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DEDUCTIVE ENERGETIC SETS IN EQUALITY ALGEBRAS

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Abstract. To contribute to the development of algebraic semantics, the concept of deductive energetic set in equality algebras is introduced, and several properties are investigated. The conditions under which a subset becomes deductive energetic in an equation algebra are explored, and its characterization is also obtained. The union and intersection of deductive energetic sets are examined. Equality homomorphic (pre) images and direct product of deductive energetic sets are addressed.

1. Introduction

The equality algebra was introduced by Jenei [2] as a new structure with two connectives, a meet operation and an equivalence, and a constant. It was motivated by the need for algebraic semantics in fuzzy type theory. Jenei identified equivalential equality algebras are term equivalent with BCK-algebras with meet. Jenei-Kóródi [3] explored the concept of filters and congruence in equality algebras underlying algebraic semantics. Developing or exploring new substructures that can be used as the basis for algebraic semantics will be a meaningful task for the development of algebraic semantics. To this end, this paper introduces the concept of deductive energetic set in equality algebras, and investigates several properties. We explore conditions for a subset to be deductive energetic in equality algebras, and obtain its characterization. We look at the union and intersection for deductive energetic sets. We deal with equality homomorphic (pre) images and direct product of deductive energetic sets.

2. Preliminaries

Definition 2.1. ([2]) By an *equality algebra*, we mean an algebraic structure $\mathcal{E}$:= $(E, \wedge, \sim, 1)$ satisfying the following conditions:

(E1) $(E, \wedge, 1)$ is a commutative idempotent integral monoid, that is, meet semilattice with the top element 1,

(E2) The operation $\sim$ is commutative,

(E3) $(\forall \mathfrak{a} \in E)(\mathfrak{a} \sim \mathfrak{a} = 1)$,

(E4) $(\forall \mathfrak{a} \in E)(\mathfrak{a} \sim 1 = \mathfrak{a})$,

(E5) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in E)(\mathfrak{a} \leq_E \mathfrak{b} \leq_E \mathfrak{c} \ \Rightarrow\ \mathfrak{a} \sim \mathfrak{c} \leq_E \mathfrak{b} \sim \mathfrak{c},\ \mathfrak{a} \sim \mathfrak{c} \leq_E \mathfrak{a} \sim \mathfrak{b})$,

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(E6) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in E)(\mathfrak{a} \sim \mathfrak{b} \leq_E (\mathfrak{a} \wedge \mathfrak{c}) \sim (\mathfrak{b} \wedge \mathfrak{c}))$,
(E7) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in E)(\mathfrak{a} \sim \mathfrak{b} \leq_E (\mathfrak{a} \sim \mathfrak{c}) \sim (\mathfrak{b} \sim \mathfrak{c}))$,

where $\mathfrak{a} \leq_E \mathfrak{b}$ if and only if $\mathfrak{a} \wedge \mathfrak{b} = \mathfrak{a}$.

In an equality algebra $\mathcal{E} := (E, \wedge, \sim, 1)$, we define two operations $\rightarrow$ and $\leftrightarrow$ on E as follows:

$$\mathfrak{a} \rightarrow \mathfrak{b} := \mathfrak{a} \sim (\mathfrak{a} \wedge \mathfrak{b}), \tag{2.1}$$

$$\mathfrak{a} \leftrightarrow \mathfrak{b} := (\mathfrak{a} \rightarrow \mathfrak{b}) \wedge (\mathfrak{b} \rightarrow \mathfrak{a}). \tag{2.2}$$

The operations $\wedge$, $\sim$, $\rightarrow$ and $\leftrightarrow$ are called *meet*, *equality operation*, *implication* and *equivalence operation*, respectively.

PROPOSITION 2.2. *(* [2,6]) Let $\mathcal{E} := (E, \wedge, \sim, 1)$ be an equality algebra. Then the following assertions are valid: For all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in E$,

$$\mathfrak{a} \rightarrow \mathfrak{b} = 1 \;\Leftrightarrow\; \mathfrak{a} \leq_E \mathfrak{b}, \tag{2.3}$$

$$1 \rightarrow \mathfrak{a} = \mathfrak{a},\; \mathfrak{a} \rightarrow 1 = 1 = \mathfrak{a} \rightarrow \mathfrak{a}, \tag{2.4}$$

$$\mathfrak{a} \leq_E (\mathfrak{a} \rightarrow \mathfrak{b}) \rightarrow \mathfrak{b}, \tag{2.5}$$

$$\mathfrak{a} \sim \mathfrak{b} \leq_E \mathfrak{a} \leftrightarrow \mathfrak{b} \leq_E \mathfrak{a} \rightarrow \mathfrak{b}, \tag{2.6}$$

$$\mathfrak{a} \leq_E (\mathfrak{a} \sim \mathfrak{b}) \sim \mathfrak{b}, \tag{2.7}$$

$$\mathfrak{a} \sim \mathfrak{b} \leq_E \mathfrak{a} \sim (\mathfrak{a} \wedge \mathfrak{b}). \tag{2.8}$$

DEFINITION 2.3. ([3]) A subset D of E is called a *filter* (or *deductive system*) of $\mathcal{E}$ if it satisfies

$$1 \in D, \tag{2.9}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in E)(\mathfrak{a} \in D, \mathfrak{a} \leq_E \mathfrak{b} \;\Rightarrow\; \mathfrak{b} \in D), \tag{2.10}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in E)(\mathfrak{a} \in D, \mathfrak{a} \sim \mathfrak{b} \in D \;\Rightarrow\; \mathfrak{b} \in D). \tag{2.11}$$

LEMMA 2.4. *(* [1,2]) Let $\mathcal{E}$ be an equality algebra. A subset D of E is a filter of $\mathcal{E}$ if and only if it satisfies (2.9) and

$$(\forall \mathfrak{a}, \mathfrak{b} \in E)(\mathfrak{a} \in D,\, \mathfrak{a} \rightarrow \mathfrak{b} \in D \;\Rightarrow\; \mathfrak{b} \in D). \tag{2.12}$$

DEFINITION 2.5. ([4]) Let $\mathcal{E} := (E, \wedge, \sim, 1)$ and $\mathcal{G} := (G, \wedge, \sim, 1)$ be equality algebras. A mapping $\Psi : E \rightarrow G$ is called an *equality homomorphism* if two operations $\wedge$ and $\sim$ in E are preserved under Ψ that is, Ψ satisfies

$$(\forall \mathfrak{a}, \mathfrak{b} \in E)\left(\Psi(\mathfrak{b} \wedge \mathfrak{a}) = \Psi(\mathfrak{b}) \wedge \Psi(\mathfrak{a}),\, \Psi(\mathfrak{b} \sim \mathfrak{a}) = \Psi(\mathfrak{b}) \sim \Psi(\mathfrak{a})\right). \tag{2.13}$$

If $\mathcal{E}$ and $\mathcal{G}$ are bounded equality algebras, we say that the equality homomorphism Ψ is *bounded*.

Note that every equality homomorphism Ψ preserves the operation $\rightarrow$ from E up to G, that is, $\Psi(\mathfrak{b} \rightarrow \mathfrak{a}) = \Psi(\mathfrak{b}) \rightarrow \Psi(\mathfrak{a})$ for all $\mathfrak{b}, \mathfrak{a} \in E$. Also, $\Psi(1) = 1$ and $\Psi(\neg\mathfrak{b}) = \neg\Psi(\mathfrak{b})$ for all $\mathfrak{b} \in E$ when Ψ is a bounded equality homomorphism (see [4, Lemma 4.1]).

3. Deductive energetic sets

In what follows, let $\mathcal{E} := (E, \wedge, \sim, 1)$ be an equality algebra unless otherwise specified.

Definition 3.1. A nonempty subset D of E is said to be *deductive energetic* in $\mathcal{E}$ if it satisfies

$$(\forall x, y \in E)\,(y \in D \;\Rightarrow\; \{x,\, x \to y\} \cap D \neq \emptyset)\,. \tag{3.1}$$

Example 3.1. Let $E := \{\xi_0, \xi_1, \xi_a, \xi_b\}$ be a set with the Hasse diagram and the equality operation $\to$ given by Figure 1 and Table 1, respectively. Then $\mathcal{E} = (E, \wedge, \to, \xi_1)$ is an equality algebra (see [5]). It is routine to verify that $D_1 = \{\xi_0, \xi_a\}$ and $D_2 = \{\xi_0, \xi_b\}$ are deductive energetic in $\mathcal{E}$.

Figure 1. Hasse Diagram

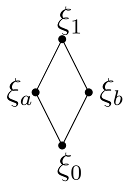

Table 1. Cayley table of the equality operation $\to$

$\to$	ξ_0	ξ_a	ξ_b	ξ_1
ξ_0	ξ_1	ξ_b	ξ_a	ξ_0
ξ_a	ξ_b	ξ_1	ξ_0	ξ_a
ξ_b	ξ_a	ξ_0	ξ_1	ξ_b
ξ_1	ξ_0	ξ_a	ξ_b	ξ_1

Proposition 3.2. *If a deductive energetic set D in $\mathcal{E}$ does not contain the top element* 1, *then it satisfies*

$$(\forall x, y \in E)\,(y \in D,\, x \leq_E y \;\Rightarrow\; x \in D)\,. \tag{3.2}$$

Proof. Let $x \leq_E y$ for $x \in E$ and $y \in D$. Then $x \to y = 1$ by (2.3), and so

$$\{x, 1\} \cap D = \{x, x \to y\} \cap D \neq \emptyset$$

by (3.1). Hence $x \in D$ since $1 \notin D$. □

Theorem 3.3. *If a nonempty subset D of E does not contain the top element* 1, *then it is deductive energetic in $\mathcal{E}$ if and only if it satisfies*

$$(\forall x, y, z \in E)\,(z \in D,\, x \leq_E y \to z \;\Rightarrow\; \{x, y\} \cap D \neq \emptyset)\,. \tag{3.3}$$

Proof. Assume that D is deductive energetic in $\mathcal{E}$ and let $x, y, z \in E$ be such that $z \in D$ and $x \leq_E y \to z$. Then $\{y,\, y \to z\} \cap D \neq \emptyset$ by (3.1), and so $y \in D$ or $y \to z \in D$. It is clear that if $y \in D$, then $\{x, y\} \cap D \neq \emptyset$. If $y \to z \in D$, then

$$\{x, 1\} \cap D = \{x, x \to (y \to z)\} \cap D \neq \emptyset$$

by (2.3) and (3.1). Since $1 \notin D$, it follows that $x \in D$. Hence $\{x, y\} \cap D \neq \emptyset$, and therefore (3.3) is valid.

Conversely, suppose that D satisfies (3.3) and let $y \in D$. Since $x \leq_E (x \to y) \to y$ for all $x \in E$ by (2.5), we have $\{x, x \to y\} \cap D \neq \emptyset$ by (3.3). Hence D is deductive energetic in $\mathcal{E}$. □

Theorem 3.4. *If a nonempty subset D of E satisfies*

$$(\forall x, y, z \in E)\,(y \to x \in D \;\Rightarrow\; \{z,\, z \to (y \to (y \to x))\} \cap D \neq \emptyset)\,, \tag{3.4}$$

then D is deductive energetic in $\mathcal{E}$.

Proof. Let $x \in D$. Then $1 \to x = x \in D$ by (2.4), and thus

$$\{z, z \to x\} \cap D = \{z,\, z \to (1 \to (1 \to x))\} \cap D \neq \emptyset$$

for all $z \in E$ by (2.4) and (3.4). Hence D is deductive energetic in $\mathcal{E}$. □

Theorem 3.5. *If D is a subset of E, then $E \setminus D$ is deductive energetic in $\mathcal{E}$ if and only if D satisfies the condition* (2.12).

Proof. Assume that $E \setminus D$ is deductive energetic in $\mathcal{E}$ and let $x, y \in E$ be such that $x \in D$ and $x \to y \in D$. If $y \notin D$, then $\{x,\, x \to y\} \cap (E \setminus D) \neq \emptyset$ by (3.1). Thus $x \in E \setminus D$ or $x \to y \in E \setminus D$, which is a contradiction. Hence $y \in D$ and consequently D satisfies the condition (2.12).

Conversely, suppose that D satisfies the condition (2.12) and let $y \in E \setminus D$. If

$$\{x, x \to y\} \cap (E \setminus D) = \emptyset$$

for some $x \in E$, then $x \in D$ and $x \to y \in D$. It follows from (2.12) that $y \in D$. This is a contradiction, and so $\{x, x \to y\} \cap (E \setminus D) \neq \emptyset$ for all $x \in E$. Therefore $E \setminus D$ is deductive energetic in $\mathcal{E}$. □

Corollary 3.6. *If a deductive energetic set $E \setminus D$ in $\mathcal{E}$ does not contain the top element* 1, *then D is a filter of $\mathcal{E}$.*

Proof. The proof is straightforward. □

Proposition 3.7. *If a deductive energetic set D in $\mathcal{E}$ does not contain the top element* 1, *then it satisfies*

$$(\forall x, y \in E)\,(y \in D \;\Rightarrow\; \{x,\, x \sim y\} \cap D \neq \emptyset)\,. \tag{3.5}$$

Proof. Let D be deductive energetic in $\mathcal{E}$ that does not contain the top element 1. Then $E \setminus D$ is a filter of $\mathcal{E}$ by Corollary 3.6. Let $y \in D$. If $\{x,\, x \sim y\} \cap D = \emptyset$ for some $x \in E$, then $x \in E \setminus D$ and $x \sim y \in E \setminus D$. Hence $y \in E \setminus D$ by (2.11), which is a contradiction. Therefore $\{x,\, x \sim y\} \cap D \neq \emptyset$ for all $x \in E$, that is, (3.5) is valid. □

Theorem 3.8. *If D_1 and D_2 are deductive energetic in $\mathcal{E}$, then so is their union.*

Proof. Assume that D_1 and D_2 are deductive energetic in $\mathcal{E}$ and let $y \in D_1 \cup D_2$. Then $y \in D_1$ or $y \in D_2$, and so $\{x, x \to y\} \cap D_1 \neq \emptyset$ or $\{x, x \to y\} \cap D_2 \neq \emptyset$ for all $x \in E$. Hence $x \in D_1$, $x \to y \in D_1$, $x \in D_2$, or $x \to y \in D_2$. It follows that $x \in D_1 \cup D_2$ or $x \to y \in D_1 \cup D_2$. Thus $\{x, x \to y\} \cap (D_1 \cup D_2) \neq \emptyset$, and therefore $D_1 \cup D_2$ is deductive energetic in $\mathcal{E}$. □

In Example 3.1, $D_1 = \{\xi_0, \xi_a\}$ and $D_2 = \{\xi_0, \xi_b\}$ are deductive energetic in $\mathcal{E}$. Note that $\{\xi_a, \xi_a \to \xi_0\} \cap \{\xi_0\} = \{\xi_a, \xi_b\} \cap \{\xi_0\} = \emptyset$. Hence $D_1 \cap D_2 = \{\xi_0\}$ is not deductive energetic in $\mathcal{E}$. This shows that in general the intersection of deductive energetic sets is not deductive energetic in $\mathcal{E}$.

We consider homomorphic (pre) images of energetic sets.

Theorem 3.9. *Let $\Psi : E \to G$ be an equality homomorphism of equality algebras $\mathcal{E} := (E, \wedge, \sim, 1)$ and $\mathcal{G} := (G, \wedge, \sim, 1)$. For every nonempty subset D of G, if D is deductive energetic in $\mathcal{G}$, then $\Psi^{-1}(D)$ is deductive energetic in $\mathcal{E}$.*

Proof. Let D be deductive energetic in $\mathcal{G}$. Then $G \setminus D$ satisfies the condition (2.12) by Theorem 3.5. Let $x, \mathfrak{a} \in E$ be such that $x \to \mathfrak{a} \in E \setminus \Psi^{-1}(D)$ and $x \in E \setminus \Psi^{-1}(D)$. Then $\Psi(x) \in G \setminus D$ and

$$\Psi(x) \to \Psi(\mathfrak{a}) = \Psi(x \to \mathfrak{a}) \in G \setminus D.$$

Hence $\Psi(\mathfrak{a}) \in G \setminus D$ by (2.12), and so $\mathfrak{a} \in \Psi^{-1}(G \setminus D) = E \setminus \Psi^{-1}(D)$, that is, $E \setminus \Psi^{-1}(D)$ satisfies the condition (2.12). Therefore $\Psi^{-1}(D)$ is deductive energetic in $\mathcal{E}$ by Theorem 3.5. □

Theorem 3.10. *Let $\Psi : E \to G$ be an onto and one-to-one equality homomorphism of equality algebras $\mathcal{E} := (E, \wedge, \sim, 1)$ and $\mathcal{G} := (G, \wedge, \sim, 1)$. For every nonempty subset B of E, if $E \setminus B$ is deductive energetic in $\mathcal{E}$, then $\Psi(E \setminus B)$ is deductive energetic in $\mathcal{G}$.*

Proof. If $E \setminus B$ is deductive energetic in $\mathcal{E}$, then B satisfies the condition (2.12) by Theorem 3.5. Let $y, \mathfrak{b} \in G$ be such that $y \in \Psi(B)$ and $y \to \mathfrak{b} \in \Psi(B)$. Then there exist $x, \mathfrak{a} \in E$ such that $\Psi(x) = y$ and $\Psi(\mathfrak{a}) = \mathfrak{b}$ since Ψ is onto. Hence $\Psi(x) = y \in \Psi(B)$ and $\Psi(x \to \mathfrak{a}) = \Psi(x) \to \Psi(\mathfrak{a}) = y \to \mathfrak{b} \in \Psi(B)$. Since Ψ is one-to-one, it follows that $x \in B$ and $x \to \mathfrak{a} \in B$. Thus $\mathfrak{a} \in B$ by (2.12), and so $\mathfrak{b} = \Psi(\mathfrak{a}) \in \Psi(B)$. This shows that $\Psi(B)$ satisfies the condition (2.12). Therefore $\Psi(E \setminus B) = \Psi(E) \setminus \Psi(B) = G \setminus \Psi(B)$ is deductive energetic in $\mathcal{G}$ by Theorem 3.5. □

4. Direct product of deductive energetic sets

We now deal with the direct product of energetic sets.

Let $\{\mathcal{E}_i := (E_i, \wedge_i, \sim_i, 1_i) \mid i \in \Lambda \subseteq \mathbb{N}\}$ be a nonempty family of equality algebras and let $\prod_{i\in\Lambda} E_i$ consist of all vectors $(x_i)_{i\in\Lambda}$, $x_i \in E_i$. Consider a structure $\prod_{i\in\Lambda} \mathcal{E}_i = \left(\prod_{i\in\Lambda} E_i, \wedge, \sim, \mathbf{1}\right)$ where $\mathbf{1} = (1_i)_{i\in\Lambda}$, and the operations $\wedge$ and $\sim$ are defined by

$$(x_i)_{i\in\Lambda} \wedge (y_i)_{i\in\Lambda} = (x_i \wedge_i y_i)_{i\in\Lambda} \text{ and } (x_i)_{i\in\Lambda} \sim (y_i)_{i\in\Lambda} = (x_i \sim_i y_i)_{i\in\Lambda},$$

respectively, for all $(x_i)_{i\in\Lambda}, (y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i$. Then $\prod_{i\in\Lambda} \mathcal{E}_i = \left(\prod_{i\in\Lambda} E_i, \wedge, \sim, \mathbf{1}\right)$ is an equality algebra. In this case, the implication $\to$ in $\prod_{i\in\Lambda} \mathcal{E}_i$ is given as follows:

$$\left(\forall (x_i)_{i\in\Lambda}, (y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i\right) ((x_i)_{i\in\Lambda} \to (y_i)_{i\in\Lambda} = (x_i \to_i y_i)_{i\in\Lambda})$$

where $\to_i$ is the implication in $\mathcal{E}_i$. Also, if $\leq_{E_i}$ is the order in $\mathcal{E}_i$, then the order $\leq_{\prod}$ in $\prod\limits_{i\in\Lambda}\mathcal{E}_i$ can be represented as follows:

$$(x_i)_{i\in\Lambda} \leq_{\prod} (y_i)_{i\in\Lambda} \ \Leftrightarrow x_i \leq_{E_i} y_i \text{ for all } i \in \Lambda. \tag{4.1}$$

THEOREM 4.1. *For every nonempty subset D_i of E_i, $i \in \Lambda$, if $E_i \setminus D_i$ is deductive energetic in $\mathcal{E}_i$, then $\prod\limits_{i\in\Lambda} E_i \setminus \prod\limits_{i\in\Lambda} D_i$ is deductive energetic in $\prod\limits_{i\in\Lambda}\mathcal{E}_i$.*

Proof. Assume that $E_i \setminus D_i$ is deductive energetic in $\mathcal{E}_i$ where $i \in \Lambda$. Then D_i satisfies the condition (2.12) by Theorem 3.5. Let $(y_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} E_i \setminus \prod\limits_{i\in\Lambda} D_i$. If

$$\{(x_i)_{i\in\Lambda},\, (x_i)_{i\in\Lambda} \to (y_i)_{i\in\Lambda}\} \cap \left(\prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i\right) = \emptyset$$

for some $(x_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} E_i$, then $(x_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} D_i$ and

$$(x_i \to_i y_i)_{i\in\Lambda} = (x_i)_{i\in\Lambda} \to (y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i.$$

Hence $x_i \in D_i$ and $x_i \to_i y_i \in D_i$, and so $y_i \in D_i$ for all $i \in \Lambda$ by (2.12). It follows that $(y_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} D_i$, a contradiction. Thus

$$\{(x_i)_{i\in\Lambda},\, (x_i)_{i\in\Lambda} \to (y_i)_{i\in\Lambda}\} \cap \left(\prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i\right) \neq \emptyset$$

for all $(x_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} E_i$, and therefore $\prod\limits_{i\in\Lambda} E_i \setminus \prod\limits_{i\in\Lambda} D_i$ is deductive energetic in $\prod\limits_{i\in\Lambda}\mathcal{E}_i$. □

PROPOSITION 4.2. *Let D_i be be a nonempty subset of E_i which does not contain the top element 1_i for all $i \in \Lambda$. If $\prod\limits_{i\in\Lambda} D_i$ is deductive energetic in $\prod\limits_{i\in\Lambda}\mathcal{E}_i$, then it satisfies*

$$(y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i,\ (x_i)_{i\in\Lambda} \leq_{\prod} (y_i)_{i\in\Lambda} \ \Rightarrow\ (x_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i \tag{4.2}$$

for all $(x_i)_{i\in\Lambda}, (y_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} E_i$.

Proof. If every nonempty subset D_i of E_i does not contain the top element 1_i for all $i \in \Lambda$, then $\prod\limits_{i\in\Lambda} D_i$ does not conain the top element $\mathbf{1}$. Let $(x_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} E_i$ and $(y_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} D_i$ be such that $(x_i)_{i\in\Lambda} \leq_{\prod} (y_i)_{i\in\Lambda}$. Then

$$(x_i)_{i\in\Lambda} \to (y_i)_{i\in\Lambda} = (x_i \to_i y_i)_{i\in\Lambda} \overset{(2.3)\,\&\,(4.1)}{=} (1_i)_{i\in\Lambda} = \mathbf{1},$$

and so

$$\{(x_i)_{i\in\Lambda},\, \mathbf{1}\} \cap \prod_{i\in\Lambda} D_i = \{(x_i)_{i\in\Lambda},\, (x_i)_{i\in\Lambda} \to (y_i)_{i\in\Lambda}\} \cap \prod_{i\in\Lambda} D_i \neq \emptyset.$$

Since $\mathbf{1} \notin \prod\limits_{i\in\Lambda} D_i$, it follows that $(x_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} D_i$. Hence (4.2) is valid. □

Theorem 4.3. *Let D_i be a nonempty subset of E_i which does not contain the top element 1_i for all $i \in \Lambda$. Then $\prod_{i\in\Lambda} D_i$ is deductive energetic in $\prod_{i\in\Lambda} \mathcal{E}_i$ if and only if it satisfies*

$$\begin{aligned}&(z_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i,\ (x_i)_{i\in\Lambda} \leq_{\prod} (y_i)_{i\in\Lambda} \to (z_i)_{i\in\Lambda}\\ &\Rightarrow\ \{(x_i)_{i\in\Lambda},\ (y_i)_{i\in\Lambda}\} \cap \prod_{i\in\Lambda} D_i \neq \emptyset\end{aligned} \tag{4.3}$$

for all $(x_i)_{i\in\Lambda}$, $(y_i)_{i\in\Lambda}$, $(z_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i$.

Proof. If every nonempty subset D_i of E_i does not contain the top element 1_i for all $i \in \Lambda$, then $\prod_{i\in\Lambda} D_i$ does not conain the top element $\mathbf{1}$. Assume that $\prod_{i\in\Lambda} D_i$ is deductive energetic in $\prod_{i\in\Lambda} \mathcal{E}_i$. Let $(x_i)_{i\in\Lambda}$, $(y_i)_{i\in\Lambda}$, $(z_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i$ be such that $(z_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i$ and $(x_i)_{i\in\Lambda} \leq_{\prod} (y_i)_{i\in\Lambda} \to (z_i)_{i\in\Lambda}$. Then

$$\begin{aligned}(x_i)_{i\in\Lambda} \to ((y_i)_{i\in\Lambda} \to (z_i)_{i\in\Lambda}) &= (x_i)_{i\in\Lambda} \to (y_i \to_i z_i)_{i\in\Lambda}\\ &= (x_i \to_i (y_i \to_i z_i))_{i\in\Lambda} = (1_i)_{i\in\Lambda} = \mathbf{1}\end{aligned}$$

and

$$\{(y_i)_{i\in\Lambda},\ (y_i)_{i\in\Lambda} \to (z_i)_{i\in\Lambda}\} \cap \prod_{i\in\Lambda} D_i \neq \emptyset,$$

and thus $(y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i$ or $(y_i)_{i\in\Lambda} \to (z_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i$. If $(y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i$, then

$$\{(y_i)_{i\in\Lambda},\ (y_i)_{i\in\Lambda} \to (z_i)_{i\in\Lambda}\} \cap \prod_{i\in\Lambda} D_i \neq \emptyset.$$

If $(y_i)_{i\in\Lambda} \to (z_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i$, then

$$\{(x_i)_{i\in\Lambda}, \mathbf{1}\} \cap \prod_{i\in\Lambda} D_i = \{(x_i)_{i\in\Lambda},\ (x_i)_{i\in\Lambda} \to ((y_i)_{i\in\Lambda} \to (z_i)_{i\in\Lambda})\} \cap \prod_{i\in\Lambda} D_i \neq \emptyset.$$

Since $\mathbf{1} \notin \prod_{i\in\Lambda} D_i$, we have $(x_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i$ and thus

$$\{(x_i)_{i\in\Lambda},\ (y_i)_{i\in\Lambda}\} \cap \prod_{i\in\Lambda} D_i \neq \emptyset.$$

Conversely, suppose that the condition (4.3) is valid for all $(x_i)_{i\in\Lambda}$, $(y_i)_{i\in\Lambda}$, $(z_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i$. Let $(y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i$. Using (2.5), we know that $x_i \leq_{E_i} (x_i \to_i y_i) \to_i y_i$ for all $i \in \Lambda$. Hence $(x_i)_{i\in\Lambda} \leq_{\prod} ((x_i)_{i\in\Lambda} \to (y_i)_{i\in\Lambda}) \to (y_i)_{i\in\Lambda}$, which implies from (4.3) that

$$\{(x_i)_{i\in\Lambda},\ (x_i)_{i\in\Lambda} \to (y_i)_{i\in\Lambda}\} \cap \prod_{i\in\Lambda} D_i \neq \emptyset.$$

Therefore $\prod_{i\in\Lambda} D_i$ is deductive energetic in $\prod_{i\in\Lambda} \mathcal{E}_i$. □

Theorem 4.4. *For every nonempty subsets D_i of E_i for $i \in \Lambda$, if $\prod_{i\in\Lambda} D_i$ satisfies*

$$\{(z_i)_{i\in\Lambda},\ (z_i)_{i\in\Lambda} \to ((y_i)_{i\in\Lambda} \to ((y_i)_{i\in\Lambda} \to (x_i)_{i\in\Lambda}))\} \cap \prod_{i\in\Lambda} D_i \neq \emptyset \tag{4.4}$$

for all $(x_i)_{i\in\Lambda}$, $(y_i)_{i\in\Lambda}$, $(z_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i$ *with* $(y_i)_{i\in\Lambda} \to (x_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i$, *then* $\prod_{i\in\Lambda} D_i$ *is deductive energetic in* $\prod_{i\in\Lambda} \mathcal{E}_i$.

Proof. Let $(x_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i$. Since

$$(1_i)_{i\in\Lambda} \to (x_i)_{i\in\Lambda} = (1_i \to_i x_i)_{i\in\Lambda} \overset{(2.4)}{=} (x_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i,$$

we have

$$\begin{aligned}&\{(z_i)_{i\in\Lambda}, (z_i)_{i\in\Lambda} \to (x_i)_{i\in\Lambda}\} \cap \prod_{i\in\Lambda} D_i \\ &= \{(z_i)_{i\in\Lambda}, (z_i)_{i\in\Lambda} \to ((1_i)_{i\in\Lambda} \to ((1_i)_{i\in\Lambda} \to (x_i)_{i\in\Lambda}))\} \cap \prod_{i\in\Lambda} D_i \overset{(4.4)}{\neq} \emptyset\end{aligned}$$

for all $(z_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i$. Hence $\prod_{i\in\Lambda} D_i$ is deductive energetic in $\prod_{i\in\Lambda} \mathcal{E}_i$. □

PROPOSITION 4.5. *Let* D_i *be a subset of* E_i *containing the top element* 1_i *for all* $i \in \Lambda$. *If* $E_i \setminus D_i$ *is deductive energetic in* $\mathcal{E}_i$ *for all* $i \in \Lambda$, *then the following assertion is valid.*

$$\begin{aligned}&(x_i)_{i\in\Lambda} \wedge (y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i \text{ or } (x_i)_{i\in\Lambda} \sim (y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i \\ &\Rightarrow \{(x_i)_{i\in\Lambda}, (y_i)_{i\in\Lambda}\} \cap \left(\prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i\right) \neq \emptyset\end{aligned}$$

for all $(x_i)_{i\in\Lambda}, (y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i$.

Proof. Let D_i be a subset of E_i containing the top element 1_i for all $i \in \Lambda$. If $E_i \setminus D_i$ is deductive energetic in $\mathcal{E}_i$, then D_i is a filter of E_i for all $i \in \Lambda$ (see Corollary 3.6). Let $(x_i)_{i\in\Lambda}, (y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i$ be such that

$$(x_i)_{i\in\Lambda} \wedge (y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i \text{ or } (x_i)_{i\in\Lambda} \sim (y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i.$$

Then $x_i \wedge_i y_i \in E_i \setminus D_i$ or $x_i \sim_i y_i \in E_i \setminus D_i$ for all $i \in \Lambda$. If

$$\{(x_i)_{i\in\Lambda}, (y_i)_{i\in\Lambda}\} \cap \left(\prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i\right) = \emptyset,$$

then $(x_i)_{i\in\Lambda}, (y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i$ and so $x_i, y_i \in D_i$ for all $i \in \Lambda$. Since

$$x_i \to_i ((x_i \sim_i y_i) \sim_i y_i) \overset{(2.3)\,\&\,(2.7)}{=} 1_i,$$

i.e., $x_i \leq_{E_i} (x_i \sim_i y_i) \sim_i y_i$ for all $i \in \Lambda$, we have

$$y_i \sim_i (x_i \sim_i y_i) \overset{(E2)}{=} (x_i \sim_i y_i) \sim_i y_i \in D_i$$

by (2.10), and thus $x_i \sim_i y_i \in D_i$ by (2.11). Since

$$x_i \sim_i y_i \leq_{E_i} x_i \sim_i (x_i \wedge_i y_i)$$

by (2.8), it follows from (2.10) and (2.11) that $x_i \wedge_i y_i \in D_i$. This is a contradiction, and hence $\{(x_i)_{i\in\Lambda}, (y_i)_{i\in\Lambda}\} \cap \left(\prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i\right) \neq \emptyset$. This completes the proof. □

PROPOSITION 4.6. *Let D_i be a nonempty subset of E_i which does not contain the top element 1_i for all $i \in \Lambda$. If $\prod_{i\in\Lambda} D_i$ is deductive energetic in $\prod_{i\in\Lambda} \mathcal{E}_i$, then it satisfies*

$$(y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i \;\Rightarrow\; \{(x_i)_{i\in\Lambda},\, (x_i)_{i\in\Lambda} \sim (y_i)_{i\in\Lambda}\} \cap \prod_{i\in\Lambda} D_i \neq \emptyset$$

for all $(x_i)_{i\in\Lambda}, (y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i$.

Proof. Note that $\prod_{i\in\Lambda} D_i$ does not contain the top element $\mathbf{1}$. Hence $\prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i$ is a filter of $\prod_{i\in\Lambda} \mathcal{E}_i$ by Corollary 3.6. Let $(y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} D_i$ and suppose that there exists $(x_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i$ such that

$$\{(x_i)_{i\in\Lambda},\, (x_i)_{i\in\Lambda} \sim (y_i)_{i\in\Lambda}\} \cap \prod_{i\in\Lambda} D_i = \emptyset.$$

Then $(x_i)_{i\in\Lambda} \notin \prod_{i\in\Lambda} D_i$ and $(x_i)_{i\in\Lambda} \sim (y_i)_{i\in\Lambda} \notin \prod_{i\in\Lambda} D_i$, and so

$$(x_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i \text{ and } (x_i)_{i\in\Lambda} \sim (y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i.$$

It follows from (2.11) that $(y_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i \setminus \prod_{i\in\Lambda} D_i$, which is a contradiction. Therefore $\{(x_i)_{i\in\Lambda},\, (x_i)_{i\in\Lambda} \sim (y_i)_{i\in\Lambda}\} \cap \prod_{i\in\Lambda} D_i \neq \emptyset$ for all $(x_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} E_i$. □

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