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ON SURJECTIVE α-AMPLIFIED ENDOMORPHISMS OF PROJECTIVE VARIETIES

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Abstract. Let X be a projective variety of dimension d over a field $\mathbf{k}$, and let $f : X \dashrightarrow X$ be a surjective endomorphism of X. In this paper, we prove that for any positive real number α, any surjective, but non-isomorphic, α-amplified endomorphism f of a projective variety X is of positive entropy and its first dynamical degree $\lambda_1(f)$ is not equal to α. We also prove that for any real number α greater than or equal to 1 any surjective α-amplified endomorphism f of a projective variety X with positive entropy such that the set of periodic points of f is Zariski dense in X should be always PCD. To be more precise, we show that, when the set of all periodic points of f_K is Zariski dense in X_K for some uncountable algebraically closed field extension K of $\mathbf{k}$, the set is actually countable.

1. Introduction

Let X be a projective variety of dimension d over a field $\mathbf{k}$, and let $f : X \dashrightarrow X$ be a dominant rational self-map. Let $\tilde{X}$ be the graph of f in the product $X \times X$, i.e., the closure of the set

$$\{(x, f(x)) \in X \times X \mid x \in X \backslash I(f)\}$$

in $X \times X$, where $I(f)$ denotes the indeterminacy locus of f. Let π_1 and π_2 be the projection of $\tilde{X}$ to the first and second projection, respectively. Let L be an ample (or, more generally, nef and big) line bundle on X. For each i $(0 \le i \le d)$, the *i-th degree of f with respect to L* is defined to be

$$\deg_{i,L}(f) = \left((\pi_2^* L)^i \cdot (\pi_1^* L)^{d-i}\right)$$

which can be written simply as $\left((f^*L)^i \cdot L^{d-i}\right)$ without the explicit specification of the birational model $\tilde{X}$ of X. The *i-th dynamical degree of f* is then defined to be

$$\lambda_i(f) = \lim_{n\to\infty} \left(\deg_{i,L}(f^n)\right)^{\frac{1}{n}} \ge 1.$$

The existence of $\lambda_i(f)$ is known (see [9] and [1]). Moreover, the value of $\lambda_i(f)$ is independent of the choice of L. Note that by definition $\lambda_0(f) = 1$.

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The notion of cohomological Lyapunov exponents can be also introduced in order to understand algebraic properties of dominant rational maps. That is, for each i $(1 \leq i \leq d+1)$, the *i-th cohomological Lyapunov exponent of f* is defined to be

$$\mu_i(f) = \lambda_i(f)/\lambda_{i-1}(f), \quad 1 \leq i \leq d,$$

and we set $\mu_{d+1}(f) = 0$ by convention. It is well known that the sequence $\{\mu_i(f)\}_{i=1}^{d+1}$ is decreasing, since the sequence $\{\lambda_i(f)\}_{i=0}^{d}$ is log-concave (see, e.g., [9], [1], [5], [10], and [11]), i.e.,

$$\lambda_i(f)^2 \geq \lambda_{i-1}(f)\lambda_{i+1}(f), \quad 1 \leq i \leq d-1.$$

If there is a unique i $(1 \leq i \leq d)$ such that

$$\mu_i(f) > 1, \quad \mu_{i+1}(f) < 1,$$

i.e., if $\lambda_i(f)$ is strictly larger than any other dynamical degree $\lambda_j(f)$ for $1 \leq j \leq d$ such that $j \neq i$, then we say that f is *i-cohomologically hyperbolic*. Also, f is said to be *cohomologically hyperbolic* if f is i-cohomologically hyperbolic for some i $(1 \leq i \leq d)$. Notice that the condition of f being cohomologically hyperbolic is equivalent to that of $\mu_j(f) \neq 1$ for all $1 \leq j \leq d$. To be more precise, if we assume that $\mu_j(f) \neq 1$ for all $1 \leq j \leq d$, then there should be a unique i $(1 \leq i \leq d)$ such that

$$\mu_i(f) > 1, \quad \mu_{i+1}(f) < 1.$$

In particular, if $\mu_j(f) > 1$ for all $1 \leq j \leq d$, then we can take $i = d$, so that $\mu_d(f) > 1$ and $\mu_{d+1}(f) = 0 < 1$. On the other hand, if $\mu_j(f) < 1$ for all $1 \leq j \leq d$, then we have

$$\mu_1(f) = \frac{\lambda_1(f)}{\lambda_0(f)} = \lambda_1(f) < 1,$$

which is a contradiction to the fact that $\lambda_k(f) \geq 1$ for all $1 \leq k \leq d$. It is generally believed that the cohomological hyperbolicity in algebraic dynamics might be regarded as a cohomological interpretation of the notion of hyperbolic dynamics in differentiable dynamical systems.

Let us denote by $\mathrm{Pic}(X)$ the group of Cartier divisors modulo linear equivalence and by $\mathrm{Pic}_0(X)$ the subgroup of the classes in $\mathrm{Pic}(X)$ which are algebraically equivalent to 0. Let

$$NS(X) = \mathrm{Pic}(X)/\mathrm{Pic}_0(X)$$

be the Néron-Severi group of X, and let $N^1(X)_{\mathbb{R}} = NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Let Z and Z' be two r-cycles on X with integer coefficients. We say that Z is weakly numerically equivalent to Z', denoted by $Z \equiv_w Z'$ (or simply, $Z \equiv Z'$ if there is no confusion), if we have

$$(Z - Z') \cdot H_1 \cdots H_r = 0$$

e for any Cartier divisors $H_1, \ldots, H_r$ on X. For each $0 \leq r \leq d$, let $N_{d-r}(X)$ be the quotient (free) $\mathbb{Z}$-module of $(d-r)$-cycles (of codimension r) with integer coefficients modulo the numerical equivalence, and let $N^r(X) = \mathrm{Hom}(N_{d-r}(X), \mathbb{Z})$. We then set

$$N_{d-r}(X)_{\mathbb{R}} = N_{d-r}(X) \otimes_{\mathbb{Z}} \mathbb{R} \text{ and } N^r(X)_{\mathbb{R}} = N^r(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Denote by $\mathrm{Amp}(X)$ (resp. $\mathrm{Big}(X)$) the ample (resp. big) cone in $N^1(X)_{\mathbb{R}}$. Then $\mathrm{Amp}(X)$ and $\mathrm{Big}(X)$ are both open and saliently convex (see, e.g., [12, Example 1.1]). Moreover, when, in addition, f is regular, they are f^*-invariant.

An endomorphism $f : X \to X$ is called *int-amplified* if there is an ample line bundle L on X such that $f^*L - L$ is ample (see [7] for more details). It is known that a surjective endomorphism f on a projective variety of dimension d is int-amplified if and only if $\mu_d(f) > 1$ (see, e.g., [8, Proposition 3.7]). Note also that any int-amplified endomorphism f has the topological degree greater than one, which implies that f is a non-isomorphic surjective endomorphism (see [7, Lemma 3.7]).

On the other hand, an endomorphism $f : X \to X$ is called *polarized* if there is an ample line bundle L on X such that $f^*L = qL$ for some integer $q \geq 2$. In this case, we have $\lambda_i(f) = q^i > 1$ for each $1 \leq i \leq d$ and $\deg(f) = q^d$. Hence any polarized endomorphism is a finite surjective morphism. In fact, any polarized endomorphism is both int-amplified and d-cohomologically hyperbolic. This is simply because we have

$$\mu_d(f) = \frac{\lambda_d(f)}{\lambda_{d-1}(f)} = \frac{q^d}{q^{d-1}} = q > 1 > \mu_{d+1}(f) = 0.$$

More generally, Xie introduced the notion of an α-quasi-amplified endomorphism for a positive real number α (see [12]). To be precise, for a positive real number α, a surjective endomorphism f of a projective variety X is said to be *α-quasi-amplified* (resp. *α-amplified*) if α does not belong to the spectrum of f^* with respect to the big cone $\mathrm{Big}(X)$ (resp. the ample cone $\mathrm{Amp}(X)$), i.e., there is an $M \in N^1(X)_{\mathbb{R}}$ such that $f^*(M) - \alpha M$ is big (resp. ample). Note that the identity map of X is always α-amplified for any $0 < \alpha < 1$. It has been asserted in [12, Theorem 1.4] that for a positive real number α, f is α-quasi-amplified if and only if

$$\alpha \notin \{\mu_j(f) \,|\, 1 \leq j \leq d\}\,.$$

(Due to some computational inaccuracies (or typographical errors) appearing in the preprint [12], it seems that at the moment some results may require further elaboration, though.)

With these being said, our first aim of this paper is to give some characterization of surjective α-amplified endomorphisms of a projective variety, as follows.

Theorem 1.1. *Let X be a projective variety of dimension d over a field $\mathbf{k}$. Let α be a positive real number, and let $f : X \to X$ be a surjective, but non-isomorphic, α-amplified endomorphism of X. Then f is of positive entropy and the first dynamical degree $\lambda_1(f)$ of f is not equal to α.*

The proof of Theorem 1.1 will be given in Section 2. It is worth mentioning that Lemma 2.1 of this paper plays an important role in the proof of Theorem 1.1 and, seemingly, gives another interesting characterization of surjective α-amplified endomorphisms of a projective variety. As far as we know, it seems that the statement of f being of positive entropy in Theorem 1.1 was not observed in the paper [12].

As an immediate consequence of Theorem 1.1, we have the following corollary.

Corollary 1.2. *Let X be a projective variety of dimension d over a field $\mathbf{k}$, and let $f : X \to X$ be a surjective, but non-isomorphic, amplified endomorphism of X. Then f is of positive entropy and thus the first dynamical degree $\lambda_1(f)$ of f is not equal to* 1.

Let K be a field extension of $\mathbf{k}$ such that K is algebraically closed. We denote by $X_K = X \times_{\mathbf{k}} K$ and $f_K : X_K \to X_K$ the induced surjective endomorphism. A surjective endomorphism $f : X \to X$ is said to be *PCD* if the set of all periodic points of f_K is countable and Zariski dense in X_K for some uncountable algebraically closed field extension K of $\mathbf{k}$ (see [6, Definition 2.2]). See also [6, Proposition 2.3] for some equivalent definitions of the property of PCD.

It is well known by a result of Fakhruddin (see [2, Theorem 5.1]) that for any amplified endomorphism f of a projective variety X, the set of all f-periodic points is Zariski dense in X. Moreover, the property of f being amplified implies that of f being PCD (see, e.g., [6, Theorem 2.5]). Indeed, note that by [6, Proposition 2.3], we may work over an uncountable infinite field. The set of uncountably many periodic points would produce a positive dimensional pointwise fixed subvariety Y possibly after some iteration, and the restriction $f|_Y$ of an amplified surjective endomorphism f of X to the subvariety Y continues to be amplified. However, by its construction $f|_Y$, is actually the identity map, which is clearly a contradiction to $f|_Y$ being amplified.

Clearly, any amplified surjective endomorphism f of a projective variety X over the complex numbers is quasi-amplified. Moreover, it follows from [12, Theorem 1.4] that f is quasi-amplified if and only if f is 1-cohomologically hyperbolic. By [12, Theorem 6.1], for any 1-cohomologically hyperbolic surjective endomorphism f of a projective variety the set of all f_K-periodic points is Zariski dense in X_K, and thus f should be PCD, as in the arguments above.

In a similar vein, our next aim of this paper is to prove the following theorem.

Theorem 1.3. *Let X be a projective variety of dimension d over a field $\mathbf{k}$. Let α be a real number greater than or equal to 1, and let $f : X \to X$ be a surjective α-amplified endomorphism of X of positive entropy. Assume that the set of periodic points of f is Zariski dense in X. Then f is always PCD.*

The proof of Theorem 1.3 will be given in Section 3. We remark that if f is a PCD endomorphism of a smooth projective variety, then f is always of positive entropy (see [6, Lemma 3.4]). On the other hand, there is an example of a $K3$ surface admitting an automorphism which is not PCD (see [6, Example 5.7], attributed to K. Oguiso), while Example 5.8 in [6], attributed to D.-Q. Zhang, shows that some $K3$ surface admits a PCD automorphism which is not amplified.

It should be also pointed out that there are some projective varieties X with an endomorphism of positive entropy whose set of periodic points is not Zariski dense in X, which are well known to the experts. For a concrete example, let E be an elliptic curve over an algebraic closed field of characteristic zero, and let $X = E \times E$. For a non-zero element $a \in E$, let $f : X \to X$ be a surjective endomorphism given by

$$f(x, y) = (x + a, 2y), \quad (x, y) \in X.$$

Then f has the dynamical degrees $\lambda_0(f) = 1$, $\lambda_1(f) = 4$, and $\lambda_2(f) = 4$, which in turn implies that we have the Lyapunov exponents $\mu_1(f) = 4$ and $\mu_2(f) = 1$. Note also that f has the topological degree equal to 2, and thus f is a non-isomorphic surjective endomorphism. Furthermore, with the choice of a suitable non-torsion element $a \in E$, f has no periodic point at all.

For another example we have learned recently, recall first that the general linear group $GL(2, \mathbb{Z})$ acts on $E \times E$ by automorphisms. Thus any element A of $GL(2, \mathbb{Z})$ gives rise to an automorphism f_A of $E \times E$, and the first dynamical degree $\lambda_1(f_A)$ is greater than 1 if and only if the matrix A has an eigenvalue of modulus greater than 1. Now, for such an A and a non-torsion element a in E, we define an automorphism g_A of $E \times (E \times E)$, as follows.

$$g_A : E \times (E \times E) \to E \times (E \times E), \quad (x, (y, z)) \mapsto (x + a, f_A(x, y)).$$

Then the first dynamical degree $\lambda_1(g_A)$ of g_A is greater than 1, since we have $\lambda_1(f_A) > 1$. But it is clear that g_A does not have any periodic points at all, since a is a non-torsion element of E.

In order to make the proofs of our results easier to understand throughout this paper, we now want to collect some important subspaces of $N^1(X)_{\mathbb{R}}$ and $N_1(X)_{\mathbb{R}}$. That is, $\mathrm{Nef}(X)$ denotes the cone of classes of nef $\mathbb{R}$-Cartier divisors in $N^1(X)_{\mathbb{R}}$, while $\mathrm{Amp}(X)$ denotes the cone of classes of ample $\mathbb{R}$-Cartier divisors in $N^1(X)_{\mathbb{R}}$ and $\mathrm{Big}(X)$ denotes the cone of classes of big $\mathbb{R}$-Cartier divisors in $N^1(X)_{\mathbb{R}}$. On the other hand, $\mathrm{PE}^1(X)$ (or $\overline{\mathrm{Eff}}(X)$) denotes the closure of the cone of classes of effective $\mathbb{R}$-Cartier divisors in $N^1(X)_{\mathbb{R}}$, and $\overline{\mathrm{NE}}(X)$ denotes the closure of the cone of classes of effective 1-cycles with real coefficients in $N_1(X)_{\mathbb{R}}$.

This paper is organized as follows. In Section 2, we prove Theorem 1.1 which characterizes surjective α-amplified endomorphisms of a projective variety which is not an automorphism. In the same section, we also give some characterization of surjective α-amplified endomorphisms of a projective variety (see Lemma 2.1 for more details). Section 3 is devoted to proving Theorem 1.3 and giving some more properties that are related to surjective α-amplified (or α-quasi-amplified) endomorphisms of projective varieties.

2. Proof of Theorem 1.1

The aim of this section is to give a proof of Theorem 1.1.

To do so, we first begin with the following lemma which is analogous to [6, Proposition 3.1].

Lemma 2.1. *Let X be a projective variety of dimension d over a field $\mathbf{k}$, and let $f : X \to X$ be a surjective endomorphism of X. Let α be a positive real number. Then the following two statements are equivalent.*

(1) *f is α-amplified.*
(2) *For any $Z \in \overline{\mathrm{NE}}(X)$, $f_*(Z) \equiv \alpha Z$ implies $Z \equiv 0$.*

Proof. For the proof, we first suppose that f is α-amplified. By definition, there is an element $D \in N^1(X)_{\mathbb{R}}$ such that $f^*(D) - \alpha D$ is ample. Thus, for any pseudo-effective 1-cycle Z which is not weakly numerically equivalent to 0, we have

$$\begin{aligned} 0 &< (f^*(D) - \alpha D) \cdot Z = f^*(D) \cdot Z - \alpha D \cdot Z = D \cdot f_*(Z) - \alpha D \cdot Z \\ &= D \cdot (f_*(Z) - \alpha Z). \end{aligned}$$

Thus $f_*(Z)$ is not weakly numerically equivalent to αZ, i.e., $f_*(Z) \not\equiv \alpha Z$.

For the converse, we suppose that f is not α-amplified. For simplicity, let V be the image of the space $N^1(X)_{\mathbb{R}}$ under the map $f^* - \alpha\mathrm{Id}$. Then, by assumption we have

$$V \cap \mathrm{Amp}(X) = \emptyset.$$

Thus there exists some 1-cycle $Z \not\equiv 0$ such that $L \cdot Z = 0$ for any $L \in V$ and such that $A \cdot Z > 0$ for any $A \in \mathrm{Amp}(X)$.

Now, if $Z \notin \overline{\mathrm{NE}}(X)$, then there exists some Cartier divisor E such that $E \cdot Z \leq 0$ and $E \cdot C > 0$ for any nonzero $C \in \overline{\mathrm{NE}}(X)$. By the Kleiman's ampleness criterion ([4, Theorem 1.18] or [3]), it follows that we have $E \in \mathrm{Amp}(X)$. But this is a contradiction, since $E \cdot Z \leq 0$. Thus $Z (\not\equiv 0)$ is actually an element of $\overline{\mathrm{NE}}(X)$.

Finally, note that for any Cartier divisor D it follows from the choice of Z that we have

$$D \cdot (f_*(Z) - \alpha Z) = f^*(D) \cdot Z - \alpha D \cdot Z = (f^*(D) - \alpha D) \cdot Z = 0.$$

Thus we have $f_*(Z) \equiv \alpha Z$, which completes the proof. □

As a consequence of Lemma 2.1, we have the following corollary.

COROLLARY 2.2. *Let X be a projective variety of dimension d over a field $\mathbf{k}$. Let α be a positive real number, and let $f : X \to X$ be a surjective α-amplified endomorphism of X. Then the first dynamical degree $\lambda_1(f)$ of f is not equal to α.*

Proof. For the proof, suppose that the first dynamical degree $\lambda_1(f)$ of f is equal to α. Then, it follows from the well-known Perron-Frobenius theorem that there exists some $Z \in \overline{\mathrm{NE}}(X) - \{0\}$ such that

$$f_*(Z) \equiv \lambda_1(f) Z \equiv \alpha Z.$$

By Lemma 2.1, it is also true that Z should be weakly numerically equivalent to 0. This is clearly a contradiction, as desired. □

The following lemma also holds (see [6, Lemma 3.2]).

LEMMA 2.3. *Let $f : X \to X$ be a surjective endomorphism of null entropy of a projective variety X over a field $\mathbf{k}$. Then all of the eigenvalues of $f^*|_{N^1(X)_{\mathbb{R}}}$ are roots of unity and f is an automorphism of X.*

Now, we are ready to give a proof of Theorem 1.1, as follows.

Proof of Theorem 1.1. For the proof, first note that if f is of null entropy, then it follows from Lemma 2.3 that the first dynamical degree $\lambda_1(f)$ of f should be equal to 1 and f is actually an automorphism of X. But clearly this is a contradiction to the assumption that f is non-isomorphic. As a consequence, f should be of positive entropy.

Finally, note that by Corollary 2.2, the first dynamical degree $\lambda_1(f)$ of f is not equal to α, as desired. This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.3 and some more properties

The main aim of this section is to give a proof of Theorem 1.3.

Recall that if f is α-amplified, then f should be also α-quasi-amplified by definition. As mentioned in Section 1, it then follows from [12, Theorem 1.4] that α does not belong to the $\mathrm{Big}(X)$-spectrum

$$\mathrm{Sp}(f^*, \mathrm{Big}(X)) = \{\mu_i(f) \,|\, i = 1, 2, \ldots, d\}$$

for f^*. Here the $\mathrm{Big}(X)$-spectrum $\mathrm{Sp}(f^*, \mathrm{Big}(X))$ for f^* means the set of all real numbers α for which f^* is not α-quasi-amplified. In particular, the first dynamical degree $\lambda_1(f) = \mu_1(f)$ of f is not equal to α (see also Theorem 1.1 and some remarks just before Theorem 1.1). Furthermore, if, in addition, f is of null entropy, then α is not equal to $\lambda_1(f) = 1$ and f is an automorphism of X.

Now, we give a proof of Theorem 1.3, as follows.

Proof of Theorem 1.3. By assumption, note first that the set of periodic points of f is Zariski dense in X. We then claim that the set of periodic points of f is actually countably infinite, as in the arguments given in Section 1 (see pages 4 and 5 of this paper for more details). Indeed, the set of uncountably many periodic points would produce a positive dimensional pointwise fixed subvariety Y possibly after some iteration, and the restriction $f|_Y$ of a surjective α-amplified endomorphism f of X to the subvariety Y continues to be surjective and α-amplified. However, by its construction $f|_Y$ is actually the identity map. But this gives rise to a contradiction.

To be more precise, since f is α-amplified, there is an element $M \in N^1(X)_{\mathbb{R}}$ such that $A_X := f^*(M) - \alpha M$ is ample on X. Let $j : Y \hookrightarrow X$ be the inclusion map. Then we have

$$\begin{aligned} j^*(A_X) &= j^*f^*(M) - \alpha j^*(M) = (f|_Y)^*(j^*(M)) - \alpha j^*(M) \\ &= (1-\alpha)j^*(M) \end{aligned} \tag{3.1}$$

which is ample on Y. For simplicity, let $A_Y := j^*(A_X)$ and $N := j^*(M)$.

If α is equal to 1, it follows from (3.1) that the ample divisor A_Y is equal to 0, which is clearly a contradiction. So, from now on we assume that α is greater than 1. Then, since $(1-\alpha)N$ is ample on Y, it is true that for any pseudo-effective 1-cycle Z which is not numerically equivalent to 0, we have

$$(1-\alpha)N \cdot Z > 0.$$

But then, since α is assumed to be greater than 1, we have $N \cdot Z < 0$ for any pseudo-effective 1-cycle Z which is not numerically equivalent to 0. This, in particular, implies that the dual element of N in $N_1(X)_{\mathbb{R}}$ is always contained in any pseudo-effective 1-cycle Z which is not numerically equivalent to 0. This is a contradiction. This completes the proof of Theorem 1.3. □

Finally, we close this section with collecting a few basic facts that are related to surjective α-amplified (or α-quasi-amplified) endomorphisms of a projective variety, as follows.

Proposition 3.1. *The following statements hold:*

(1) *Let $\pi : X \to Y$ be a finite surjective morphism of projective varieties X and Y over a field $\mathbf{k}$. Let $f : X \to X$ and $g : Y \to Y$ be surjective endomorphisms such that $g \circ \pi = \pi \circ f$. If f is α-amplified (resp. α-quasi-amplified), then g is also α-amplified (resp. α-quasi-amplified).*
(2) *Let $f : X \to X$ and $g : Y \to Y$ be surjective endomorphisms of projective varieties X and Y over a field $\mathbf{k}$, respectively. Let α be a positive real number. Then $f \times g : X \times Y \to X \times Y$ is α-amplified (resp. α-quasi-amplified) if and only if f and g are α-amplified (resp. α-quasi-amplified).*

Proof. (1) The proof of the statement (1) can be found in [12, Corollary 1.6].

(2) The proof of the statement (2) can be easily adapted from that of [6, Lemma 3.7]. For the sake of reader's convenience, we quickly give a proof.

Indeed, suppose that $A_X := f^*(N) - \alpha N$ and $A_Y := g^*(M) - \alpha M$ are ample (resp. big) for some $N \in N^1(X)_{\mathbb{R}}$ and $M \in N^1(Y)_{\mathbb{R}}$. Let

$$D = p_X^*(N) + p_Y^*(M),$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ denote the natural projections onto the first and second factor, respectively. Then it is easy to obtain

$$\begin{aligned}(f \times g)^*(D) - \alpha D &= p_X^* f^*(N) + p_Y^* g^*(M) - \alpha p_X^*(N) - \alpha p_Y^*(M) \\ &= (p_X^* f^*(N) - \alpha p_X^*(N)) + (p_Y^* g^*(M) - \alpha p_Y^*(M)) \\ &= p_X^*(A_X) + p_Y^*(A_Y)\end{aligned}$$

Since $p_X^*(A_X) + p_Y^*(A_Y)$ is ample (resp. big) on $X \times Y$, $f \times g$ is α-amplified (resp. α-quasi-amplified).

Conversely, suppose first that $B := (f \times g)^*(D) - \alpha D$ is big for some $D \in N^1(X)_{\mathbb{R}}$. Then there exists an ample class $A \in N^1(X)_{\mathbb{R}}$ and an effective class $E \in N^1(X)_{\mathbb{R}}$ such that $B = A + E$. Since $X \times \{y\}$ is not contained in the support of E for a general element $y \in Y$, $B|_{X \times \{y\}}$ is big and thus f is α-quasi-amplified. Similarly, g is also α-quasi-amplified. So we are done.

It is not difficult to see that the above arguments with $E = 0$ can be also used to prove that, when $f \times g$ is α-amplified, f and g are also α-amplified. This completes the proof of Proposition 3.1. $\square$

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