kappa_1 d_2)(\mathrm{Re}(\mu G_\beta(\eta^*,\eta^*)) - \mu G(\eta^*,\eta^*) + \gamma'(\eta^*)) \\ & +\kappa_1 d_1\big(\mathrm{Re}(-\tfrac{\mu}{\eta^*}\hat{G}_\beta(\eta^*,\eta^*)) + \tfrac{\mu}{\eta^*}\hat{G}(\eta^*,\eta^*) + \hat{\gamma}'(\eta^*)\big) \\ & -(\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0))(\mathrm{Re}(J_\beta(\eta^*,\eta^*)) - J(\eta^*,\eta^*) + \alpha'(\eta^*)) \\ & +\kappa_2 d_4(\mathrm{Re}(-\tfrac{1}{\eta^*}\hat{J}_\beta(\eta^*,\eta^*)) + \tfrac{1}{\eta^*}\hat{J}(\eta^*,\eta^*) + \hat{\alpha}'(\eta^*)). \end{aligned}$$

There is a critical point τ^* provided the existence of β since the right hand side of (39) is positive by Lemma 4.4.

We now define

$$\begin{aligned} T(\beta) &= (-4+\kappa_1 d_2)(\mathrm{Re}(\mu G_\beta(\eta^*,\eta^*)) - \mu G(\eta^*,\eta^*) + \gamma'(\eta^*)) + \kappa_1 d_1\big(\mathrm{Re}(-\tfrac{\mu}{\eta^*}\hat{G}_\beta(\eta^*,\eta^*)) \\ &\quad +\tfrac{\mu}{\eta^*}\hat{G}(\eta^*,\eta^*) + \hat{\gamma}'(\eta^*)\big) + \kappa_2 d_4(\mathrm{Re}(-\tfrac{1}{\eta^*}\hat{J}_\beta(\eta^*,\eta^*)) + \tfrac{1}{\eta^*}\hat{J}(\eta^*,\eta^*) + \hat{\alpha}'(\eta^*)) \\ &\quad -(\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0))(\mathrm{Re}(J_\beta(\eta^*,\eta^*)) - J(\eta^*,\eta^*) + \alpha'(\eta^*)). \end{aligned}$$

Using Lemma 4.4, we have $T'(\beta) > 0$ for $\beta > 0$ and

$$T(0) = (-4+\kappa_1 d_2)\gamma'(\eta^*) + \kappa_1 d_1 \hat{\gamma}'(\eta^*) - (\kappa_2 d_3 + 4a'(\alpha(\eta^*) - s_0))\alpha'(\eta^*) + \kappa_2 d_4 \hat{\alpha}'(\eta^*)$$

which is negative by Theorem 3.1. Moreover,

$$\begin{aligned}\lim_{\beta\to\infty} T(\beta) &= (-4+\kappa_1 d_2)(-\mu G(\eta^*,\eta^*)+\gamma'(\eta^*))+\tfrac{\kappa_1 d_1}{\eta^*}\hat{\gamma}(\eta^*)+\kappa_1 d_1\hat{\gamma}'(\eta^*)\\ &\quad -(\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0)-\kappa_2\tfrac{d_4}{\eta^*})(-J(\eta^*,\eta^*)+\alpha'(\eta^*))+\kappa_2 d_4\hat{\alpha}'(\eta^*)\\ &= \big(\sqrt{1+\mu}\,(4-\kappa_1 d_2)+\tfrac{\kappa_1 d_1}{\eta^*}\big)\,\hat{\gamma}(\eta^*)+\kappa_1 d_1\hat{\gamma}'(\eta^*)\\ &\quad +\big(\kappa_2(d_3-\tfrac{d_4}{\eta^*})+4a'(\alpha(\eta^*)-s_0)\big)\hat{\alpha}(\eta^*))+\kappa_2 d_4\hat{\alpha}'(\eta^*)\end{aligned}$$

since $\mu G(\eta^*,\eta^*)-\gamma'(\eta^*)=\sqrt{1+\mu}\hat{\gamma}(\eta^*)$ and $\alpha'(\eta^*)-J(\eta^*,\eta^*)=\hat{\alpha}(\eta^*)$. Hence $\lim_{\beta\to\infty} T(\beta)>0$ under the assumptions $4+\frac{\kappa_1 d_1}{\eta^*}>0$ and $\kappa_2 d_3+4a'(\alpha(\eta^*)-s_0)>\kappa\frac{d_4}{\eta^*}$. □

The following theorem summarizes the results above. We now summarize the foregoing developments in the form of the following theorem.

Theorem 4.6. *Suppose that* $ks_0>a_0$, $\frac{\mu}{1+\mu}+\tanh(k(\frac{1}{2}-s_0)+a_0)>0$, $\mu\kappa_1\chi'(\gamma(\eta))<\kappa_2\,\xi'(\alpha(\eta))$, $C'(\gamma(\eta);a(\alpha(\eta)-s_0))+\kappa_1\chi''(\gamma(\eta))\,\hat{\gamma}(\eta)<0$ *and* $C'(\gamma(\eta);a(\alpha(\eta)-s_0))a'(\alpha(\eta)-s_0)-\kappa_2\xi''(\alpha(\eta))<0$ *for all* $\eta>0$. *Then (7) and (5) have at least one stationary solution* (u^*,v^*,w^*,s^*,η^*) *where* $u^*=v^*=w^*=s^*=0$ *and* (V^*,p^*,W^*,q^*,η^*) *for all* τ, *respectively. Moreover, assume that* $4>\kappa_1\chi''(\gamma(\eta^*))\,\hat{\gamma}(\eta^*)$ *and* $4a'(\alpha(\eta^*)-s_0)+\kappa_2\xi''(\alpha(\eta^*))>\frac{\kappa_2}{\eta^*}\xi'(\alpha(\eta^*))$. *Then there exists a unique* τ^* *such that the linearization* $-\widetilde{A}+\tau^*B$ *has a purely imaginary pair of eigenvalues. The point* $(0,0,0,0,\eta^*,\tau^*)$ *is then a Hopf point for (7), and there exists a* C^0*-curve of nontrivial periodic orbits for (7) and (5), bifurcating from* $(0,0,0,0,\eta^*,\tau^*)$ *and* $(V^*,p^*,W^*,q^*,\eta^*,\tau^*)$, *respectively.*

We proved the existence of stationary solutions and Hopf bifurcation under the above assumptions. Specially, the condition $\mu\kappa_1\chi'(\gamma(\eta))<\kappa_2\,\xi'(\alpha(\eta))$ is fundamentally governed by the interplay between attraction and repulsion.

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