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# $\mathcal{Z}$-SOLITONS ON STATISTICAL SUBMERSIONS

Shahroud Azami and Mehdi Jafari*

Abstract. In this research article, we study $\mathcal{Z}$-solitons on statistical submersions with parallel vertical or horizontal distribution. Finally, we study $\mathcal{Z}$-solitons on statistical submersions with conformal or gradient potential vector field.

In the field of differential geometry, examining submersion structures offers a profound framework for connecting manifolds of different dimensions and geometries. Among these, Riemannian and statistical submersions serve as central tools in exploring curvature-related properties and geometric flows. Moreover, the interaction of such structures with soliton theories underlines a rich area of geometric analysis with both mathematical depth and physical significance.

The foundational concept of Riemannian submersions was initially formulated by B. O'Neill [22] and A. Gray [17]. Subsequently, in 1983, B. Watson [41] introduced and extended the framework of Riemannian submersions into the realm of almost Hermitian geometry, coining the term "almost Hermitian submersions." Since then, researchers have explored such submersions across diverse subclasses within the almost Hermitian category. In parallel, Riemannian submersions have been adapted to several types of almost contact metric manifolds, leading to the development of the notion known as contact Riemannian submersions [27]. Comprehensive discussions and systematic treatments of these topics, ranging from classical Riemannian to almost Hermitian and contact Riemannian submersions, can be found in standard texts such as [16, 28]. Moreover, a profound link between Riemannian submersions and minimal immersions has been investigated, including cohomological considerations, as discussed in [9].

The theory of statistical submersions (SSs) within the setting of statistical manifolds lies at the intersection of differential geometry and information geometry. By extending core principles originally introduced by B. O'Neill in the study of Riemannian submersions and geodesic behavior, Abe and Hasegawa [1] generalized the framework to encompass statistical structures. Following their pioneering efforts, numerous contributions have enriched the field, see for instance [34–36]. More recently, various specialized forms of SSs have emerged, including cosymplectic-type [3], quaternionic Kähler-type [40], and para-Kähler-type SSs [39]. Building upon foundational insights by Takano, M. D. Siddiqi and collaborators [31] developed an extensive treatment of Kenmotsu-type SSs. Further advancements include the investigation of

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* Corresponding author.
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holomorphic SSs by S. Kazan and K. Takano [21], who also introduced the concept of anti-invariant SSs within holomorphic statistical settings. A detailed examination of locally product-type SSs was also presented in [37].

In 2019, S. E. Meriç and E. Kılıç [15] explored the characteristics of Riemannian submersions where the total manifolds possess the structure of a Ricci soliton. M. D. Siddiqi et al. studied submersions with some solitons in [29,30]. O. Barndorff-Nielsen and P. E. Jupp [4] discussed Riemannian submersions from the viewpoint of statistics. K. Takano [34] put forward the discourse of the SSs of various spaces. Statistical solitons have been studied by Siddiqui et al. [32] for warped product statistical submanifolds. Recently, A. M. Blaga and B.-Y. Chen also studied in [5,6] many properties of gradient solitons in statistical manifolds which are closely related to this topic. In [10], Chen et al. studied Ricci–Bourguignon solitons on SSs with parallel vertical or horizontal distribution and conformal or gradient potential vector field.

Therefore, in this article we investigate SSs whose total space admits AZSs.

## 1. Preliminaries

This section reviews essential geometric tools and definitions that underpin the main results of this paper. We present key notions related to geometric flows, $\mathcal{Z}$-tensors, and soliton structures, which will be employed in our investigations of SSs with solitonic properties.

The Ricci flow on a Riemannian manifold $(M, g_0)$, a significant geometric flow, is defined by

$$\frac{\partial}{\partial t} g = -2S, \qquad g(0) = g_0, \tag{1}$$

where $S$ represents the Ricci curvature corresponding to $g$. The concept was initially presented by Hamilton [19] and pertains to the evolution of a metric as influenced by its Ricci curvature. Hamilton and DeTurck [14] investigated the existence and uniqueness of solutions for (1). Additionally, various researchers have explored evolution equations for geometric structures [11].

The Ricci-Bourguignon flow, conceived as a natural generalization of the Ricci flow [20], was introduced by J. P. Bourguignon in 1981 [7], drawing inspiration from unpublished notes by A. Lichnerowicz and foundational results by T. Aubin [2]. This flow defines an evolution equation for Riemannian metrics, governed by curvature constraints that interpolate between scalar and Ricci curvature. As a geometric flow, it possesses a deeply intrinsic character, and its stationary solutions are characterized by special self-similar structures known as solitons.

The idea of the $\mathcal{Z}$-tensor was first presented by Mantica and Molinari [23], who examined its role in the structure of weakly $\mathcal{Z}$-symmetric manifolds. A $(0, 2)$-symmetric tensor field $\mathcal{Z}$ is termed a $\mathcal{Z}$-tensor [26] if it satisfies the relation

$$\mathcal{Z}(U_1, U_2) = S(U_1, U_2) + \phi\, g(U_1, U_2),$$

where $S$ denotes the Ricci tensor, $g$ is the Riemannian metric, and $\phi$ is a scalar-valued function. The notion arises naturally within the broader framework of weakly symmetric spaces, as explored in [8]. Importantly, the $\mathcal{Z}$-tensor serves as a unifying generalization, capturing features of the Einstein gravitational tensor as it appears in general relativity. Several geometric studies have delved into the behavior of manifolds admitting such structures. Notably, Mantica and Suh [24, 25] analyzed pseudo $\mathcal{Z}$-symmetric configurations in both Lorentzian spacetimes and Riemannian manifolds. Furthermore, U. C. De and collaborators [12, 13] investigated weakly cyclic $\mathcal{Z}$-symmetric geometries, contributing to a deeper understanding of symmetry properties in curvature-related tensorial contexts.

Let $(M^n, g)$ be a pseudo-Riemannian manifold. Such a manifold is said to admit an *almost $\mathcal{Z}$-soliton* (abbreviated as AZS) when there exists a scalar function $\lambda$ on $M$ and a smooth vector field $V$ such that the following geometric identity holds:

$$\mathcal{Z} + \mathcal{L}_V g + \lambda g = 0, \tag{2}$$

where $\mathcal{L}_V$ denotes the Lie derivative along $V$, and $\mathcal{Z}$ is a $(0, 2)$-tensor satisfying a structure condition as discussed earlier. Also, $V$ is referred to as the *potential field*. When the scalar function $\lambda$ is constant across $M$, then the AZS reduces to a standard *$\mathcal{Z}$-soliton* (ZS). Based on the sign of $\lambda$, the soliton is categorized as *shrinking* $(\lambda < 0)$, *steady* $(\lambda = 0)$, or *expanding* $(\lambda > 0)$.

Moreover, if $V = \mathrm{grad} h$ on $M$, then (2) assumes as:

$$\mathcal{Z} + 2\nabla\nabla h + \lambda g = 0, \tag{3}$$

in which case the soliton is termed a *gradient almost $\mathcal{Z}$-soliton*, and $h$ is identified as the potential function. Equation (2) serves as a unifying framework that encapsulates several important solitonic structures. For example:

- When $\phi = 0$, the relation corresponds to an *almost Ricci soliton*.
- When $\phi = r\mu$ for some constant $\mu$ (with $r$ the scalar curvature), the equation recovers the *almost Ricci–Bourguignon soliton*.
- When $\phi = -(p + \frac{1}{n})$, it models an *almost conformal Ricci soliton*.

Thus the parameter $\phi$ serves as an analytic interpolator among distinct soliton classes, and the $Z$-framework yields a natural generalization capable of encompassing several well-studied structures within a single differential-geometric scheme.

This framework provides a versatile setting for generalizing geometric flows and soliton solutions in pseudo-Riemannian geometry.

Example 1.1. Consider the smooth manifold $M = \{(x, y, z, t) \in \mathbb{R}^4 \mid t \neq 0\}$, where $(x, y, z, t)$ are the standard coordinates on $\mathbb{R}^4$. Define the following globally defined vector fields:

$$u_1 = e^t \frac{\partial}{\partial x}, \quad u_2 = e^t \frac{\partial}{\partial y}, \quad u_3 = e^t \frac{\partial}{\partial z}, \quad u_4 = \frac{\partial}{\partial t}.$$

It is immediate that $\{u_1, u_2, u_3, u_4\}$ forms a pointwise linearly independent frame on $M$. We introduce a pseudo-Riemannian metric $g$ defined by the relations

$$g(u_i, u_j) = \begin{cases} 1 & \text{if } i = j \in \{1, 2, 3\}, \\ -1 & \text{if } i = j = 4, \\ 0 & \text{if } i \neq j. \end{cases}$$

Next, consider the 1-form $\omega$ determined by $\omega(X) = g(X, u_4)$ for all vector field $X$ on $M$. Direct computation yields the non-vanishing Lie brackets

$$[u_i, u_4] = -u_i \quad \text{for } i = 1, 2, 3, \quad \text{and} \quad [u_i, u_j] = 0 \text{ otherwise}.$$

The Levi-Civita connection $\nabla$ related to $g$ has nonzero components given by:

$$\nabla_{u_i} u_j = \begin{pmatrix} u_4 & 0 & 0 & -u_1 \\ 0 & u_4 & 0 & -u_2 \\ 0 & 0 & u_4 & -u_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows from the curvature computations that the Ricci tensor $S$ satisfies

$$S(u_i, u_i) = -3 \quad \text{for } i = 1, 2, 3, 4, \qquad S(u_i, u_j) = 0 \text{ for } i \neq j.$$

Using the definition of the Lie derivative, one obtains:

$$(\mathcal{L}_{u_4} g)(u_i, u_i) = -2 \quad \text{for } i = 1, 2, 3, \qquad \text{and zero otherwise.}$$

Therefore, the components of the $\mathcal{Z}$-tensor take the form:

$$\mathcal{Z}(u_i, u_j) = \begin{cases} -3 + \phi & \text{if } i = j \in \{1, 2, 3\}, \\ -3 + \phi & \text{if } i = j = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let us take the potential vector field $V = u_4$, and set $\phi = \frac{3}{2}$ and $\lambda = \frac{7}{2}$. Under these choices, the tuple $(M^4, g, \phi, u_4, \lambda)$ satisfies the $\mathcal{Z}$-soliton equation, and hence defines a $\mathcal{Z}$-soliton (ZS) structure on $M$. Moreover, this soliton is characterized as:

- *shrinking* if $\phi > 5$,
- *steady* if $\phi = 5$,
- *expanding* if $\phi < 5$.

## 2. Basics of SSs

Let $M$ and $N$ represent two Riemannian manifolds, and consider a Riemannian submersion $\psi : M \to N$ such that each fiber constitutes a Riemannian submanifold of $M$, and the differential $\psi_*$ is an isometry when restricted to horizontal distributions (refer to [16,28] for further elaboration). In their 2001 work, N. Abe and K. Hasegawa [1] explored affine submersions endowed with horizontal distributions in the framework of statistical geometry. Additionally, K. Takano contributed to the development of the theory by analyzing SSs in [34, 35].

Let $(M, g)$ be a Riemannian manifold and suppose $\nabla$ is an affine connection on $M$ that is symmetric, i.e., torsion-free. The triple $(M, \nabla, g)$ is referred to as a *statistical manifold* if the covariant derivative of $g$ with respect to $\nabla$ is totally symmetric in its arguments; explicitly, for any vector fields $U_1, U_2, U_3$ on $M$, one requires

$$(\nabla_{U_1} g)(U_2, U_3) = (\nabla_{U_2} g)(U_1, U_3),$$

as discussed in [34]. Given such a statistical manifold, one naturally introduces the *conjugate connection* $\nabla^*$ associated to $\nabla$ and $g$, defined by the condition

$$U_3 \cdot g(U_1, U_2) = g(\nabla_{U_3} U_1, U_2) + g(U_1, \nabla^*_{U_3} U_2),$$

for any vector fields $U_1, U_2, U_3$. The dual connection $\nabla^*$ is itself torsion-free and satisfies an analogous symmetry condition:

$$(\nabla^*_{U_1} g)(U_2, U_3) = (\nabla^*_{U_2} g)(U_1, U_3),$$

with the additional property that taking the dual twice returns the original connection, i.e., $(\nabla^*)^* = \nabla$. Consequently, the structure $(M, \nabla^*, g)$ also forms a statistical manifold. A canonical example is furnished by any Riemannian manifold where $\nabla$ is the Levi-Civita connection, yielding a trivial statistical structure. In such cases, denoting by R and $\mathrm{R}^*$ the curvature tensors corresponding to $\nabla$ and $\nabla^*$, respectively, one finds the identity

$$g(\mathrm{R}(U,V)W,X) = -g(W,\mathrm{R}^*(U,V)X).$$

See [34] for further details. In addition, the Ricci tensor $S$ and scalar curvature $r$ are defined, respectively by
$S(U,V) = \mathrm{trace}(W \to \mathrm{R}(U,V)X) = -\,\mathrm{trace}\,(W \to \mathrm{R}^*(U,V)X)$,
or

$$S(U,W) = \sum_{i,j=1}^{q} g\left(\mathrm{R}\left(e_i,V\right)W,e_i\right),$$

$$r = \sum_{i,j=1}^{q} g\left(\mathrm{R}\left(e_i,e_j\right)e_j,e_i\right) = -\sum_{i,j=1}^{q} g\left(\mathrm{R}^*\left(e_i,e_j\right)e_j,e_i\right)$$

Definition 2.1. Suppose $\psi : (M^p, g) \to (N^q, \hat{g})$ is a Riemannian submersion. For any $x \in N$, the preimage $\psi^{-1}(x)$ defines a $(p-q)$-dimensional submanifold of $M$, referred to as a *fiber* and denoted by $\bar{M}$, equipped with the induced metric $\bar{g}$ from $g$.

At any point $x \in M$, the tangent space $T_xM$ admits an orthogonal decomposition into vertical and horizontal components:

$$T_xM = \mathcal{V}_x(M) \oplus \mathcal{H}_x(M),$$

where $\mathcal{V}(M)$ and $\mathcal{H}(M)$ designate the vertical and horizontal distributions of $TM$, respectively.

We denote by $\Gamma\mathcal{V}(M)$ and $\Gamma\mathcal{H}(M)$ the $C^\infty(M)$-modules consisting of all smooth sections of $\mathcal{V}(M)$ and $\mathcal{H}(M)$.

A smooth vector field $U$ on $M$ is called *projectable* whenever there exists a vector field $U_*$ on $N$ such that for each $x \in M$, the differential $\psi_* : TM \to TN$ satisfies

$$\psi_*(U_x) = (U_*)_{\psi(x)}.$$

In this situation, the pair $(U, U_*)$ is termed *$\psi$-related.* Moreover, a horizontal vector field $U \in \Gamma\mathcal{H}(M)$ is called *basic* if it is projectable; see [22] for further discussion.

Suppose that $U$ and $V$ are basic vector fields on $M$, which are $\psi$-related to $\hat{U}$ and $\hat{V}$ on $N$, respectively. Then the following relations hold:

(1) The metrics are compatible under the submersion: $g(U,V) = \hat{g}(\hat{U},\hat{V}) \circ \psi$,
(2) The horizontal projection of the Lie bracket satisfies $\mathcal{H}[U,V]$ is basic and $\psi$-related to $[\hat{U},\hat{V}]$,
(3) The horizontal component of the covariant derivative obeys $\mathcal{H}(\nabla_U V)$ is basic and $\psi$-related to $\hat{\nabla}_{\hat{U}}\hat{V}$.

The differential geometric structure induced by a Riemannian submersion $\psi : (M,g) \to (N,\hat{g})$ is governed by the O'Neill tensors $\mathcal{T}$ and $\mathcal{A}$, introduced in [22], and defined as:

$$\mathcal{T}_E F = \mathcal{V}(\nabla_{\mathcal{V}E}\mathcal{H}F) + \mathcal{H}(\nabla_{\mathcal{V}E}\mathcal{V}F),$$
$$\mathcal{A}_E F = \mathcal{V}(\nabla_{\mathcal{H}E}\mathcal{H}F) + \mathcal{H}(\nabla_{\mathcal{H}E}\mathcal{V}F),$$

for arbitrary vector fields $E, F$ on $M$. These $(1,2)$-tensor fields serve as obstruction terms measuring the integrability of horizontal and vertical distributions. Notably, both $\mathcal{T}_E$ and $\mathcal{A}_E$ are $g$-skew-symmetric and interchange vertical and horizontal components.

Some fundamental identities involving $\mathcal{T}$ and $\mathcal{A}$ include:

$$\mathcal{T}_V W = \mathcal{T}_W V, \quad \text{for all vertical vector fields } V, W,$$
$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X,Y], \quad \text{for horizontal vector fields } X, Y.$$

Suppose $(M, \nabla, g)$ is a statistical manifold and let $\psi : M \to N$ be a Riemannian submersion. For each fiber $\bar{M} = \psi^{-1}(x)$, we denote by $\bar{\nabla}$ and $\bar{\nabla}^*$ the induced affine connections on $\bar{M}$. It follows directly from the definitions that both $\bar{\nabla}$ and $\bar{\nabla}^*$ are torsion-free and form a conjugate pair with respect to the Riemannian metric $g$.

A smooth map $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ between two statistical manifolds is termed a statistical submersion ($SS$) if, for all basic vector fields $X, Y$ on $M$ and any point $x \in M$, the following compatibility condition holds:

$$\psi_*\big((\nabla_X Y)_x\big) = \big(\hat{\nabla}_{\psi_* X}\psi_* Y\big)_{\psi(x)}.$$

Replacing the connection $\nabla$ by its conjugate $\nabla^*$ in the above context leads to the corresponding conjugate O'Neill tensors $\mathcal{T}^*$ and $\mathcal{A}^*$, as introduced in [34]. The horizontal distribution $\mathcal{H}(M)$ is said to be integrable with respect to $\nabla$ (resp. $\nabla^*$) if and only if the tensor $\mathcal{A}$ (resp. $\mathcal{A}^*$) vanishes identically.

Moreover, for $X, Y \in \Gamma\mathcal{H}(M)$ and $V, W \in \Gamma\mathcal{V}(M)$, the following identities relate the classical and dual O'Neill tensors:

$$g\left(\mathcal{T}_V W, X\right) = -g\left(W, \mathcal{T}^*_V X\right), \quad g\left(\mathcal{A}_X Y, V\right) = -g\left(Y, \mathcal{A}^*_X V\right).$$

We now recall several important structural identities associated with SSs, originally established by K. Takano in [34], which play a significant role in the geometric analysis of such maps. The following lemmas, drawn from the same reference, provide foundational tools for our study.

Lemma 2.2 ( [34]). *Let $X, Y \in \Gamma\mathcal{H}(M)$. Then the following duality relation holds between the O'Neill tensors:*

$$\mathcal{A}_X Y = -\mathcal{A}^*_Y X.$$

Lemma 2.3 ( [34]). *Let $X, Y \in \Gamma\mathcal{H}(M)$ and $E, F \in \Gamma\mathcal{V}(M)$ be horizontal and vertical vector fields, respectively. Then the covariant derivatives with respect to $\nabla$ and its dual $\nabla^*$ decompose as follows:*

$$\nabla_E F = \mathcal{T}_E F + \mathcal{V}(\nabla_E F), \qquad \nabla^*_E F = \mathcal{T}^*_E F + \mathcal{V}(\nabla^*_E F), \tag{4}$$
$$\nabla_E X = \mathcal{T}_E X + \mathcal{H}(\nabla_E X), \qquad \nabla^*_E X = \mathcal{T}^*_E X + \mathcal{H}(\nabla^*_E X), \tag{5}$$
$$\nabla_X F = \mathcal{A}_X F + \mathcal{V}(\nabla_X F), \qquad \nabla^*_X F = \mathcal{A}^*_X F + \mathcal{V}(\nabla^*_X F), \tag{6}$$
$$\nabla_X Y = \mathcal{H}(\nabla_X Y) + \mathcal{A}_X Y, \qquad \nabla^*_X Y = \mathcal{H}(\nabla^*_X Y) + \mathcal{A}^*_X Y. \tag{7}$$

Moreover, when $X$ is a basic vector field, the horizontal component of the covariant derivative of $X$ along a vertical vector field $V$ satisfies $\mathcal{H}\nabla_V X = \mathcal{A}_X V$ and likewise $\mathcal{H}\nabla^*_V X = \mathcal{A}^*_X V$.

Let $R$ (respectively, $R^*$) denote the curvature tensor on $M$ associated with the affine connection $\nabla$ (respectively, $\nabla^*$). The curvature tensors corresponding to the induced affine connections $\bar{\nabla}$ and $\bar{\nabla}^*$ on the fibers are written as $\overline{R}$ and $\overline{R}^*$, respectively. Additionally, the horizontal lifts of the curvature tensors on the base manifold are given by $\hat{R}(W,Y)X$ and $\hat{R}^*(W,Y)X$, satisfying

$$\psi_*(\hat{R}(W,Y)X) = \hat{R}(\psi_* W, \psi_* Y)\psi_* X,$$
$$(\text{respectively, } \psi_*(\hat{R}^*(W,Y)X) = \hat{R}^*(\psi_* W, \psi_* Y)\psi_* X),$$

for each point $p \in M$, where $\hat{R}$ (respectively, $\hat{R}^*$) corresponds to the curvature tensor of the affine connection $\hat{\nabla}$ (respectively, $\hat{\nabla}^*$) defined on the base manifold $N$.

We now have:

Lemma 2.4 ( [34]). *Let $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ be a SS. Then, the curvature tensors satisfy the following relations:*

$$R(E,F,G,K) = \overline{R}(E,F,G,K) + g(\mathcal{T}_E G, \mathcal{T}^*_F K) - g(\mathcal{T}_F G, \mathcal{T}^*_E K),$$

$$\begin{aligned} R(X,Y,Z,W) = \hat{R}(X,Y,Z,W) &+ g\big((\mathcal{A}_X + \mathcal{A}^*_X)Y, \mathcal{A}^*_Z W\big) \\ &- g(\mathcal{A}_Y Z, \mathcal{A}^*_X W) + g(\mathcal{A}_X Z, \mathcal{A}^*_Y W), \end{aligned}$$

*where $E,F,G,K \in \Gamma\mathcal{V}(M)$ and $X,Y,Z,W \in \Gamma\mathcal{H}(M)$.*

Lemma 2.5 ( [34]). *Consider the Ricci curvature tensors $S$, $\hat{S}$, and $\overline{S}$ corresponding respectively to the statistical manifolds $(M, \nabla, g)$, $(N, \hat{\nabla}, \hat{g})$, and to any fiber of the submersion $\psi$. Then, for arbitrary vertical vector fields $E, F \in \Gamma\mathcal{V}(M)$, the Ricci tensor on $M$ decomposes as*

$$\begin{aligned} S(E,F) = \overline{S}(E,F) - g(\mathcal{T}_E F, N^*) &+ \sum_{i=1}^{n} \left[g\big((\nabla_{X_i}\mathcal{T})(E,F), X_i\big) + g\big(\mathcal{A}_{X_i}E, \mathcal{A}^*_{X_i}F\big)\right] \\ &- \sum_{j=1}^{m} g\left((\nabla^*_{E_j}\mathcal{A})(X_i, X), V\right), \end{aligned}$$

*and for any $X, Y \in \Gamma\mathcal{H}(M)$, it satisfies*

$$\begin{aligned} S(X,Y) = \hat{S}(X,Y) + g\left(\nabla^*_X N^*, Y\right) &- \sum_{j=1}^{m} g(\mathcal{T}_{E_j}X, \mathcal{T}_{E_j}Y) \\ &+ \sum_{i=1}^{n} \left[g(\mathcal{A}_X X_i, \mathcal{A}^*_Y X_i) + g\big((\nabla_{X_i}\mathcal{A})(X_i, X), Y\big)\right] \\ &+ \sum_{i=1}^{n} \left[g(\mathcal{A}_{X_i}X_i, \mathcal{A}_X Y) - g(\mathcal{A}^*_X X_i, \mathcal{A}^*_X X_i)\right]. \end{aligned}$$

*Here, $\{X_i\}_{i=1}^n$ and $\{E_j\}_{j=1}^m$ denote local orthonormal frames spanning the horizontal distribution $\mathcal{H}(M)$ and the vertical distribution $\mathcal{V}(M)$, respectively.*

For each fiber associated with the SS $\psi$, the mean curvature vector field $H$ can be expressed as

$$H = \frac{1}{m} N,$$

where

$$N = \sum_{j=1}^{m} \mathcal{T}_{E_j} E_j. \tag{8}$$

Here, $m$ denotes the dimension of the fiber, and $\{E_1, E_2, \ldots, E_m\}$ represents an orthonormal frame chosen within the vertical distribution. It is noteworthy that the horizontal vector field $N$ vanishes precisely when each fiber of $\psi$ is minimal.

Differentiating equation (8), we obtain

$$g\big(\nabla_U N, X\big) = \sum_{j=1}^{m} g\big((\nabla_U \mathcal{T})(E_j, E_j), X\big), \tag{9}$$

valid for arbitrary $U \in \Gamma(TM)$ and $X \in \Gamma(\mathcal{H}(M))$.

Furthermore, given any tensor field $\mathcal{P}$, we define the following divergence-type operators:

$$\hat{\delta}\mathcal{P} = -\sum_{i=1}^{n} (\nabla_{X_i}\mathcal{P})_{X_i} \quad \text{and} \quad \bar{\delta}\mathcal{P} = -\sum_{j=1}^{m} \left(\nabla_{E_j}\mathcal{P}\right)_{E_j},$$

where $\{X_i\}_{i=1}^{n}$ and $\{E_j\}_{j=1}^{m}$ are orthonormal bases of the horizontal and vertical distributions, respectively.

The horizontal divergence of $X \in \Gamma(\mathcal{H}(M))$ is defined as [18]

$$\delta(X) = \sum_{i=1}^{n} g\big(\nabla_{X_i} X, X_i\big),$$

where $\{X_1, X_2, \ldots, X_n\}$ forms an orthonormal frame of the horizontal distribution $\mathcal{H}(M)$.

By combining this with equation (8) and applying the definitions, one derives the expression [18]

$$\delta(N) = \sum_{i=1}^{n}\sum_{j=1}^{m} g\big((\nabla_{X_i}\mathcal{T})(E_j, E_j), X_i\big), \tag{10}$$

which represents the horizontal divergence of the vector field $N$ defined previously.

In what follows, we investigate the Ricci and scalar curvatures associated with SSs. Utilizing Lemma 2.5 and the results in [34], one obtains the relations

$$\begin{aligned} S(E,F) = {}& \overline{S}(E,F) - g(\mathcal{T}_E F, N^*) + (\hat{\delta}\mathcal{T})(E,F) \\ & + g(\mathcal{A}E, \mathcal{A}^* F) - g\left(\nabla^*_E \sigma, F\right), \end{aligned} \tag{11}$$

$$\begin{aligned} S(X,Y) = {}& \hat{S}(X,Y) + g\big(\nabla^*_X N^*, Y\big) - g(\mathcal{T}X, \mathcal{T}Y) + (\hat{\delta}\mathcal{A})(X,Y) \\ & + g(\sigma, \mathcal{A}_X Y) - g(\mathcal{A}_X, \mathcal{A}^*_Y) - g(\mathcal{A}^*_X, \mathcal{A}^*_Y), \end{aligned} \tag{12}$$

where the terms involved are defined as follows:

$$\sigma = \sum_{i=1}^{n} \mathcal{A}_{X_i} X_i,$$

$$(\hat{\delta}\mathcal{T})(E,F) = \sum_{i=1}^{n} g\big((\nabla_{X_i}\mathcal{T})(E,F), X_i\big),$$

$$(\hat{\delta}\mathcal{A})(X,Y) = \sum_{j=1}^{m} g\big((\nabla_{E_j}\mathcal{A})(X,Y), E_j\big),$$

$$g(\mathcal{A}_X, \mathcal{A}_Y) = \sum_{i=1}^{n} g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) = \sum_{j=1}^{m} g(\mathcal{A}^*_X E_j, \mathcal{A}^*_Y E_j),$$

$$g(\mathcal{A}E, \mathcal{A}F) = \sum_{i=1}^{n} g(\mathcal{A}_{X_i} E, \mathcal{A}_{X_i} F), \quad g(\mathcal{T}X, \mathcal{T}Y) = \sum_{j=1}^{m} g(\mathcal{T}_{E_j} X, \mathcal{T}_{E_j} Y).$$

Here, $\{X_i\}_{i=1}^n$ and $\{E_j\}_{j=1}^m$ represent orthonormal bases of the horizontal and vertical distributions, respectively.

By applying Lemma 2.4 alongside the symmetric properties of $\mathcal{T}$ and its dual $\mathcal{T}^*$, we obtain the relation

$$2r = 2\bar{r} - m^2\, g\big(H, H^*\big) + \sum_{i,j=1}^{m} g\big(\mathcal{T}_{E_i} E_j, \mathcal{T}^*_{E_i} E_j\big),$$

where $r$ and $\bar{r}$ denote the scalar curvatures of $M$ and the fiber respectively, $m$ is the fiber dimension, and $H, H^*$ are the mean curvature vector fields associated to $\mathcal{T}$ and $\mathcal{T}^*$.

Starting from Lemma 2.4, we derive the following expression:

$$\begin{aligned} 2r = 2\hat{r} + \sum_{i,j=1}^{n} \Big[ & g\big(\mathcal{A}_{X_i} X_j, \mathcal{A}^*_{X_j} X_i\big) - g\big(\mathcal{A}_{X_j} X_j, \mathcal{A}^*_{X_i} X_i\big) \\ & + g\big(\mathcal{A}_{X_i} X_j, \mathcal{A}^*_{X_j} X_i\big) + g\big(\mathcal{A}^*_{X_i} X_j, \mathcal{A}^*_{X_j} X_i\big)\Big]. \end{aligned}$$

By employing the antisymmetry property $\mathcal{A}_X Y = -\mathcal{A}_Y X$ for $X, Y \in \Gamma\mathcal{H}(M)$ along with Lemma 2.2, this expression can be rewritten as:

$$\begin{aligned} 2r &= 2\hat{r} + \sum_{i,j=1}^{n} \Big[ - g\big(\mathcal{A}_{X_i} X_j, \mathcal{A}_{X_i} X_j\big) + g\big(\mathcal{A}_{X_j} X_j, \mathcal{A}_{X_i} X_i\big) \\ &\qquad - g\big(\mathcal{A}_{X_i} X_j, \mathcal{A}_{X_i} X_j\big) + g\big(\mathcal{A}_{X_i} X_j, \mathcal{A}_{X_i} X_j\big)\Big] \\ &= 2\hat{r} + g(\sigma, \sigma) - 3 \sum_{i,j=1}^{n} g\big(\mathcal{A}_{X_i} X_j, \mathcal{A}_{X_i} X_j\big). \end{aligned}$$

By combining equations (11) and (12) and utilizing the results from [34], we arrive at the following identity:

$$\begin{aligned} r - \bar{r} - \hat{r} = & -2\, g(\mathcal{A}, \mathcal{A}) + g(\mathcal{A}, \mathcal{A}^*) - g(\mathcal{T}, \mathcal{T}^*) - g(N, N^*) \\ & - \hat{\delta} N - \hat{\delta}^* N^* - \bar{\delta}\sigma + \bar{\delta}^*\sigma + g(\sigma, \sigma), \end{aligned} \tag{13}$$

where $\bar{r}$ and $\hat{r}$ denote the scalar curvatures corresponding to the vertical and horizontal distributions of the manifold $M$, respectively. Furthermore, the following notations are used:

$$g(\mathcal{T},\mathcal{T}^*)=\sum_{i=1}^{n} g\big(\mathcal{T}X_i,\mathcal{T}^*X_i\big),\quad g(\mathcal{A},\mathcal{A})=\sum_{i=1}^{n} g\big(\mathcal{A}_{X_i},\mathcal{A}_{X_i}\big),$$

$$g(\mathcal{A},\mathcal{A}^*)=\sum_{i=1}^{n} g\big(\mathcal{A}_{X_i},\mathcal{A}^*_{X_i}\big).$$

## 3. AZSs along SSs

This section deals with AZS of a SS $\psi:(M,\nabla,g)\to(N,\hat{\nabla},\hat{g})$ between statistical manifolds. Moreover, we discuss the nature of the fibers of such submersions.

Let $\psi:(M,\nabla,g)\to(N,\hat{\nabla},\hat{g})$ denote a SS between two statistical manifolds. The vertical distribution $\mathcal{V}$ induced by $\psi$ is referred to as $\nabla$-parallel if, for any smooth vector field $W$ on $M$ and every section $E$ of the vertical bundle, the covariant derivative $\nabla_W E$ is again a section of $\mathcal{V}$, namely

$$\nabla_W E\in\Gamma(\mathcal{V}).$$

Similarly, the horizontal distribution $\mathcal{H}$ is said to be invariant under parallel transport with respect to $\nabla$ whenever, for any vector field $W$ on $M$ and all horizontal fields $X$, the derivative $\nabla_W X$ remains within the horizontal distribution:

$$\nabla_W X\in\Gamma(\mathcal{H}).$$

One can similarly extend these notions of parallelism to the dual connection $\nabla^*$.

Invoking Lemma 2.3 together with the above definitions, we immediately infer the following characterization:

Lemma 3.1 ( [10]). *Suppose that $\psi:(M,\nabla,g)\to(N,\hat{\nabla},\hat{g})$ defines a SS between two statistical manifolds. Then, the vertical distribution $\mathcal{V}$ is $\nabla$-parallel (respectively, $\nabla^*$-parallel) if and only if the horizontal projections of the tensors $\mathcal{T}_E F$ and $\mathcal{A}_X F$ (respectively, $\mathcal{T}^*_E F$ and $\mathcal{A}^*_X F$) vanish identically for every pair of vertical vector fields $E,F\in\Gamma(\mathcal{V})$ and each horizontal field $X\in\Gamma(\mathcal{H})$.*

*Likewise, the horizontal distribution $\mathcal{H}$ is preserved under parallel transport with respect to $\nabla$ (or $\nabla^*$) if and only if the vertical components of $\mathcal{T}_E X$ and $\mathcal{A}_X Y$ (respectively, $\mathcal{T}^*_E X$ and $\mathcal{A}^*_X Y$) vanish for all vector fields $X,Y\in\Gamma(\mathcal{H})$ and $E\in\Gamma(\mathcal{V})$.*

### Case 1: The vertical distribution is parallel

Let $\psi:(M,\nabla,g)\to(N,\hat{\nabla},\hat{g})$ be a SS and $(M,g,\phi,V,\lambda)$ an AZS on a Riemannian manifold $(M,g)$. Then equation (2) gives

$$(\mathcal{L}_V g)(E,F)+S(E,F)+(\lambda+\phi)g(E,F)=0 \tag{14}$$

for $E,F\in\Gamma\mathcal{V}(M)$. On the other hand, from the definition of $\mathcal{L}_V$, we get

$$(\mathcal{L}_V g)(E,F)=g(\nabla_E V,F)+g(\nabla_F V,E). \tag{15}$$

Combining (14) and (15) gives

$$0 = g\left(\nabla_E V, F\right) + g\left(\nabla_F V, E\right) + S(E,F) + (\lambda+\phi) g(E,F). \tag{16}$$

Combining this with the equation in Lemma 2.5, we find

$$0 = \bar{g}\big(\bar{\nabla}_E V, F\big) + \bar{g}\big(\bar{\nabla}_F V, E\big) + \overline{S}(E,F) + (\lambda+\phi)\bar{g}(E,F) + \mathcal{K}, \tag{17}$$

where

$$\begin{aligned}\mathcal{K} = &- g\left(\mathcal{T}_E F, N^*\right) + \sum_{i=1}^{n}\left[g\left(\left(\nabla_{X_i}\mathcal{T}\right)(E,F), X_i\right) + g\left(\mathcal{A}_{X_i}E, \mathcal{A}^*_{X_i}F\right)\right]\\ &- \sum_{j=1}^{m} g\left(\left(\nabla^*_{E_j}\mathcal{A}\right)(X_i, X), V\right)\end{aligned} \tag{18}$$

and $\{X_i\}_{1\leq i\leq n}$ is an orthonormal basis of the horizontal distribution $\mathcal{H}$.
Now, let us assume that the vertical distribution $\mathcal{V}$ of $\psi$ is parallel, then it follows from Lemma 3.1 implies $\mathcal{T}_E F = \mathcal{A}_X F = 0$ for any $E, F \in \Gamma\mathcal{V}(M)$ and $X \in \Gamma\mathcal{H}(M)$. Further, we also have $N = 0$ from (8); and hence

$$g\left(\left(\nabla_{X_i}\mathcal{T}\right)(E,F), X_i\right) = 0$$

from (9). Therefore, (18) reduces to

$$\mathcal{K} = -\sum_{j=1}^{m} g\left(\left(\nabla^*_{E_j}\mathcal{A}\right)(X_i, X), V\right). \tag{19}$$

Now in light of Lemma 3.1, we have

Remark 3.2. Consider a SS $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$. The following statements are mutually equivalent:
(i) The vertical distribution $\mathcal{V}$ is parallel,
(ii) The horizontal distribution $\mathcal{H}$ is parallel,
(iii) The fundamental tensors $\mathcal{T}$ and $\mathcal{A}$ along with their dual counterparts $\mathcal{T}^*$ and $\mathcal{A}^*$ vanish identically.

Moreover, by adopting Lemmas 2.4 and 2.5 in [34], we give the following relations:

$$\begin{aligned} g\left(\left(\nabla_E\mathcal{A}\right)(X,Y), V\right) &= g\left(\left(\nabla_E\mathcal{A}\right)_X Y, V\right)\\ &= -g\left(Y, \left(\nabla^*_E\mathcal{A}^*\right)_X V\right) - g\left(\mathcal{A}_Y V, J_E X\right)\end{aligned} \tag{20}$$

where $J = \nabla - \nabla^*$.

$$\left(\nabla_X\mathcal{A}\right)_E Y = -\mathcal{A}_{\mathcal{A}_X E} Y, \quad \left(\nabla^*_X\mathcal{A}^*\right)_E Y = -\mathcal{A}^*_{\mathcal{A}^*_X E} Y. \tag{21}$$

Now, applying (20) and (21) in (19), we turn up

$$\mathcal{K} = \sum_{j=1}^{m}\left[g\left(X, \mathcal{A}^*_{\mathcal{A}^*_{E_j}X_i} V\right) - g\left(\mathcal{A}_X V, \mathcal{T}_{E_j} X_i\right) + g\left(\mathcal{A}_X V, \mathcal{T}^*_{E_j} X_i\right)\right].$$

Applying Remark 3.2 leads to the conclusion from the above equation that $\mathcal{K} = 0$ and equation (17) becomes

$$\bar{g}\big(\bar{\nabla}_E V, F\big) + \bar{g}\big(\bar{\nabla}_F V, E\big) + \overline{S}(E,F) + (\lambda+\phi)\bar{g}(E,F) = 0,$$

Consequently, we get:

THEOREM 3.3. *Let $(M, g, \phi, V, \lambda)$ be an AZS where the potential vector field (PVF) $V$ lies in the vertical distribution. Consider a SS*

$$\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g}).$$

*If the vertical distribution $\mathcal{V}$ remains invariant under parallel transport induced by the connection $\nabla$ (or, analogously, under $\nabla^*$), then it follows that every fiber of the submersion $\psi$ is naturally endowed with an AZS structure, which fulfills the condition*

$$\bar{g}\big(\bar{\nabla}_E V, F\big) + \bar{g}\big(\bar{\nabla}_F V, E\big) + \overline{S}(E, F) + (\lambda + \phi)\bar{g}(E, F) = 0,$$

*for all vertical vector fields $E, F$.*

Similarly, for the dual connection case, we have:

THEOREM 3.4 (Dual Case). *Let $(M, g, \phi, V, \lambda)$ denote an AZS structure equipped with a PVF $V$ tangent to the vertical distribution. Consider a SS*

$$\psi : (M, \nabla^*, g) \to (N, \hat{\nabla}^*, \hat{g}).$$

*If the vertical distribution $\mathcal{V}$ remains invariant under parallel transport with respect to $\nabla$ (or equivalently, under $\nabla^*$), then each fiber of the map $\psi$ inherits an AZS structure satisfying*

$$\bar{g}\big(\bar{\nabla}^*_E V, F\big) + \bar{g}\big(\bar{\nabla}^*_F V, E\big) + \overline{S}^*(E, F) + (\lambda + \phi)\bar{g}(E, F) = 0,$$

*for all vector fields $E, F$ tangent to the fibers.*

**Case 2: The horizontal distribution is parallel**

THEOREM 3.5. *Let $(M, g, \phi, V, \lambda)$ be an AZS manifold equipped with a PVF $V$, and consider a SS*

$$\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g}).$$

*Suppose that the horizontal distribution $\mathcal{H}$ is parallel with respect to $\nabla$ (or, alternatively, $\nabla^*$). Then the following statements hold:*

*(a) If the PVF $V$ lies in the vertical distribution, the base manifold $(N, \hat{\nabla}, \hat{g})$ is Einstein.*

*(b) If $V$ is horizontal, then the base manifold $(N, \hat{\nabla}, \hat{g})$ itself forms an AZS with PVF $V' = \psi_* V$.*

*Proof.* Starting from equation (2), Lemma 2.5, and relation (13), we deduce that for any horizontal vector fields $X, Y \in \Gamma\mathcal{H}(M)$ with their $\psi$-related counterparts $\hat{X}, \hat{Y} \in \Gamma TN$, the following holds:

$$\begin{aligned}
&g\left(\nabla_X V, Y\right) + g\left(\nabla_Y V, X\right) + \hat{S}(\hat{X}, \hat{Y}) + g\left(\nabla^*_X N^*, Y\right) \\
&\quad - \sum_{j=1}^{m} g\left(\mathcal{T}_{E_j} X, \mathcal{T}_{E_j} Y\right) + \sum_{i=1}^{n} g\left(\mathcal{A}_X X_i, \mathcal{A}^*_Y X_i\right) \\
&\quad + \sum_{i=1}^{n} g\left(\left(\nabla_{X_i} \mathcal{A}\right)\left(X_i, X\right), Y\right) + \sum_{i=1}^{n} \left[g\left(\mathcal{A}_{X_i} X_i, \mathcal{A}_X Y\right) - g\left(\mathcal{A}^*_X X_i, \mathcal{A}^*_X X_i\right)\right] \\
&\quad + (\lambda + \phi) g(X, Y) = 0.
\end{aligned} \tag{22}$$

Since horizontal distribution $\mathcal{H}$ is parallel with respect to $\nabla$, for any horizontal vector fields $X, Y \in \Gamma\mathcal{H}(M)$, we have:

$$\sum_{i=1}^{n} g\big((\nabla_{X_i}\mathcal{A})(X_i, X), Y\big) + \sum_{i=1}^{n} \left[g(\mathcal{A}_{X_i}X_i, \mathcal{A}_X Y) - g(\mathcal{A}^*_X X_i, \mathcal{A}^*_X X_i)\right]$$
$$- \sum_{j=1}^{m} g(\mathcal{T}_{E_j}X, \mathcal{T}_{E_j}Y) + \sum_{i=1}^{n} g\left(\mathcal{A}_X X_i, \mathcal{A}^*_Y X_i\right) = 0.$$

Invoking equations (8) through (10) and applying Lemma 3.1 simplifies the above expression to

$$g\left(\nabla_X V, Y\right) + g\left(\nabla_Y V, X\right) + \hat{S}(\hat{X}, \hat{Y}) + (\lambda + \phi)g(X, Y) = 0. \tag{23}$$

**Case (a):** If the vector field $V$ is vertical, then by using relation (6), we rewrite (23) as

$$g\left(\mathcal{A}_X V, Y\right) + g\left(\mathcal{A}_Y V, X\right) + \hat{S}(\hat{X}, \hat{Y}) + (\lambda + \phi)g(X, Y) = 0. \tag{24}$$

Since the horizontal distribution $\mathcal{H}$ is parallel, it follows that

$$\hat{S}(\hat{X}, \hat{Y}) + (\lambda + \phi)g(X, Y) = 0, \tag{25}$$

which characterizes $(N, \hat{\nabla}, \hat{g})$ as an Einstein statistical manifold.

**Case (b):** When $V$ is horizontal, equation (23) can be equivalently expressed as

$$(\mathcal{L}_V g)(X, Y) + \hat{S}(\hat{X}, \hat{Y}) + (\lambda + \phi)g(X, Y) = 0, \tag{26}$$

indicating that $(N, \hat{\nabla}, \hat{g})$ forms an AZS with the PVF $V' = \psi_* V$. $\square$

Theorem 3.6. *Let $(M, g, \phi, V, \lambda)$ represent an AZS configuration on the manifold $M$, where $V$ denotes the associated PVF. Assume that*

$$\psi : (M, \nabla^*, g) \to (N, \hat{\nabla}^*, \hat{g})$$

*defines a SS between statistical manifolds. If the horizontal distribution $\mathcal{H}$ is preserved under parallel translation with respect to $\nabla$ (or, equivalently, with respect to $\nabla^*$), then the following assertions are valid:*

1. *When $V$ lies entirely in the vertical distribution, the target manifold $(N, \hat{\nabla}^*, \hat{g})$ satisfies the Einstein condition.*
2. *When $V$ is horizontal, the induced structure on $(N, \hat{\nabla}^*, \hat{g})$ defines an AZS, where the potential field is given by $V' = \psi_* V$.*

## 4. AZSs on SSs with a conformal potential field

A vector field $\zeta$ defined on $(M, g)$ is termed a *conformal* whenever there exists a smooth real-valued function $\theta : M \to \mathbb{R}$ such that the metric tensor $g$ satisfies

$$\mathcal{L}_\zeta g = 2\theta g. \tag{27}$$

In the special case when $\theta$ identically vanishes, the field $\zeta$ becomes a Killing vector field, preserving the Riemannian metric under its flow.

Theorem 4.1. *Let* $(M, g, \phi, \zeta, \lambda)$ *be an AZS with the PVF* $\zeta \in \Gamma(TM)$ *being conformal. Consider a SS*

$$\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g}).$$

*If the conformal PVF* $\zeta$ *lies entirely in the vertical distribution, then each fiber of* $\psi$ *is an Einstein manifold. Moreover, its scalar curvature satisfies*

$$\bar{r} = -m(\theta + \lambda + \phi),$$

*where* $m$ *denotes the dimension of the fiber.*

*Proof.* Since $(M, g, \phi, \zeta, \lambda)$ is an AZS, for any $E, F \in \Gamma T(M)$, adopting Theorem 3.3, we have

$$\bar{g}\left(\bar{\nabla}_F V, E\right) + \bar{g}\left(\bar{\nabla}_E V, F\right) + \overline{S}(E, F) + (\lambda + \phi)\bar{g}(E, F) = 0. \tag{28}$$

Now, by applying (13), (15), (27), and (28) we find

$$\overline{S}(E, F) + (\theta + \lambda + \phi)\,\bar{g}(E, F) = 0. \tag{29}$$

Consequently, it follows that each fiber of $\psi$ is Einstein. Now, by contracting (29), we find

$$\bar{r} = -m(\theta + \lambda + \phi), \tag{30}$$

This serves to finalize the theorem's proof. □

Corollary 4.2. *Suppose* $(M, g, \phi, \zeta, \lambda)$ *is an AZS where the PVF* $\zeta \in \Gamma(TM)$ *is Killing. Consider a SS*

$$\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g}).$$

*If the Killing vector field* $\zeta$ *belongs to the vertical distribution, then each fiber of* $\psi$ *forms an Einstein manifold whose scalar curvature is given by*

$$\bar{r} = -m(\lambda + \phi),$$

*where* $m$ *denotes the fiber dimension.*

Next, for the dual setting, we get:

Theorem 4.3. *Let* $(M, g, \phi, \zeta, \lambda)$ *be an AZS with a conformal PVF* $\zeta \in \Gamma(TM)$*, and let*

$$\psi : (M, \nabla^*, g) \to (N, \hat{\nabla}^*, \hat{g})$$

*be a SS. If the conformal vector field* $\zeta$ *is vertical, then every fiber of* $\psi$ *is an Einstein manifold with scalar curvature*

$$\bar{r}^* = -m(\theta + \lambda + \phi).$$

Corollary 4.4. *Let* $(M, g, \phi, \zeta, \lambda)$ *be an AZS with Killing potential field* $\zeta \in \Gamma T(M)$ *and* $\psi : (M, \nabla^*, g) \to \left(N, \hat{\nabla}^*, \hat{g}\right)$ *a SS. If the Killing vector field* $\zeta$ *is vertical, then each fiber of* $\psi$ *is an Einstein manifold with scalar curvature* $\bar{r}^* = -m(\lambda + \phi)$.

Now, using (30), and above theorems, we obtain the following:

Corollary 4.5. *If* $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ *is a SS with vertical conformal potential field* $\zeta \in \Gamma T(M)$ *and if any fiber admits an AZS* $(M, g, \phi, \zeta, \lambda)$*, then the AZS of any fiber with scalar curvature* $\bar{r}$ *is expanding, or steady, or shrinking according to* $-\frac{\bar{r}}{m} - \phi > \theta$*, or* $-\frac{\bar{r}}{m} - \phi = \theta$*, or* $-\frac{\bar{r}}{m} - \phi < \theta$*, respectively.*

Corollary 4.6. *If $\psi : (M, \nabla^*, g) \to \left(N, \hat{\nabla}^*, \hat{g}\right)$ is a SS with vertical conformal potential field $\zeta \in \Gamma T(M)$ and any fiber admits an AZS $(M, g, \phi, \zeta, \lambda)$, then the AZS of any fiber with scalar curvature $\bar{r}^*$ is expanding, or steady or shrinking according to $-\frac{\bar{r}^*}{m} - \phi > \theta$, or $-\frac{\bar{r}^*}{m} - \phi = \theta$, or $-\frac{\bar{r}^*}{m} - \phi < \theta$, respectively.*

## 5. Gradient AZSs on SSs

This section focuses on gradient almost $\zeta$-solitons (AZS) associated with SSs, where the PVF is given by the gradient of a smooth function, i.e., $V = \operatorname{grad}(\Psi)$.

Consider a SS

$$\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$$

equipped with a vertical PVF $V = \operatorname{grad}(\Psi)$, and suppose that the vertical distribution $\mathcal{V}$ is parallel with respect to $\nabla$ (or equivalently $\nabla^*$). Under these assumptions, the Ricci tensor on each fiber satisfies

$$\overline{S}(E, F) = -(\lambda + \phi)\bar{g}(E, F) - \left[\bar{g}(\bar{\nabla}_E V, F) + \bar{g}(\bar{\nabla}_F V, E)\right], \tag{31}$$

for any $E, F \in \Gamma\mathcal{V}(M)$.

By tracing equation (31), we obtain the scalar curvature of the fibers:

$$\bar{r} = -m(\lambda + \phi) - \operatorname{div}(V),$$

where $m = \dim \mathcal{V}$ denotes the dimension of the fiber.

As a direct consequence of Theorem 3.3, the following holds:

Theorem 5.1. *Let $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ be a SS with vertical PVF $V = \operatorname{grad}(\Psi)$. If the vertical distribution $\mathcal{V}$ is parallel, then each fiber supports a gradient AZS whose potential function $\Psi$ satisfies the Poisson-type equation*

$$\Delta\Psi = \bar{r} + m(\lambda + \phi),$$

*where the Laplacian is given by $\Delta\Psi = -\operatorname{div}(\operatorname{grad}\Psi)$.*

Analogously, for the dual connection case, we have the following statement:

Theorem 5.2. *Consider the SS*

$$\psi : (M, \nabla^*, g) \to (N^*, \hat{\nabla}^*, \hat{g}),$$

*with vertical potential field $V = \operatorname{grad}(\Psi)$. Provided the vertical distribution $\mathcal{V}$ is parallel, each fiber admits a gradient AZS satisfying*

$$\Delta^*\Psi = \bar{r}^* + m(\lambda + \phi),$$

*where $\Delta^*$ denotes the Laplacian corresponding to the dual connection.*

## 6. Some examples

We recall some examples from [33] as follows:

Example 6.1. Consider the statistical manifold $(M, \nabla, g)$, where $M = \mathbb{R}^6$ and the Riemannian metric $g$ is given in standard Cartesian coordinates by $g = \sum_{i,j=1}^{6} dx_i \otimes dx_j$. Let $\{e_j = \partial/\partial x_j\}_{j=1}^{6}$ denote the global coordinate frame on $M$, and define the affine connection $\nabla$ by:

$$\begin{aligned}
\nabla_{e_i} e_i &= e_6, \quad \text{for } i = 1, \ldots, 5,\\
\nabla_{e_j} e_j &= 0, \quad \text{for } j = 1, \ldots, 5,\\
\nabla_{e_i} e_6 &= e_i, \quad \text{for } i = 1, \ldots, 5,\\
\nabla_{e_6} e_k &= 0, \quad \text{for } k = 1, \ldots, 6.
\end{aligned}$$

With this structure, $(M, \nabla, g)$ turns out to be a statistical manifold of constant curvature, and its scalar curvature is calculated to be $-20$.

The scalar curvature $-20$ indicates that $(\mathbb{R}^6, \nabla, g)$ is Einstein. By Theorem 3.3, any vertical gradient AZS with potential field in the vertical distribution exists on the fibers of a SS built from this manifold.

Example 6.2. Define a smooth map

$$\psi : (\mathbb{R}^6, \nabla, g) \longrightarrow (\mathbb{R}^3, \hat{\nabla}, \hat{g}),$$

based on the statistical manifold $(\mathbb{R}^6, \nabla, g)$ described in Example 6.1. The mapping $\psi$ is given by

$$\psi(x_1, \ldots, x_6) = (y_1, y_2, y_3),$$

with

$$y_1 = \frac{x_1 + x_2}{\sqrt{2}}, \quad y_2 = \frac{x_3 + x_4}{\sqrt{2}}, \quad y_3 = \frac{x_5 + x_6}{\sqrt{2}}.$$

The Jacobian matrix $J_\psi$ of this transformation takes the form

$$J_\psi = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

whose rank is clearly 3, matching the dimension of the codomain $\mathbb{R}^3$. Since $\psi$ satisfies the criteria of a SS, it qualifies as one.

To determine the vertical and horizontal distributions associated with $\psi$, we compute

$$\mathcal{V}(M) = \operatorname{Span}\left\{V_1 = \frac{1}{\sqrt{2}}(-\partial_{x_1} + \partial_{x_2}), \quad V_2 = \frac{1}{\sqrt{2}}(-\partial_{x_3} + \partial_{x_4}), \quad V_3 = \frac{1}{\sqrt{2}}(-\partial_{x_5} + \partial_{x_6})\right\},$$

$$\mathcal{H}(M) = \operatorname{Span}\left\{H_1 = \frac{1}{\sqrt{2}}(\partial_{x_1} + \partial_{x_2}), \quad H_2 = \frac{1}{\sqrt{2}}(\partial_{x_3} + \partial_{x_4}), \quad H_3 = \frac{1}{\sqrt{2}}(\partial_{x_5} + \partial_{x_6})\right\}.$$

Moreover, the differential $\psi_*$ maps the horizontal frame as

$$\psi_*(H_i) = \partial_{y_i}, \quad \text{for } i = 1, 2, 3,$$

and the inner products are preserved under this pushforward:

$$g(H_i, H_i) = \hat{g}(\psi_*(H_i), \psi_*(H_i)).$$

This confirms that $\psi$ is compatible with the metrics on horizontal distributions, thereby validating its role as a SS.

The vertical and horizontal distributions are explicitly given. Since $\mathcal{T} = 0$ and $\mathcal{A} = 0$, the horizontal distribution is parallel. Thus, Theorems 3.3 and 3.5 apply, implying the base manifold is Einstein and each fiber admits a gradient AZS.

EXAMPLE 6.3. Let ( $M_1, \nabla, g$ ) and ( $M_2, \hat{\nabla}, \hat{g}$ ) be two statistical manifolds. Consider the natural projection $\pi_2 : (M_1, \nabla, g) \times \left(M_2, \hat{\nabla}, \hat{g}\right) \to \left(M_2, \hat{\nabla}, \hat{g}\right)$. Then $\pi_2$ shall be a SS whose vertical distribution and whose horizontal distribution both parallel. In recent article [38], it was proved that $(\mathbb{R}^4 = \{(x, y, z, w) \mid x, y, z, w \in \mathbb{R}\}, \nabla, g)$ is a statistical manifold, where $g = \exp^{2y}(-\partial_1^2 + \partial_3^2) + \partial_2^2 + \partial_4^2$ and affine connection $\nabla$ is defined as:

$$\nabla_{\partial_1}\partial_1 = -\nabla_{\partial_3}\partial_3 = \exp^{2x}\partial_1 + \exp^{2y}\partial_2 + \exp^{2z}\partial_3,$$
$$\nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = -\nabla_{\partial_3}\partial_4 = -\nabla_{\partial_4}\partial_3 = \partial_1,$$
$$\nabla_{\partial_1}\partial_3 = \nabla_{\partial_3}\partial_1 = -\exp^{2z}\partial_1 + \exp^{2x}\partial_3 + \exp^{2y}\partial_4,$$
$$\nabla_{\partial_1}\partial_4 = \nabla_{\partial_4}\partial_1 = \nabla_{\partial_2}\partial_3 = \nabla_{\partial_3}\partial_2 = \partial_3,$$
$$\nabla_{\partial_2}\partial_2 = \nabla_{\partial_2}\partial_4 = \nabla_{\partial_4}\partial_2 = \nabla_{\partial_4}\partial_4 = 0.$$

Note that $\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \partial_3 = \frac{\partial}{\partial z}$ and $\partial_4 = \frac{\partial}{\partial w}$.

EXAMPLE 6.4. Let a statistical manifold $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ with $\hat{g} = \exp^{2y}(-\partial_{1*}^2 + \partial_{3*}^2) + \partial_{2*}^2$ and affine connection $\hat{\nabla}$ as follows [38]:

$$\hat{\nabla}_{\partial_{1*}}\partial_{1*} = -\hat{\nabla}_{\partial_{3*}}\partial_{3*} = \exp^{2x}\partial_{1*} + \exp^{2y}\partial_{2*} + \exp^{2z}\partial_{3*},$$
$$\hat{\nabla}_{\partial_{1*}}\partial_{2*} = \hat{\nabla}_{\partial_{2*}}\partial_{1*} = \partial_{1*},$$
$$\hat{\nabla}_{\partial_{1*}}\partial_{3*} = \hat{\nabla}_{\partial_{3*}}\partial_{1*} = -\exp^{2x}\partial_{1*} + \exp^{2x}\partial_{3*},$$
$$\hat{\nabla}_{\partial_{2*}}\partial_{2*} = 0, \quad \hat{\nabla}_{\partial_{2*}}\partial_{3*} = \hat{\nabla}_{\partial_{3*}}\partial_{2*} = \partial_{3*}$$

where $\partial_{1*} = \frac{\partial}{\partial x}, \quad \partial_{2*} = \frac{\partial}{\partial y}$ and $\partial_{3*} = \frac{\partial}{\partial z}$.
Now, we define a SS $\psi : (\mathbb{R}^4, \nabla, g) \to \left(\mathbb{R}^3, \hat{\nabla}, \hat{g}\right)$ as

$$\psi(x, y, z, w) = (x, y, z).$$

Then for $\partial_1, \partial_2, \partial_3 \in \mathcal{H}(\mathbb{R}^4)$ and $\partial_4 \in \mathcal{V}(\mathbb{R}^4)$, it is easy to obtain that $\mathcal{T}_{\partial_4}\partial_i = 0, i = 1, 2, 3, 4$. Therefore, $\psi$ is with isometric fiber from $\mathcal{T} = 0$.

EXAMPLE 6.5. We take an example from [38] which proves that $(\mathbb{R}^2 = \{(y, w) \mid y, w \in \mathbb{R}\}, \hat{\nabla}, \hat{g})$ is a statistical manifold with $\hat{g} = \partial_{2*}^2 + \partial_{4*}^2$ and affine connection $\hat{\nabla}$ [38]:

$$\hat{\nabla}_{\partial_{2*}}\partial_{2*} = -\hat{\nabla}_{\partial_{4*}}\partial_{4*} = 0,$$
$$\hat{\nabla}_{\partial_{2*}}\partial_{4*} = \hat{\nabla}_{\partial_{4*}}\partial_{2*} = 0.$$

Now, we have a SS $\psi : (\mathbb{R}^4, \nabla, g) \to \left(\mathbb{R}^2, \hat{\nabla}, \hat{g}\right)$ as

$$\psi(x, y, z, w) = (y, w).$$

For $\partial_2, \partial_4 \in \mathcal{H}(\mathbb{R}^4)$ and $\partial_1, \partial_3 \in \mathcal{V}(\mathbb{R}^4)$, we find that $\mathcal{A} = 0$ and hence $\mathcal{H}(\mathbb{R}^4)$ is integrable.

For the final three examples, we check the assumptions: fibers are equipped with $\nabla$- or $\nabla^*$-parallel distributions, torsion vanishes, and the horizontal distribution is integrable. Consequently, each fiber forms an AZS or Einstein manifold, illustrating explicitly the abstract theorems and showing how these examples generalize or complement results from [38].

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**Shahroud Azami**
Department of Pure Mathematics, Faculty of Science,
Imam Khomeini International University, Qazvin, Iran.
*E-mail*: azami@sci.ikiu.ac.ir

**Mehdi Jafari**
Department of Mathematics, Payame Noor University,
PO BOX 19395-4697, Tehran, Iran.
*E-mail*: m.jafarii@pnu.ac.ir