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ITERATED WEIGHTED PROJECTIVE SPACE FIBRATIONS AND TORIC ORBIFOLDS

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Abstract. We generalize classical generalized Bott towers to orbifolds using weighted projectivizations of line bundles, which we call *weighted projective towers*. From the perspective of toric topology, such a space can be constructed from a product of simplices with a rational characteristic function on it. However, such a construction gives an orbifold fibration in general. Our main theorem provides explicit criteria for when a toric orbifold over a product of simplices admits a structure of a weighted projective tower.

1. Introduction

Taking projectivization of a vector bundle over a topological space induces a fibration having projective spaces as its fibers. If the base space is a toric manifold and the vector bundle is given by a Whitney sum of line bundles, then the resulting projectivization is again a toric manifold. For instance, if the base space X is a point, then the projectivization of the Whitney sum of $(n+1)$-many complex line bundles on X becomes the complex projective space $\mathbb{C}\mathrm{P}^n$. One can also perform the same construction with $X = \mathbb{C}\mathrm{P}^n$ and the Whitney sum of $(m+1)$-many complex line bundles on $\mathbb{C}\mathrm{P}^n$, which defines a $\mathbb{C}\mathrm{P}^m$-fibration over $\mathbb{C}\mathrm{P}^n$. By iterating this construction, we have a sequence of fibrations

$$B_k \xrightarrow{\pi_k} B_{k-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{pt\},$$

such that B_0 is a point and B_i is the projectivization of the Whitney sum of (n_i+1)-many complex line bundles over B_{i-1}. We call the above sequence a *generalized Bott tower* and each B_i a *generalized Bott manifold* [11, 12].

In this paper, we extend the construction of generalized Bott towers into the realm of orbifolds, exploring two complementary approaches. Firstly, we generalize the construction by considering *weighted projectivizations* of Whitney sums of line bundles. This modification naturally leads to sequences of iterated fibrations whose fibers are weighted projective spaces, which yields what we define as a *weighted projective tower* (Definition 2.2). We refer to [3, Chapter 3] and [7, Section 6] for the weighted projectivization of a vector bundle, which is simply called a weighed projective bundle. In short, the object of Definition 2.2 can be understood as a sequence of weighted projective bundles whose initial term is a point.

Secondly, we examine toric topological constructions of these spaces. In general, a *toric orbifold* can be constructed from a combinatorial source, called a characteristic

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pair (P, λ), consisting of a simple polytope P and a rational characteristic function $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^{\dim P}$; see (4) for the condition of λ. Here the polytope P determines a smooth manifold, called a *moment-angle manifold* $\mathcal{Z}_P$ [9, Chapter 4], and λ determines an almost free torus action on $\mathcal{Z}_P$. The toric orbifold $X(P, \lambda)$ is then defined to be the quotient of $\mathcal{Z}_P$ by this almost free torus action. We refer to [1, Chapter 1] for the general theory of orbifolds and [18] for the axiomatic and constructive definition of a toric orbifold. If the characteristic function λ satisfies a stronger condition: linearly independency is replaced by an integral basis, then λ defines a free action so that the resulting space is a smooth manifold.

For the case of generalized Bott towers, the corresponding moment-angle manifold is a product of odd-dimensional spheres and the characteristic function determines a free torus action that can be described as a triangular vector matrix [11, Section 3]. In this paper, we consider a similar vector matrix which might come from a rational characteristic function, hence it determines an almost free torus action on a product of odd dimensional spheres. It turns out that the corresponding toric orbifold has a structure of iterated orbifold fibrations, namely fibers have additional quotient by the local group of each orbifold chart; see Subsection 4.1, which is a subtle distinction between orbifold fibration and ordinary fibration.

The main contribution of this paper, stated in Theorem 4.4, provides a necessary and sufficient condition for a toric orbifold over a product of simplices associated with a vector matrix (corresponding to a characteristic function) to admit the structure of a weighted projective tower. We achieve this result by bridging the bundle theoretic weighted projectivizations and toric topological constructions.

2. Weighted projective bundles

Given a primitive vector $\chi = (\chi_0, \dots, \chi_n) \in \mathbb{Z}^n$ consisting of natural numbers, we consider S^1-action on an odd dimensional sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ defined by

$$t \circ (z_0, \dots, z_n) = (t^{\chi_0} z_0, \dots, t^{\chi_n} z_n) \tag{1}$$

for $t \in S^1$ and $(z_0, \dots, z_n) \in S^{2n+1}$. The quotient space S^{2n+1}/S^1 for the action (1), which we denote by $\mathbb{C}\mathrm{P}^n_\chi$, is called the *weighted projective space* with weight χ. We shall write a point of $\mathbb{C}\mathrm{P}^n_\chi$ as $[z_0, \dots, z_n]$, the equivalent class of $(z_0, \dots, z_n)$ for the action (1). Following the result of [6, Theorem 1.1], all weighted projective spaces of complex dimension 1 are homeomorphic to the ordinary projective space $\mathbb{C}\mathrm{P}^1$. Hence, throughout the paper, we always assume that $n \geq 2$.

For a complex vector bundle E of rank $n+1$ over a topological space X, one can define a *weighted projective bundle* $\mathbb{P}(E; \chi)$ over X with fiber $\mathbb{C}\mathrm{P}^n_\chi$ by taking fiberwise weighted projectivization with respect to the weight χ. When $\chi = (1, \dots, 1)$, the associated weighted projective bundle is the usual projectivization of a vector bundle.

To define the main object of this paper, we begin with taking the base space X as a weighted projective space $\mathbb{C}\mathrm{P}^n_\chi$ with weight $\chi = (\chi_0, \dots, \chi_n) \in \mathbb{N}^{n+1}$ and E as a Whitney sum of line bundles over $\mathbb{C}\mathrm{P}^n_\chi$. To discuss this space more rigorously, we first briefly review from [3, Section 3] the description of line bundles over a weighted projective space.

Consider the following continuous map

$$\phi\colon \mathbb{C}\mathrm{P}^n_\chi \to \mathbb{C}\mathrm{P}^n$$

defined by $\phi([z_0, \dots, z_n]) = [z_0^{\eta_0}, \dots, z_n^{\eta_n}]$, where $\eta_i = \operatorname{lcm}\{\chi_0, \dots, \chi_n\}/\chi_i$. Then, one can define the *canonical line bundle* over $\mathbb{C}\mathrm{P}^n_\chi$ as the pull-back $\mathcal{O}_\chi(1) := \phi^*\mathcal{O}(1)$ of the canonical line bundle $\mathcal{O}(1)$ over $\mathbb{C}\mathrm{P}^n$ via the map ϕ. We note that the canonical line bundle $\mathcal{O}_\chi(1)$ over $\mathbb{C}\mathrm{P}^n_\chi$ can also be represented by the following line bundle

$$S^{2n+1} \times_{S^1} \mathbb{C} \to \mathbb{C}\mathrm{P}^n_\chi, \tag{2}$$

with S^1-action on $S^{2n+1} \times \mathbb{C}$ given by $t \cdot (z_0, \dots, z_n, w) = (t^{\chi_0} z_0, \dots, t^{\chi_n} z_n, t^\ell w)$, where $\ell = \operatorname{lcm}\{\chi_0, \dots, \chi_n\}$.

Indeed, the canonical line bundle $\mathcal{O}(1)$ over $\mathbb{C}\mathrm{P}^n$ can be represented by

$$\mathcal{O}(1) = S^{2n+1} \times_{S^1} \mathbb{C} \to \mathbb{C}\mathrm{P}^n$$

with S^1-action on $S^{2n+1} \times \mathbb{C}$ given by $t \cdot (z_0, \dots, z_n, w) = (tz_0, \dots, tz_n, tw)$. Therefore, writing $\mathcal{L}_\chi$ as the total space of the bundle (2), one can see that there is an isomorphism

$$\Phi \colon \mathcal{L}_\chi \to \mathcal{O}_\chi(1) = \mathbb{C}\mathrm{P}^n_\chi \times_{\mathbb{C}\mathrm{P}^n} \mathcal{O}(1)$$

defined by

$$\Phi([z_0, \dots, z_n, w]) = \big([z_0, \dots, z_n], [z_0^{\eta_0}, \dots, z_n^{\eta_n}, w^\ell]\big)$$

with $\ell := \operatorname{lcm}\{\chi_0, \dots, \chi_n\}$ and $\eta_i := \ell/\chi_i$ as above. The readers are referred to [3, Section 1-(c)] for more details.

Proposition 2.1. *Let $\mathcal{O}_\chi(k) := S^{2n+1} \times_{S^1} \mathbb{C}$ be the quotient space with respect to the S^1-action on $S^{2n+1} \times \mathbb{C}$ given by*

$$t \cdot (z_0, \dots, z_n, w) = (t^{\chi_0} z_0, \dots, t^{\chi_n} z_n, t^k w).$$

The projection $\pi_k \colon \mathcal{O}_\chi(k) \to \mathbb{C}\mathrm{P}^n_\chi$ is a line bundle over $\mathbb{C}\mathrm{P}^n_\chi$ if and only if k is a multiple of $\ell = \operatorname{lcm}\{\chi_0, \dots, \chi_n\}$. In this case, $\mathcal{O}_\chi(k) \cong \phi^*\mathcal{O}(1)^{\otimes k/\ell}$.

Proof. For each $\mathbf{z} = [z_0, \dots, z_n] \in \mathbb{C}\mathrm{P}^n_\chi$, we write $I_\mathbf{z} := \{i \in \{0, \dots, n\} \mid z_i \neq 0\}$. Then, the local group $G_\mathbf{z}$ of an orbifold chart around $\mathbf{z}$ is the cyclic group of order $\gcd\{\chi_i \mid i \in I_\mathbf{z}\}$. Therefore the fiber $\pi_k(\mathbf{z}) = \mathbb{C}/G_\mathbf{z}$ is a genuine vector space if and only if k is a multiple of $\gcd\{\chi_i \mid i \in I_\mathbf{z}\}$. Hence, the first assertion follows as this condition must hold for all points in $\mathbb{C}\mathrm{P}^n_\chi$.

As for the second claim, consider the map

$$\Phi_k \colon \mathcal{O}_\chi(k) \to \phi^*\left(\mathcal{O}(1)^{k/\ell}\right)$$

defined by $\Phi_k([z_0; \dots; z_n, w]) = ([z_0, \dots, z_n], [z_0^{\eta_0}, \dots, z_n^{\eta_n}, w])$. It is well-defined because

$$\begin{aligned}
\Phi_k([t^{\chi_0} z_0, \dots, t^{\chi_n} z_n, t^k w]) &= \big([t^{\chi_0} z_0, \dots, t^{\chi_n} z_n], [t^{\chi_0\eta_0} z_0^{\eta_0}, \dots, t^{\chi_n\eta_n} z_n^{\eta_n}, t^k w]\big) \\
&= \big([t^{\chi_0} z_0, \dots, t^{\chi_n} z_n], [t^\ell z_0^{\eta_0}, \dots, t^\ell z_n^{\eta_n}, (t^\ell)^{k/\ell} w]\big) \\
&= ([z_0, \dots, z_n], [z_0^{\eta_0}, \dots, z_n^{\eta_n}, w]) \\
&= \Phi_k([z_0, \dots, z_n, w]).
\end{aligned}$$

By the definition of a pull-back bundle, any element of $\phi^*\left(\mathcal{O}(1)^{k/\ell}\right)$ can be represented by $([z_0, \dots, z_n], [z_0^{\eta_0}, \dots, z_n^{\eta_n}, w])$. Hence the inverse of Φ_k^{-1} is just the map sending an element $([z_0, \dots, z_n], [z_0^{\eta_0}, \dots, z_n^{\eta_n}, w]) \in \phi^*\left(\mathcal{O}(1)^{k/\ell}\right)$ to $[z_0, \dots, z_n, w] \in \mathcal{O}_\chi(k)$. □

Now, we introduce our main object, an iterated weighted projective fibration.

Definition 2.2. A *weighted projective tower* of height k is a sequence of fibrations

$$\text{wPT}(k) \xrightarrow{\pi_k} \text{wPT}(k-1) \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} \text{wPT}(1) \xrightarrow{\pi_1} \text{wPT}(0) = \{pt\}, \tag{3}$$

where

$$\text{wPT}(i) := \mathbb{P}(E_0 \oplus \cdots \oplus E_{n_i}; \chi(i)_0, \ldots, \chi(i)_{n_i})$$

is the weighted projectivization of the Whitney sum of complex line bundles $E_0, \ldots, E_{n_i}$ over $\text{wPT}(i-1)$ with respect to the weight $\chi(i) := (\chi(i)_0, \ldots, \chi(i)_{n_i}) \in \mathbb{Z}^{n_i+1}$.

We note that when $\chi(i) = (1, \ldots, 1) \in \mathbb{N}^{n_i+1}$ for all i, the associated weighted projective tower is a generalized Bott tower [11].

Remark 2.3. It is well-known from [17] that the integral cohomology of a weighted projective space is concentrated in even degrees and torsion free. Hence, iterated applications of Leray–Serre spectral sequences conclude that the integral cohomology of $\text{wPT}(i)$ is also concentrated in even degrees and torsion free.

A weighted projective space $\mathbb{C}\text{P}^{n_1}_{\chi(1)}$ is equipped with a well-behaved action of the n_1-dimensional compact torus T^{n_1} which lift to the total space of a Whitney sum of line bundles. Also, each fiber $\mathbb{C}\text{P}^{n_2}_{\chi(2)}$ of the fibration

$$\pi_2\colon \text{wPT}(2) \to \text{wPT}(1) = \mathbb{C}\text{P}^{n_1}_{\chi(1)}$$

is equipped with a T^{n_2}-action. Therefore, it follows that $\text{wPT}(2)$ is a complex (n_1+n_2)-dimensional orbifold equipped with an action of the torus $T^{n_1+n_2}$. Inductively, one can conclude that $\text{wPT}(k)$ is an orbifold of complex dimension $\sum_{i=1}^k n_i$ equipped with an action of the compact torus of real dimension $\sum_{i=1}^k n_i$.

In the next section, we examine the space $\text{wPT}(i)$ from the perspective of toric topology. For general background on toric topology, we refer the reader to [9].

3. Toric orbifolds

We begin with summarizing the definition of an *orbifold*, which follows from [20] and [1]. Let X be a paracompact Hausdorff topological space. An n-dimensional *orbifold chart* on X is a triple $(V, G, \phi\colon V \to U)$, where

- V is an open subset of $\mathbb{R}^n$;
- G is a finite subgroup of the orthogonal group $O(n)$ acting smoothly and effectively on V;
- $\phi\colon V \to U$ is a G-equivariant map for some open subset U of X and G acts trivially on U, which induces a homeomorphism $\tilde{\phi}\colon V/G \to U \subseteq X$.

In particular, given a point $p \in X$, if (V, G, ϕ) is an orbifold chart such that $\pi_p^{-1}(p)$ is a single point, then (V, G_p, ϕ_p) is called an *orbifold chart around* p and G_p is called the *local group* at p.

Two orbifold charts $(V_1, G_1, \phi_1\colon V_1 \to U_1)$ and $(V_2, G_2, \phi_2\colon V_2 \to U_2)$ with $U_1 \cap U_2 \neq \emptyset$ are called *locally compatible*, if for each $q \in U_1 \cap U_2$, there exists an orbifold chart $(W, G_q, \phi_q\colon W \to U_{12} \subset U_1 \cap U_2)$ around q with smooth embeddings $\iota_1\colon W \hookrightarrow V_1$ and $\iota_2\colon W \hookrightarrow V_2$ such that $\phi_1 \circ \iota_1 = \phi_q$ and $\phi_2 \circ \iota_2 = \phi_q$. Finally, a paracompact Hausdorff space X is called an *n-dimensional orbifold* if it is covered by a collection of locally compatible orbifold charts.

Remark 3.1. A collection $\mathcal{U} = \{(V_\alpha, G_\alpha, \phi_\alpha \colon V_\alpha, \to U_\alpha)\}$ of locally compatible orbifold charts is called an *orbifold atlas* on X. Two orbifold atlases $\mathcal{U}_1$ and $\mathcal{U}_2$ are said to be *equivalent* if they have a common refinement. The equivalence class $[\mathcal{U}]$ of $\mathcal{U}$ is called the *orbifold structure* on X; see [20]

The following proposition provides a natural source of constructing orbifolds.

Proposition 3.2. [1, Section 1.1] Let M be a smooth manifold and K a compact Lie group acting smoothly, effectively and almost freely on M. Then, the quotient space M/K is equipped with an orbifold structure.

In toric topology, such a construction of an orbifold arises from a combinatorial source (P, λ) consisting of an n-dimensional simple convex polytope P and a function

$$\lambda \colon \mathcal{F}(P) = \{F_1, \dots, F_m\} \to \mathrm{Hom}(S^1, T^n) \cong \mathbb{Z}^n$$

assigning each facet $F_i \in \mathcal{F}(P)$ of P a primitive vector $\lambda(F_i) \in \mathbb{Z}^n$ such that for each vertex $v \in V(P)$,

$$\{\lambda(F_{i_1}), \dots, \lambda(F_{i_n})\} \text{ is linearly independent, whenever } \bigcap_{k=1}^{n} F_{i_k} = v. \tag{4}$$

Such a function λ is called a *rational characteristic function* on P, see [14, Section 7] and [8, Section 3].

For an n-dimensional simple polytope P, we denote by K_P the $(n-1)$-dimensional simplicial complex dual to ∂P. Let $[m] := \{1, \dots, m\}$ be the vertex set of K_P, which bijectively corresponds to facets $\mathcal{F}(P) = \{F_1, \dots, F_m\}$ of P. For each simplex $\sigma \in K_P$ and its vertex set $V(\sigma) \subset [m]$, we assign a topological space

$$(D^2, S^1)^\sigma := \prod_{i=1}^{m} W_i, \quad \text{where } W_i = \begin{cases} D^2 & \text{if } i \in V(\sigma); \\ S^1 & \text{if } i \notin V(\sigma). \end{cases}$$

This assignment leads us to a functor

$$(D^2, S^1)^{K_P} \colon \textsc{cat}(K_P) \to \textsc{top}$$

from the the category $\textsc{cat}(K_P)$ of simplices of K_P, where the morphisms are inclusions, to the category $\textsc{top}$ of topological spaces. Now, the moment-angle complex $\mathcal{Z}_P$ is

$$\mathcal{Z}_P = \mathrm{colim}(D^2, S^1)^{K_P} = \bigcup_{\sigma \in K_P} (D^2, S^1)^\sigma \subset D^{2m}. \tag{5}$$

Lemma 3.3. [9, Corollary 6.2.5] Let P be an n-dimensional simple polytope with m facets. Then, the moment-angle complex $\mathcal{Z}_P$ corresponding to P is a smooth $(m+n)$-dimensional manifold.

We notice that the action of m-dimensional torus T^m on D^{2m} by the coordinate multiplication induces a T^m-action on $\mathcal{Z}_P$ as each piece $(D^2, S^1)^\sigma$ is invariant under T^m-action. In particular, the orbit space of $\mathcal{Z}_P$ with respect to this T^m-action is combinatorially equivalent to P, see [9, Chapter 4, 6] for more details.

Remark 3.4. In general, given a simplicial complex K on $[m]$ and a sequence of topological pair $\{(X_i, A_i)\}_{i=1}^m$, one can define a *polyhedral functor* [4, 5],

$$(\underline{X}, \underline{A})^K \colon \textsc{cat}(K) \to \textsc{top}$$

by assigning $\sigma \in K$ a topological space $(\underline{X}, \underline{A})^\sigma := \prod_{i=1}^m W_i$, where $W_i = X_i$ if $i \in V(\sigma)$, and $W_i = A_i$ otherwise. The *polyhedral product* of K and $\{(X_i, A_i)\}_{i=1}^m$ is defined by

$$\mathcal{Z}_K(\underline{X}, \underline{A}) = \operatorname{colim}(\underline{X}, \underline{A})^{K_P} = \bigcup_{\sigma \in K_P} (\underline{X}, \underline{A})^\sigma \subset X^m.$$

The moment-angle complex $\mathcal{Z}_P$ defined above (5) is an example of polyhedral products for the case where $(X_i, A_i) = (D^2, S^1)$ for all $i = 1, \dots, m$.

Example 3.5. Let P be the 2-dimensional simplex Δ^2. Then the associated moment-angle complex is

$$\begin{aligned}\mathcal{Z}_P &= (D^2 \times D^2 \times S^1) \cup (D^2 \times S^1 \times D^2) \cup (S^1 \times D^2 \times D^2) \\ &= \partial(D^2 \times D^2 \times D^2) \cong S^5.\end{aligned}$$

In general, when $P = \Delta^n$, one can see that the moment-angle manifold is

$$\mathcal{Z}_P = \partial(\underbrace{D^2 \times \cdots \times D^2}_{(n+1)\text{-times}}) = S^{2n+1}.$$

For a rational characteristic function $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^n$, we consider an $(n \times m)$-matrix Λ whose i-th column is $\lambda(F_i)$. Regarding Λ as a linear map $\Lambda\colon \mathbb{Z}^m \to \mathbb{Z}^n$, we have the following short exact sequence of tori:

$$1 \longrightarrow \ker(\exp\Lambda) \hookrightarrow T^m \xrightarrow{\exp\Lambda} T^n \longrightarrow 1. \tag{6}$$

Lemma 3.6. *Given a linear map $\Lambda\colon \mathbb{Z}^m \to \mathbb{Z}^n$ corresponding to a rational characteristic function λ and the induced short exact sequence* (6)*, the kernel of* $\exp\Lambda$ *is isomorphic to* $T^{m-n} \times \operatorname{coker}\Lambda$.

Proof. Consider the map $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}\colon \mathbb{R}^m \to \mathbb{R}^n$, which we simply denote by $\Lambda_{\mathbb{R}}$. Observe that

$$\ker(\exp\Lambda) = \{(e^{2\pi i x_1}, \dots, e^{2\pi i x_m}) \mid \mathbf{x} := (x_1, \dots, x_m) \in \mathbb{R}^m,\ \Lambda(\mathbf{x}) \in \mathbb{Z}^n\}.$$

Given an element $\mathbf{u} \in \mathbb{Z}^n$, we write

$$T^{m-n}_{\mathbf{u}} = \{(e^{2\pi i x_1}, \dots, e^{2\pi i x_m}) \mid \mathbf{x} := (x_1, \dots, x_m) \in \mathbb{R}^m,\ \Lambda_{\mathbb{R}}(\mathbf{x}) = \mathbf{u}\} \subset \ker(\exp\Lambda).$$

Since $(e^{2\pi i x_1}, \dots, e^{2\pi i x_m}) = (e^{2\pi i(x_1+y_1)}, \dots, e^{2\pi(x_m+y_m)})$ for all $(y_1, \dots, y_m) \in \mathbb{Z}^m$, we have $T^{m-n}_{\mathbf{u}} = T^{m-n}_{\mathbf{v}}$ if and only if $\mathbf{v} = \mathbf{u} + \Lambda(\mathbf{y})$ for some $\mathbf{y} \in \mathbb{Z}^m$, which is equivalent to saying that $\mathbf{u} - \mathbf{v} \in \operatorname{im}\Lambda$. In particular, if $T^{m-n}_{\mathbf{u}} \neq T^{m-n}_{\mathbf{v}}$ as sets, then they must be disjoint due to the linearity. Hence, the result follows. □

The examples below show two different cases for the number of connected components of $\ker(\exp\Lambda)$, which can be detected from the surjectivity of Λ by Lemma 3.6.

Example 3.7. Let $P = \Delta^2$ be a 2 dimensional simplex. We consider the following two linear maps corresponding to rational characteristic functions on P.

1. $\Lambda_1\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ represented by $\begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$. Notice that it is surjective and

$$\ker(\exp\Lambda_1\colon T^3 \to T^2) = \{(t, t^2, t^3) \mid t \in S^1\}.$$

Hence, the resulting quotient space

$$S^5/S^1,\ (z_1, z_2, z_3) \sim (tz_1, t^2 z_2, t^3 z_3)$$

is a weighted projective space $\mathbb{C}P^2_{(1,2,3)}$.

2. $\Lambda_2\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ represented by $\begin{bmatrix} -5 & 1 & 1 \\ -4 & -1 & 2 \end{bmatrix}$. Notice that $\operatorname{coker}\Lambda_2 \cong \mathbb{Z}/3\mathbb{Z}$. Hence, we have $\ker(\exp\Lambda_2) \cong S^1 \times \mathbb{Z}/3\mathbb{Z}$ by Lemma 3.6. Indeed,

$$\ker(\exp\Lambda_2) = \{(t,t^2,t^3)\} \sqcup \{(t,\mu^2 t^2, \mu t^3)\} \sqcup \{(t,\mu t^2,\mu^2 t^3)\},$$

where $t \in S^1$ and μ is the 3rd root of unity. The resulting quotient space $S^5/\ker(\exp\Lambda_2)$ is called a *fake weighed projective space* with weight $(1,2,3)$. We refer to [2, 10, 16] for the definition and details about fake weighted projective spaces.

PROPOSITION 3.8. *Let λ be an rational characteristic function on a simple polytope P, and Λ the associated rational characteristic matrix. Then,* $\ker(\exp\Lambda)$ *acts almost freely on $\mathcal{Z}_P$.*

Proof. The claim can be proved similarly as in [9, Proposition 5.4.6 (a) and Proposition 7.3.13]. For a point $z = (z_0,\dots,z_m) \in \mathcal{Z}_P$, assume that $z_{i_1} = \cdots = z_{i_k} = 0$ for some $\{i_1,\dots,i_k\} \subset [m]$. Then, $z \in (D^2,S^1)^\sigma$, where $\sigma = \{i_1,\dots,i_k\}$. Considering T^m-action on $\mathcal{Z}_P$, the coordinate subtorus

$$T^\sigma := \{(t_1,\dots,t_m) \in T^m \mid t_j = 1 \text{ if } j \notin \{i_1,\dots,i_k\}\} \leq T^m$$

fixes z. Hence, the stabilizer of z with respect to the the action of $\ker(\exp\Lambda)$ on $\mathcal{Z}_P$ is

$$T^\sigma \cap \ker(\exp\Lambda) = \ker(T^\sigma \hookrightarrow T^m \xrightarrow{\exp\Lambda} T^n),$$

which is finite because of the condition (4). □

So far we have defined a smooth manifold $\mathcal{Z}_P$ and almost free action of $\ker(\exp\Lambda)$ on it from a given rational characteristic pair (P,λ). Proposition 3.2 leads us to construct the following orbifold

$$X(P,\lambda) := \mathcal{Z}_P/\exp(\ker\Lambda).$$

Recall that $\mathcal{Z}_P$ equips with natural T^m-action. We call $X(P,\lambda)$ together with the action of residual torus $T^m/\ker(\exp\Lambda) \cong T^n$ the *toric orbifold* from an rational characteristic pair (P,λ).

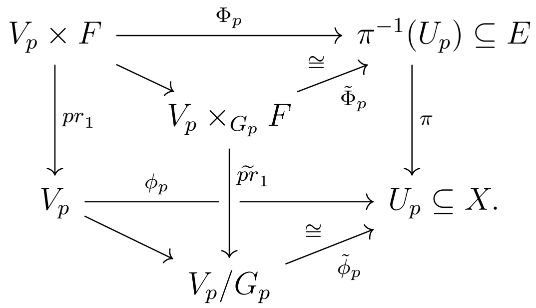
4. Iterated orbifold fibrations

4.1. Orbifold fiber bundles. We first outline the description of an orbifold fibration. One can also refer to [1, Chapter 2] for more rigorous definition using the language of groupoids, but here we follow the classical definition of an orbifold vector bundle (see for instance [19, Section 2]), which can be easily extended to an orbifold fiber bundle in general.

An orbifold fiber bundle is a triple $(F,\ \pi\colon E \to X)$, where X is an orbifold, F and E are topological spaces, and $\pi\colon E \to X$ a continuous surjection together with a *compatible local trivializations* explained as follows. For each point p in X, there exists an orbifold chart $(V_p, G_p, \phi_p\colon V_p \to U_p)$ around p such that

- G_p-action on V_p extends to $V_p \times F$,

- there exists a G_p-equivariant map $\Phi_p\colon V_p \times F \to \pi^{-1}(U_p)$ which induces a homeomorphism $\tilde{\Phi}_p\colon V_p \times_{G_p} F \to \pi^{-1}(U_p)$ such that the following diagram commutes;



We call a triple $(V_p \times F, G_p, \Phi_p\colon V_p \times F \to \pi^{-1}(U_p))$ a *local trivialization* of the orbifold fiber bundle (X, F, E, π) around p. Finally, the *compatibility* of two local trivializations $(V_p \times F, G_p, \Phi_p\colon V_p \times F \to \pi^{-1}(U_p))$ and $(V_q \times F, G_q, \Phi_q\colon V_q \times F \to \pi^{-1}(U_q))$ with $U_q \cap U_q \neq \emptyset$ is similarly defined as the local compatibility of orbifold charts.

4.2. Toric orbifolds over a product of simplices. In this subsection, we consider the product of simplices $P = \prod_{i=1}^{k} \Delta^{n_i}$ and corresponding moment-angle manifold. Observe that P is of dimension $n_1 + \cdots + n_k$ and has $n_1 + \cdots + n_k + k$ facets. We write a facet of P as follows;

$$F_{(r,s)} = \Delta^{n_1} \times \cdots \times \Delta_s^{n_r - 1} \times \cdots \times \Delta^{n_k}, \tag{7}$$

where $\Delta_s^{n_r-1}$ denotes the facet of Δ^{n_r} which is opposite to the vertex indexed by s, for $1 \leq r \leq k$ and $0 \leq s \leq n_r$.

Recall from Example 3.5 that the moment-angle manifold over Δ^n is S^{2n+1}. The following proposition concludes that the corresponding moment-angle manifold is the product of odd spheres:

$$\mathcal{Z}_P = \prod_{i=1}^{k} S^{2n_i+1} \subset (D^2)^{\sum_{i=1}^{k}(n_i+1)}. \tag{8}$$

Proposition 4.1. [9, Proposition 4.1.3] Let P be the product of two simple polytopes P_1 and P_2. Then, $\mathcal{Z}_P \cong \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}$.

We note that the moment-angle manifold $\mathcal{Z}_P$ as in (8) is equipped with the natural action of $T^{\sum_{i=1}^{k}(n_i+1)}$ by coordinate multiplication.

Next, we consider an integer matrix S of size $\sum_{i=1}^{k}(n_i+1) \times k$. For convenience, we write the indices of rows of S by

$$(1,0), \dots, (1,n_1), (2,0) \dots, (2,n_2), \dots, (k,0), \dots, (k,n_k), \tag{9}$$

which matches to the index (7) of the facets of P. We notice that the map

$$\exp S\colon T^k \to T^{\sum_{i=1}^{k}(n_i+1)}$$

yields the T^k-action on $\mathcal{Z}_P$. For notational convenience, we write

$$\mathbf{c}_{ij} := \left[c_{(i,0),j}, \dots c_{(i,n_i),j}\right]^t \in \mathbb{Z}^{n_i+1}$$

and express S as a matrix $(\mathbf{c}_{ij})_{1 \leq i,j \leq k}$ consisting of vector entries, which we simply call a *vector matrix*.

Lemma 4.2. *Let the vector matrix* $S = (\mathbf{c}_{ij})_{1\le i,j\le k}$ *be lower triangular and each diagonal entry* $\mathbf{c}_{ii}$ *is primitive for each* $i = 1,\dots,k$. *Then, there exists a surjective characteristic function*

$$\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^{\sum_{i=1}^k n_i}$$

such that the associated linear map $\Lambda\colon \mathbb{Z}^{\sum_{i=1}^k n_i+1} \to \mathbb{Z}^{\sum_{i=1}^k n_i}$ *satisfies* $\operatorname{im}(\exp S) = \ker(\exp \Lambda)$ *and it acts almost freely on* $\mathcal{Z}_P$ *for* $P = \prod_{i=1}^k \Delta^{n_i}$.

Proof. Notice that for each primitive vector $\mathbf{c}_{ii} = [c_{(i,0),i},\dots,c_{(i,n_i),i}] \in \mathbb{Z}^{n_i+1}$ for $i = 1,\dots,k$, there exists a matrix, say M_i, of size $n_i \times (n_i+1)$ such that $M_i \cdot \mathbf{c}_{ii} = \mathbf{0}$, all column vectors of M_i are primitive and M_i, as a linear map from $\mathbb{Z}^{n_i+1}$ to $\mathbb{Z}^{n_i}$, is surjective. We note that the n_i-dimensional simplex Δ^{n_i} together with M_i yields an rational characteristic pair for the weighted projective space $\mathbb{C}P^{n_i}_{\mathbf{c}_{ii}}$, see [15, Section 2.2] and [13, Section 3.1] for a canonical choice of M_i.

We prove the claim by the induction on k. The discussion above verifies the case where $k = 1$. Writing $S_\ell := (\mathbf{c}_{ij})_{1\le i,j\le \ell}$, a lower triangular vector matrix with primitive diagonal vectors, we assume that there exists a rational characteristic function

$$\lambda_\ell\colon \mathcal{F}(\Delta^{n_1}\times\cdots\times\Delta^{n_\ell}) \to \mathbb{Z}^{\sum_{i=1}^\ell n_i}$$

satisfying the hypothesis for $\ell \le k-1$.

Now, we claim that the following matrix satisfies the assumption:

$$\Lambda_k := \left[\begin{array}{ccc|c} & & & 0\\ & \Lambda_{k-1} & & \vdots \\ & & & 0 \\ \hline B_1 & \cdots & B_{k-1} & M_k \end{array}\right],$$

where M_k is defined as above, Λ_{k-1} is given by the induction hypothesis and $B_1,\dots,B_{k-1}$ are determined by solving the following systems of linear equations inductively:

$$\begin{aligned} B_{k-1}\cdot \mathbf{c}_{k-1,k-1} &= -M_k\cdot \mathbf{c}_{k,k}\\ B_{k-2}\cdot \mathbf{c}_{k-2,k-2} &= -B_{k-1}\cdot \mathbf{c}_{k-1,k-2} - M_k\cdot \mathbf{c}_{k,k-1}\\ &\vdots\\ B_1\cdot \mathbf{c}_{1,1} &= -B_2\cdot \mathbf{c}_{2,1}\cdots - B_{k-1}\cdot \mathbf{c}_{k-1,1} - M_k\cdot \mathbf{c}_{k,1}. \end{aligned} \tag{10}$$

Since unknowns are entries in $B_1,\dots,B_{k-1}$ and $\mathbf{c}_{11},\dots,\mathbf{c}_{k-1,k-1}$ are primitive vectors, underdetermined linear system (10) has at least one solution. Finally, we index the columns of Λ_k by (9) and define an rational characteristic function

$$\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^{\sum_{i=1}^k n_i}$$

by assigning $\lambda(F_{(i,s)})$ with the column vector of Λ_k indexed by (i,s) for $1\le i\le k$ and $0 \le s \le n_i$.

It remains to check that the above λ satisfies the condition (4). Let

$$v_{r_1,\dots,r_k} := \bigcap_{i=1}^k \big(F_{(i,0)}\cap\cdots\cap F_{(i,r_i-1)}\cap F_{(i,r_i+1)}\cap\cdots\cap F_{(i,n_i)}\big) \tag{11}$$

denote a vertex of P. Then, for fixed i, the set of vectors

$$\big\{\lambda(F_{(i,0)}),\dots,\lambda(F_{(i,r_i-1)}),\lambda(F_{(i,r_i+1)}),\dots,\lambda(F_{(i,n_i)})\big\} \tag{12}$$

is linearly independent due to M_i. Since Λ_k is block upper triangular, we conclude the collection of characteristic vectors of (12) for $i = 1, \dots k$ is linearly independent.

Finally, we observe that $\Lambda_k \cdot S_k = \mathbf{0}$ by the relation (10), which implies that $\mathrm{im}(\exp S_k) \subseteq \ker(\exp \Lambda_k)$. Furthermore, The linear map

$$\Lambda_k \colon \mathbb{Z}^{\sum_{i=1}^{k}(n_i+1)} \to \mathbb{Z}^{\sum_{i=1}^{k} n_i}$$

is surjective, because each $M_i \colon \mathbb{Z}^{n_i+1} \to \mathbb{Z}^{n_i}$ is surjective for $i = 1, \dots, k$ and Λ_k is a block lower triangular matrix with block diagonals $M_1, \dots, M_k$. Hence we conclude that $\mathrm{im}(\exp S) = \ker(\exp \Lambda)$ by Lemma 3.6. The second assertion follows from Proposition 3.8. □

Let P and be the product of simplices as above and S be a vector matrix of size-$(k \times k)$ satisfying the condition of Lemma 4.2. In this case, we write

$$X_S := X(P, \lambda) = \prod_{j=1}^{k} S^{2n_j+1} / \mathrm{im}(\exp S), \tag{13}$$

where λ is a characteristic function on P as defined in Lemma 4.2.

Let $P_{k-1} := \prod_{i=1}^{k-1} \Delta^{n_i}$ and $S_{k-1} = (\mathbf{c}_{ij})_{1 \leq i,j \leq k-1}$ the $(k-1) \times (k-1)$ vector submatrix of S. Consider an orbifold

$$X_{S_{k-1}} := \prod_{j=1}^{k-1} S^{2n_j+1} / \mathrm{im}(\exp S_{k-1}).$$

Since the canonical projection

$$\pi_k \colon \prod_{j=1}^{k} S^{2n_j+1} \to \prod_{j=1}^{k-1} S^{2n_j+1} \tag{14}$$

is equivariant with respect to the map $\theta_k \colon T^{\sum_{j=1}^{k} n_j+1} \to T^{\sum_{j=1}^{k-1} n_j+1}$, the map π_k of (14) descents to the map

$$\widetilde{\pi}_k \colon X_S \to X_{S_{k-1}}.$$

Similarly, we consider $\widetilde{\pi}_i \colon X_{S_{i+1}} \to X_{S_i}$ for $i = 1, \dots, k-1$, which yields a sequence

$$X_S \xrightarrow{\widetilde{\pi}_k} X_{S_{k-1}} \xrightarrow{\widetilde{\pi}_{k-1}} \cdots \xrightarrow{\widetilde{\pi}_3} X_{S_2} \xrightarrow{\widetilde{\pi}_2} X_{S_1} \xrightarrow{\widetilde{\pi}_1} X_0 = \{pt\}, \tag{15}$$

such that for each i, the map $\widetilde{\pi}_i \colon X_{S_i} \to X_{S_{i-1}}$ is an orbifold fibration. To be more precise, a point $p \in X_{S_{i-1}}$, we have

$$\widetilde{\pi}^{-1}(p) \cong (S^{2n_i+1}/S^1)/G_p,$$

where G_p is the local group of p and S^1-action on S^{2n_i+1} is given by

$$t \cdot (z_0, \dots, z_{n_i}) = (t^{c_{(i,0),i}} z_0, \dots, t^{c_{(i,n_i),i}} z_{n_i}).$$

Hence, the fiber is given by $\mathbb{CP}^{n_i}_{\mathbf{c}_{ii}}/G_p$. In particular, $X_{S_1} = \mathbb{CP}^{n_1}_{\mathbf{c}_{11}}$ and the projection $\widetilde{\pi}_1 \colon X_{S_1} \to X_{S_0}$ is a trivial fibration over a point.

Let $\omega \colon \prod_{j=1}^{k} S^{2n_j+1} \twoheadrightarrow X_S$ be the orbit map with respect to the action of $\mathrm{im}(\exp S_k)$. Then, for each vertex v of P, its preimage $\omega^{-1}(v)$ is a fixed point. If $v = v_{r_1,\dots,r_k}$ as in (11), then the order of local group is equal to the absolute value of the determinant of the matrix consisting of (12) for $i = 1, \dots, k$ as column vectors. In this case, it agrees with the product $\prod_{i=1}^{k} c_{(i,r_i),i}$. The same argument can be applied to each orbit map $\omega_i \colon \prod_{j=1}^{i} S^{2n_j+1} \twoheadrightarrow X_{S_i}$. So, we collect these numbers for each i as follows:

$$\textsc{Sing}_i := \left\{ \prod_{j=1}^{i} c_{(j,r_j),j} \colon 0 \leq r_j \leq n_j \right\},$$

which represents the collection of the order of local group at each fixed points in X_{S_i} for each $i = 1, \ldots, k$. We note that the order of the local group of a non-fixed point divides some elements of $\textsc{Sing}_i$ because the local group at a point in an orbit with respect to the torus action always injects into the local group at a fixed point in the closure of that orbit. See [18, Section 2.1] for more details.

Example 4.3. Consider the case when $k = 2$ and $n_1 = n_2 = 2$ with

$$S_2 = \left[\begin{array}{c|c} 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ \hline a & 1 \\ b & 4 \\ c & 5 \end{array}\right]$$

for some integers $a, b, c \in \mathbb{Z}$. Following the result of Lemma 4.2 and computations given in its proof, one can take a characteristic function λ whose matrix presentation is given by

$$\left[\begin{array}{ccc|ccc} -2 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 \\ \hline 4a-b & 0 & 0 & -4 & 1 & 0 \\ 5a-c & 0 & 0 & -5 & 0 & 1 \end{array}\right].$$

The corresponding toric orbifold X_{S_2} is the total space of an orbifold fibration over $\mathbb{C}P^2_{(1,2,3)}$ whose fiber is $\mathbb{C}P^2_{(1,4,5)}/G_p$ for the local group G_p of $p \in \mathbb{C}P^2_{(1,2,3)}$. If p is a fixed point, then the order of local group G_p for $p \in X_{S_2}$ is an element of $\textsc{Sing}_2 = \{1, 4, 5, 2, 8, 10, 3, 12, 15\}$.

Theorem 4.4. *The sequence* (15) *associated with S as in Lemma 4.2 has the structure of weighed projective tower of* (3) *if and only if for each i, the entries in $\mathbf{c}_{ij}$ for $i > j$ are divisible by* $\operatorname{lcm}(\textsc{Sing}_{i-1})$.

Proof. For each $i \geq 2$, consider the diagram

$$\begin{array}{ccc} \prod_{j=1}^{i} S^{2n_j+1} & \xrightarrow{\pi_i} & \prod_{j=1}^{i-1} S^{2n_j+1} \\ \big\downarrow{\scriptstyle \omega_i} & & \big\downarrow{\scriptstyle \omega_{i-1}} \\ X_{S_i} & \xrightarrow{\widetilde{\pi}_i} & X_{S_{i-1}}, \end{array}$$

where the vertical maps are the orbit maps corresponding to the actions of $\operatorname{im}(\exp S_i)$ and $\operatorname{im}(\exp S_{i-1})$, respectively. The restricted action of $\operatorname{im}(\exp S_{i-1}) \cong T^{i-1}$ on $\prod_{j=1}^{i} S^{2n_j+1}$ gives rise to

$$\prod_{j=1}^{i-1} S^{2n_j+1} \times_{T^{i-1}} \mathbb{C}^{n_i+1} \twoheadrightarrow X_{S_{i-1}} \tag{16}$$

which can be thought of as an orbifold complex vector bundle whose fiber at $p \in X_{S_{i-1}}$ is $\mathbb{C}^{n_i+1}/G_p$ for the local group G_p.

Note that the T^{i-1}-action on $\mathbb{C}^{n_i+1} \setminus \{\mathbf{0}\}$ is given by the map $T^{i-1} \to T^{n_i+1}$ defined by

$$(t_1, \dots, t_{i-1}) \mapsto \left(\prod_{j=1}^{i-1} t_j^{c_{(i,0),j}}, \prod_{j=1}^{i-1} t_j^{c_{(i,1),j}}, \dots, \prod_{j=1}^{i-1} t_j^{c_{(i,n_i),j}} \right).$$

Hence, the fiber of the orbifold vector bundle (16) is a genuine vector space if and only if each $c_{(i,0),j}, \dots c_{(i,n_i),j}$ for $j < i$ is divisible by the order of local group. The computation for this claim is essentially same as the proof of Proposition 2.1 using the canonical line bundle over the weighted projectivization of a vector bundle [3, Section 2], instead of the canonical line bundle over a weighted projective space.

Recall that the collection of the orders of local groups of points in $X_{S_{i-1}}$ agrees with the set $\mathrm{S{\scriptsize ING}}_{i-1}$ together with all divisors of elements in $\mathrm{S{\scriptsize ING}}_{i-1}$. Hence, we conclude that the orbifold vector bundle (16) is a genuine vector bundle if and only if the entries of $\mathbf{c}_{ij}$ for $i > j$ are divisible by $\operatorname{lcm}(\mathrm{S{\scriptsize ING}}_{i-1})$. In this case, the vector bundle (16) can be decomposed into the Whitney sum of line bundles because T^{i-1} is abelian. The quotient by the S^1-subgroup of T^i parametrized by the last column of S_i yields the weighted projectivization of the bundle (16) with respect to the weight $\mathbf{c}_{ii}$. Hence, the result follows. □

Following the result of Theorem 4.4, for instance the matrix S_2 of Example 4.3 defines a weighed projective tower $\mathrm{wPT}(2) \to \mathrm{wPT}(1)$ if and only if a, b, c are divisible by 6.

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