Korean J. Math. **34** (2026), No. 2, pp. 147–164
https://dx.doi.org/10.11568/kjm.2026.34.2.147

# CONTINUOUS COMODULES

Nikken Prima Puspita*, Indah Emilia Wijayanti, and Budi Surodjo

Abstract. Let $R$ be a commutative ring with unity and $C$ an $R$-coalgebra. The ring $R$ is clean if every elements $r$ in $R$ is the sum of a unit and an idempotent element of $R$. An $R$-module $M$ is clean if the endomorphism ring of $M$ over $R$ is clean. Moreover, every continuous module is clean. We adapt this idea to comodules and coalgebras. A $C$-comodule $M$ is called a clean comodule if the $C$-comodule endomorphisms of $M$ are clean. We introduce continuous comodules and proved that every continuous comodules is a clean comodules.

## 1. Introduction

Let $R$ be a commutative ring with unity. The notion of a clean ring or module motivates us to bring this concept to the coalgebra and comodule case. Nicholson [1] introduced clean rings in 1977; the ring $R$ is said to be clean if every element of $R$ can be expressed as the sum of a unit and an idempotent element. Furthermore, clean rings are a subclass of exchange rings [2, 3].

We denote the set of endomorphisms of an $R$-module $R$ by $End_R(R)$, which is isomorphic to the ring $R$. Consequently, $R$ is clean if and only if the ring $End_R(R)$ is clean. For a general $R$-module $M$, several authors have studied the cleanness of the endomorphisms of $R$-modules $M$ (denoted by $End_R(M)$). In 1998, Nicholson and Varadarajan [4] showed that the ring of a linear transformation of a countable linear vector space is clean. The equivalent results hold for arbitrary vector spaces over a field and any vector space over a division ring [5], [6].

In 2006, Camillo et al. [7] introduced the clean module. A clean module is an $R$-module $M$ for which the ring of $End_R(M)$ is clean. We recall results in [7] and [8] regarding the necessary and sufficient conditions of clean modules [7, Propositions 2.2 and 2.3]. In [7], the authors also gave some examples of clean modules and proved that the endomorphism ring of a continuous module is clean. In 1969, Sweedler proposed a coalgebra over a field. Later, this ground field was generalized to any ring [9]. A comodule over a coalgebra is the dual of a module over a ring.

Received August 21, 2025. Revised February 8, 2026. Accepted February 8, 2026.
2010 Mathematics Subject Classification: 16T15, 16D10.
Key words and phrases: Coalgebra, Comodule, Clean Comodule, Continuous Comodule.
∗ Corresponding author.
† This research is supported by the Directory of Research and Community Service, The Ministry of Higher Education, Sciences and Thechnology, Indonesia by the grant of Fundamental research No. 359-038/UN7.D2.1/PP/V/2025.

Throughout this work, $(C, \Delta, \varepsilon)$ is a coassociative and counital $R$-coalgebra, and $(M, \varrho^M)$ is a right $C$-comodule. Moreover, we define $C^* = Hom_R(C, R)$. For arbitrary $f, g \in C^*$, the convolution product in $C^*$ (denoted by $*$) is defined as follows:

$$f * g = \mu \circ (f \otimes g) \circ \Delta. \tag{1}$$

The set $C^*$ is an $R$-algebra (ring) over addition and convolution products as in Equation (1) [9]. We refer to $(C^*, +, *)$ as a dual $R$-algebra of $C$. The structure of $C^*$ implies some relationship between the categories of $C$-comodules and $C^*$-modules. Any right (respectively, left) $C$-comodule $M$ is a left (respectively, right) module over the dual algebra $C^*$ via the scalar multiplication

$$\rightharpoonup: C^* \otimes_R M \to M, g \otimes m \mapsto (I_M \otimes_R g) \circ \varrho^M(m), \tag{2}$$

that is, $g \otimes m \mapsto (I_M \otimes_R g) \circ \varrho^M(m) = \sum m_{\underline{0}} g(m_{\underline{1}}) \in M$ [9]. Furthermore, the category of right $C$-comodules ($\mathbf{M}^C$) is a subcategory of the left $C^*$-modules (${}_{C^*}\mathbf{M}$).

In this paper, we assume that $C$ does not always satisfy the $\alpha$-condition. However, $End^C(M)$ is a subring of ${}_{C^*}End(M)$. Although $M$ is a clean $C^*$-module, this does not imply that the ring of $End^C(M)$ is clean. For example, $\mathbb{Z}$ is a subring of $\mathbb{R}$. Hence, $\mathbb{R}$ is a clean ring, but $\mathbb{Z}$ is not clean. On this basis, as a dualization of clean modules, clean comodules are defined in the following way.

Definition 1.1. Let $(C, \Delta, \varepsilon)$ be an $R$-coalgebra. A right (respectively, left) $C$-comodule $M$ is said to be a clean comodule if the ring $End^C(M)$ is clean.

Definition 1.1 implies that $M \in \mathbf{M}^{\mathbf{C}}$ is clean if for any $f \in End^C(M), f = u + e$, where $u$ is a unit and $e$ is an idempotent element of $End^C(M)$. Moreover, since any $R$-coalgebra $C$ is a comodule over itself, $C$ is a clean coalgebra over $R$ if the ring of $End^C(C)$ is clean.

In Section 2, we start our investigation with some propositions concerning the necessary and sufficient conditions of clean comodules. We study the continuous and quasi-continuous modules of [10]. In 2006, Camillo et al. [7] proved that every quasi-continuous module is clean. By using this fact, we define continuous and quasi-continuous in comodule structures. In Section 3, as our main result, we prove that a continuous or quasi-continuous comodule is clean.

## 2. Necessary and Sufficient Conditions for Clean Comodules

We recall [7, Lemma 2.1 and Proposition 2.2], which provide necessary and sufficient conditions for modules to be clean. We will obtain similar results for clean comodules by changing the endomorphisms of $R$-modules to $C$-comodules. Moreover, we need to make sure that the set of kernels is also a subcomodule. Our goal is to derive some necessary and sufficient conditions for comodules to be clean.

Lemma 2.1. *Let $C$ be an $R$-coalgebra that is flat as an $R$-module, and let $M$ be a right $C$-comodule. Consider $f, e \in End^C(M)$, where $e$ is an idempotent element, $A = Ker(e)$, and $B = Im(e)$. Then $f - e$ is a unit in $End^C(M)$ if and only if there exists a $C$-comodule decomposition $M = X \oplus Y$ such that $f(A) \subseteq X$, $1 - f(B) \subseteq Y$, and both $f \colon A \to X$ and $1 - f \colon B \to Y$ are isomorphisms.*

*Proof.* It can be seen that $End^C(M) \subseteq End_R(M)$. Take $f, e \in End^C(M)$, where $e$ is an idempotent, $A = Ker(e)$, and $B = Im(e)$.

($\Rightarrow$) Suppose that $f - e = u$ or $f = u + e$ for some unit $u \in End^C(M)$ (i.e., $f$ is clean in $End^C(M)$). We prove that there exists a $C$-comodule decomposition $M = X \oplus Y$ such that $f(A) \subseteq X$, $(1-f)(B) \subseteq Y$, and both $f\colon A \to X$ and $1 - f\colon B \to Y$ are $C$-comodule isomorphisms. Suppose that $e \in End^C(M)$ is an idempotent and $1 = I_M$, where $A = Ker(e)$ and $B = Im(e)$. Since $End^C(M)$ is a subring of $End_R(M)$, $e \in End_R(M)$. As this is an idempotent $R$-module endomorphism and by using [14, see p. 58]

$$Ker(e) = Im(1-e), \tag{3}$$

we have that

$$Im(e) = Ker(1-e) \tag{4}$$

and

$$M = Im(1-e) \oplus Im(e) = Ker(e) \oplus Im(e) = A \oplus B. \tag{5}$$

The decomposition in Equation (5) is an $R$-module decomposition. Since $C$ is flat, $e$ is a $C$-pure morphism; moreover, $A = Ker(e)$ is a $C$-subcomodule of $M$.

We now need to prove that $1 - e$ is a $C$-comodule morphism. The commutativity of $C$-comodule morphisms of $e$ means that $\varrho^M \circ e = (e \otimes I_C) \circ \varrho^M$. Moreover, for every $m \in M$, we have

$$\begin{aligned}
\varrho^M \circ (1-e)(m) &= \varrho^M \circ (1(m) - e(m)) \\
&= (\varrho^M \circ 1)(m) - (\varrho^M \circ e)(m) \\
&= ((1 \otimes I_C) \circ \varrho^M)(m) - ((e \otimes I_C) \circ \varrho^M)(m) \\
&\qquad \text{(since 1 and } e \text{ are } C\text{-comodule morphisms)} \\
&= (((1 \otimes I_C) - (e \otimes I_C)) \circ \varrho^M)(m) \\
&= (((1-e) \otimes I_C) \circ \varrho^M)(m).
\end{aligned}$$

Hence, $\varrho^M \circ (1-e) = ((1-e) \otimes I_C) \circ \varrho^M$, or in other words, $(1-e)$ is a $C$-comodule morphism. Consequently, $Ker(1-e)$ is a $C$-subcomodule, which implies that

$$M = Ker(e) \oplus Ker(1-e) = Ker(e) \oplus Im(e) = A \oplus B$$

is a $C$-comodule decomposition. Let $f\colon M \to M$ be a $C$-comodule morphism, $A = Ker(e)$, and $B = Im(e)$. Suppose that $f = u + e \in S$, and let $X = u(A)$ and $Y = u(B)$. We break the remainder of the forward proof into five steps:

1. For $f|_A$, we have

$$\begin{aligned}
f(A) &= (e+u)(A) \\
&= e(A) + u(A) \\
&= 0 + u(A) \qquad (\text{since } A = Ker(e)) \\
&= X.
\end{aligned}$$

   Hence, $f(A) \subseteq X$.

2. Furthermore, for $1 - f|_B$,
$$\begin{aligned}1 - f(B) &= B - f(B)\\ &= B - (u + e)(B)\\ &= B - u(B) - e(B)\\ &= (1 - e)(B) - u(B) \qquad \text{(since } B = Im(e) = Ker(1 - e))\\ &= -u(B)\\ &= -Y.\end{aligned}$$
This means that $1 - f(B) \subseteq -Y \subseteq Y$.
3. We now prove that $f\colon A \to X$ is a $C$-comodule isomorphism. Since $f = u + e$,
$$f(1 - e) = (e + u)(1 - e) = (e - e + u - ue) = u(1 - e).$$
Because $(1 - e)(M) = Im(1 - e) = Ker(e)$,
$$\begin{aligned}f((1 - e)(M)) &= u((1 - e)(M))\\ \Leftrightarrow f(Ker(e)) &= u(Ker(e))\\ \Leftrightarrow f(A) &= u(A)\\ \Leftrightarrow &= X.\end{aligned}$$
This means that $f_A \simeq X$, so $f\colon A \to X$ is an isomorphism. In particular, $f\colon A \to X$ is a $C$-comodule isomorphism.
4. We now prove that $1 - f$ is a $C$-comodule isomorphism. We have already shown that if $f$ is a $C$-comodule morphism, then so is $1 - f$. Moreover, we have
$$(1 - f)e = e - fe = ee - fe = (e - f)e = (e - (e + u))e = -ue.$$
Since $e(M) = Im(e) = B$, then
$$\begin{aligned}((1 - f)e)(M) &= -ue(M)\\ (1 - f)(e(M)) &= -u(e(M))\\ \Leftrightarrow (1 - f)(B) &= -u(B) = -Y \subseteq Y.\end{aligned}$$
Since $u$ is a unit, $(1 - f)|_B$ is an isomorphism. In particular, $1 - f\colon B \to Y$ is an isomorphism.
5. Consequently, since $f\colon A \to X$ and $1-f\colon B \to Y$ are $C$-comodule isomorphisms, we have a decomposition
$$M = A \oplus B \simeq X \oplus Y.$$

($\Leftarrow$) Conversely, we prove that $u := f - e$ is a unit in $End^C(M)$ as follows:
$$\begin{aligned}u(M) &= u(A + B)\\ &= (f - e)(A + B)\\ &= f(A) - e(A) - f(B) + e(B)\\ &= f(A) - 0 - f(B) + ((f - u)(B)) \qquad \text{(since } A = Ker(e) \text{ and } f = e + u)\\ &= f(A) - u(B)\\ &= f(A) + (1 - f)(B) \qquad \text{(since } (-u)(B) = (1 - f)(B)).\end{aligned}$$
Hence, $f\colon A \to X$ and $1 - f\colon B \to Y$ are isomorphisms. Therefore,
$$u(M) = f(A) + (1 - f)(B) = X + Y = I_M(M).$$

It can be seen that $u = I_M$ is a unit in $S = End^C(M)$. In particular, $f - e = u$ is a unit in $End^C(M)$. $\square$

From Lemma 2.1, we obtain the following proposition, which is useful in proving our main theorem.

Proposition 2.2. *Let $C$ be a flat $R$-module and an $R$-coalgebra, and let $M$ be a right $C$-comodule. An element $f \in End^C(M)$ is clean if and only if there exist $C$-comodule decompositions $M = A \oplus B = X \oplus Y$ such that $f(A) \subseteq X$, $(1-f)(B) \subseteq Y$, and both $f\colon A \to X$ and $1 - f\colon B \to Y$ are isomorphisms.*

*Proof.* ($\Leftarrow$) From Lemma 2.1, we have $M = A \oplus B$ as a $C$-comodule decomposition. Take $X = u(A)$ and $Y = u(B)$; then $f|_A\colon A \to X$ and $(1 - f)|_B\colon B \to Y$ are isomorphisms. In particular, there exists a $C$-comodule decomposition $M = A \oplus B \simeq X \oplus Y$.

($\Rightarrow$) Conversely, consider the decomposition

$$M = A \oplus B.$$

There exists a projection map

$$\pi_B\colon M \to B, a + b \mapsto b \tag{6}$$

and an inclusion map

$$\iota_B\colon B \to M \tag{7}$$

such that $f_B = \iota_B \circ \pi_B$ is an idempotent in $End_R(M)$. Put

$$Y = Im(f_B), X = Ker(f_B),$$

$X = A$, and $Y = B$. Now we need to check that $\iota_B \circ \pi_B\colon M \to M$ in $End^C(M)$. It can be seen that the inclusion map $\iota_B$ is a $C$-comodule morphism. For every $m = a + b \in M$, we have

$$\begin{aligned}
(\pi_B \otimes I_C) \circ \varrho^M(m) &= (\pi_B \otimes I_C) \circ \varrho^M(a + b)\\
&= (\pi_B \otimes I_C)(\varrho^M(a)) + (\pi_B \otimes I_C)(\varrho^M(b))\\
&= (\pi_B \otimes I_C)(\sum a_0 \otimes a_1) + (\pi_B \otimes I_C)(\sum b_0 \otimes b_1)\\
&= (\sum \pi_B(a_0) \otimes I_C(a_1)) + (\sum \pi_B(b_0) \otimes I_C(b_1))\\
&= 0 + (\sum \pi_B(b_0) \otimes I_C(b_1)) \qquad \text{(since } a_0 \in A\text{)}\\
&= \sum b_0 \otimes b_1\\
&= \varrho^M|_B(b) \qquad \text{(since } B \text{ is a } C\text{-subcomodule of } M\text{)}\\
&= \pi_B \circ \varrho^M|_B(m).
\end{aligned}$$

Thus, $\pi_B$ is a $C$-comodule morphism; moreover, $e = \iota_B \circ \pi_B$ is an idempotent in $End^C(M)$.

Now, we have that there exists a $C$-comodule decomposition $M = X \oplus Y$ such that $f(A) \subseteq X$, $(1 - f)(B) \subseteq Y$, and $f\colon A \to X$ and $1 - f\colon B \to Y$ are isomorphisms. As implied by Lemma 2.1, $f$ is a clean element in $End^C(M)$, so $M$ is a clean $C$-comodule. $\square$

With Lemma 2.1 and Proposition 2.2, we have obtained necessary and sufficient conditions for clean $C$-comodules. In [11], [12], [13], some results regarding cleanness in comodule theory are discussed.

## 3. Essential Subcomodules and Continuous Comodules

Before studying the continuous comodule, we need to understand some preliminary structures, which are motivated by a similar situation in module theory. We define an essential subcomodule below.

Definition 3.1. Let $M$ be a comodule over an $R$-coalgebra $C$. A $C$-subcomodule $N$ is called an essential subcomodule of $M$ (and $M$ is called an essential extension of $N$) if for any nonzero $C$-subcomodule $L \subseteq M$, we have $L \cap N \neq \{0\}$. A $C$-subcomodule $N$ is called *closed* if it does not have a proper essential extension in $M$. If $N'$ is a closed subcomodule of $M$ and $N$ is an essential subcomodule of $N'$, then we call $N'$ *a closure of* $N$ *in* $M$.

Definition 3.1 means that for every nonzero $C$-subcomodule $L \subseteq M$ with $L \cap N = \{0\}$, $L = \{0\}$. Throughout, we denote an essential $C$-subcomodule $N$ of $M$ by $N \subseteq^e M$. In module theory, essential $R$-submodules have the following property. Let $K$ and $L$ be submodules of $M$ with $K \subseteq L \subseteq M$. Then, $K \subseteq_e M$ if and only if $K \subseteq_e L$ and $L \subseteq_e M$. We derive a similar property for the comodule structure.

Lemma 3.2. *Let $M$ be a comodule over an $R$-coalgebra $C$, and let $K$ and $N$ be $C$-subcomodules of $M$, where $K \subseteq N \subseteq M$. Then $K \subseteq^e M$ if and only if $K \subseteq^e N$ and $N \subseteq^e M$.*

*Proof.* ($\Rightarrow$) Let $K \subseteq^e M$. This means that for any nonzero $C$-subcomodule $H \subseteq M$, we have $K \cap H \neq \{0\}$. Suppose that $H'$ is a non-zero subcomodule of $N$. Since $H'$ is also a subcomodule of $M$ and $K \subseteq^e M$, $K \cap H' \neq \{0\}$. Thus, $K \subseteq^e N$. Moreover, for any nonzero $C$-subcomodule $H \subseteq M$, since $K \subseteq^e M$, we have $H \cap K \neq \{0\}$. Hence, $\{0\} \neq H \cap K \subseteq K \subseteq N$. Thus, $H \cap N \neq \{0\}$, so $N \subseteq^e M$.

($\Leftarrow$) Now suppose that $K \subseteq^e N$ and $N \subseteq^e M$. We prove that $K \subseteq^e M$. Given an arbitrary $C$-subcomodule $L \subseteq M$ with $K \cap L = \{0\}$, since $K \subseteq^e N$ and $K \cap L = \{0\}$, $L \cap N = \{0\}$. On the other hand, $N \subseteq^e M$ and $L$ is a $C$-subcomodule of $M$, which implies $L = 0$. In particular, $K \subseteq^e M$. □

We can now obtain a modification of [15, Remark 19.4].

Lemma 3.3. *Let $M$ be a $C$-comodule. The following assertions hold:*

*(1) Every $C$-subcomodule of $M$ has at least one closure in $M$.*

*(2) If $G$ and $N$ are two $C$-subcomodules of $M$, where $G \cap N = 0$, then $G$ has at least one closed complement $H$ in $M$ that contains $N$.*

*(3) If $N_1$ and $N_2$ are two subcomodules of $M$ such that $M_1 \subseteq N_1$, $M_2 \subseteq N_2$, and $M_1 \cap M_2 = 0$, then $M$ is an essential extension of $M_1 \oplus M_2$, and $M_1 \cap K \neq 0$ for every subcomodule $K$ of $M$ that properly contains $M_2$.*

*Proof.*

(1) Let $N$ be a $C$-subcomodule of $M$ and

$$\xi = \{H_i | H_i \text{ is a } C\text{-subcomodule of } M,\ N \subseteq^e H_i\}.$$

The set $\xi$ is not empty and contains the union of any ascending chain of its elements. By Zorn's Lemma, there exists a maximal element of $\xi$, $H = \cup_{i\in I} H_i \in \xi$, such that $N \subseteq^e H$ and $H$ does not have a proper essential extension in $M$. Therefore, $H$ is a closure of $N$ in $M$.

(2) Let

$$\xi' = \{H_i | H_i \text{ is a subcomodule of } M,\ N \subseteq H_i, \text{ and } G \cap H_i = \{0\} \text{ for any } i\}.$$

The set $\xi'$ is not empty, since $N \in \xi'$. Moreover, it contains the union of any ascending chain of its elements.

(a) By Zorn's Lemma, $\xi'$ contains a maximal element $H$ such that $N \subseteq H$, $G \cap H = \{0\}$, and $G \oplus H \subseteq M$.

(b) We want to show $G \oplus H \subseteq^e M$. Let $X$ be a $C$-subcomodule of $M$; then we have two cases:

(i) If $X \in \xi'$, then $N \subseteq X \subseteq H$, since $H$ is the maximal element. Thus, $X \cap (G + H) = X \cap G + X \cap H \neq \{0\}$.

(ii) If $X$ is not in $\xi'$, then $G \cap X \neq \{0\}$. Consequently, $X \cap (G + H) \neq \{0\}$.

Thus, in either case, $G \oplus H \subseteq^e M$.

(c) For any $C$-subcomodule $K$ of $M$ that properly contains $H$, we have $G \cap K \neq \{0\}$, since $H$ is a maximal element in $\xi'$.

Assume that $H$ is not closed, that is, $K$ is a subcomodule of $M$ such that $H \subseteq^e K$ and $K$ properly contains $H$. Hence, $G \cap K \neq \{0\}$. Moreover, since $H \subseteq^e K$ and $G \cap K$ is a $C$-subcomodule of $K$, we have $(G \cap K) \cap H = G \cap (K \cap H) = G \cap H \neq \{0\}$. This contradicts $G \cap H = \{0\}$. Consequently, $H$ is closed and $N \subseteq H$.

(3) Let $N_1$ and $N_2$ be two subcomodules of $M$.

(a) By Lemma 3.3, there is a closed subcomodule $M_2$ of $M$ such that $M_2 \subseteq N_2$ and $N_1 \cap M_2 = \{0\}$. Hence, $M$ is an essential extension of $N_1 \oplus M_2$, and $M_1 \cap K \neq \{0\}$ for every $C$-subcomodule $K$ of $M$ that properly contains $M_2$.

(b) By (1), $N_1$ has at least one closure $M_1$ in $M$. Hence, $N_1 \subseteq^e M_1$ and $N_1 \cap M_2 = \{0\}$. Since $M_1 \cap M_2$ is a $C$-subcomodule of $M_1$ and $M_1 \cap M_2 \cap N_1 = \{0\}$, $M_1 \cap M_2 = \{0\}$.

(c) Moreover, we already know that $N_1 \oplus M_2 \subseteq^e M$ and $N_1 \oplus M_2 \subseteq M_1 \oplus M_2 \subseteq M$. By Lemma 3.2, we have $M_1 \oplus M_2 \subseteq^e M$.

□

Now we adapt the concept of continuous modules to the category of right $C$-comodules $\mathbf{M^C}$. We start by defining the following statements:

($CM_1$): For every subcomodule $A \in M$, there exists a direct summand $K$ of $M$ such that $A \subseteq^e K$.

($CM_2$): If a subcomodule $A$ of $M$ is isomorphic to a summand of $C$-subcomodule $M$, then $A$ is a summand of $M$.

($CM_3$): If $N_1$ and $N_2$ are summands of $C$-comodule $M$ such that $N_1 \cap N_2 = \{0\}$, then $N_1 \oplus N_2$ is a summand of $M$.

We use these to define continuous and quasi-continuous comodules.

Definition 3.4. A comodule $M$ over an $R$-coalgebra $C$ is called CS if it satisfies ($CM_1$), and $M$ is called a continuous $C$-comodule if it satisfies ($CM_1$) and ($CM_2$); $M$ is called a quasi-continuous $C$-comodule if it satisfies ($CM_1$) and ($CM_3$).

A quasi-continuous module has special characteristics related to the idempotent element. The following lemma is a consequence of Lemma 3.3.

LEMMA 3.5. *Let $C$ be a flat $R$-module and a coalgebra over $R$. If $M$ is a quasi-continuous $C$-comodule, then every idempotent endomorphism of any $C$-subcomodule $M$ can be extended to an idempotent endomorphism of $M$.*

*Proof.* Let $N$ be a $C$-subcomodule of a quasi-continuous module $M$, and let $e \in End^C(N)$ with $C$ flat. Then, by using the same argument as in Lemma 2.1, we have that

$$N = Im(e) \oplus Im(I_N - e)$$

is a $C$-subcomodule decomposition. Let $N_1 = Im(e)$ and $N_2 = Im(I_N - e)$ such that $N = N_1 \oplus N_2$. By Lemma 3.3, there exist closed subcomodules $Q_1, Q_2 \subseteq M$ such that $N_1 \subseteq Q_1$, $N_2 \subseteq Q_2$, $Q_1 \cap Q_2 = \{0\}$, and $Q_1 \oplus Q_2 \subseteq^e M$. Since $M$ is quasi-continuous, we have the following:

1. From $(CM_1)$, there are direct summands $K_1$ and $K_2$ of $M$ such that $Q_1 \subseteq^e K_1$ and $Q_2 \subseteq^e K_2$.
2. Since $Q_1$ and $Q_2$ are closed, $Q_1 = K_1$ and $Q_2 = K_2$ are direct summands of $M$.
3. From $(CM_3)$, $Q_1 \oplus Q_2$ is also a summand of $M$, so $M = Q_1 \oplus Q_2 \oplus Q_3$ for a $C$-subcomodule $Q_3$ of $M$.

Take $M = Q_1 \oplus Q_2 \oplus Q_3$ for a $C$-subcomodule $Q_3 \subseteq M$. Then, there is a projection map

$$\pi_{Q_1} \colon M \to Q_1$$

where the kernel of $\pi_{Q_1}$ is $Q_2 \oplus Q_3$ and the inclusion $\iota_{Q_1} \colon Q_1 \to M$ is a $C$-comodule morphism. Now we construct the map

$$e' = \iota_{Q_1} \circ \pi_{Q_1} \colon M \to Q_1 \to M, m \mapsto \iota_{Q_1} \circ \pi_{Q_1}(m)$$

and derive some properties:

1. The map $e'$ is idempotent; that is, for any $m = q_1 + q_2 + q_3 \in M$, where $q_i \in Q_i$ for $i = 1, 2, 3$, we have
$$\begin{aligned} e' \circ e'(m) &= e'(e'(q_1 + q_2 + q_3)) \\ &= e'(\pi_{Q_1}(q_1 + q_2 + q_3)) = e'(q_1) = e'(q_1 + 0 + 0) = q_1 \end{aligned}$$
and $e'(m) = e'(q_1 + q_2 + q_3) = q_1$.
2. We prove that $e'$ is a $C$-comodule morphism, that is, $\varrho^M \circ e' = (e' \otimes I_C) \circ \varrho^M$. Take any $m = q_1 + q_2 + q_3 \in M$, where $q_i \in Q_i$ for $i = 1, 2, 3$; then we have
$$\begin{aligned} \varrho^M \circ e'(m) &= \varrho^M \circ e'(q_1 + q_2 + q_3) = \varrho^M \circ \iota_{Q_1} \circ \pi_{Q_1}(q_1 + q_2 + q_3) \\ &= \varrho^M(q_1) = \sum q_{1\underline{0}} \otimes q_{1\underline{1}} \end{aligned}$$

and

$$\begin{aligned}(e' \otimes I_C) \circ \varrho^M(m) &= (e' \otimes I_C) \circ \varrho^M(q_1 + q_2 + q_3) \\ &= (e' \otimes I_C) \circ (\varrho^M(q_1) + \varrho^M(q_2 + q_3)) \\ &= ((e' \otimes I_C) \circ \varrho^M)(q_1) + ((e' \otimes I_C) \circ \varrho^M)(q_2 + q_3) \\ &= (e' \otimes I_C) \circ \varrho^M(q_1) + 0 \\ &\qquad \text{(since } Q_2 + Q3 \text{ is a } C\text{-subcomodule of } M \text{ and also} \\ &\qquad \text{the kernel of } \pi_{Q_1}) \\ &= (\iota_{Q_1} \circ \pi_{Q_1} \otimes I_C)(\sum q_{1\underline{0}} \otimes q_{1\underline{1}}) \\ &= \sum q_{1\underline{0}} \otimes q_{1\underline{1}} \qquad \text{(since } Q_1 \text{ is a } C\text{-subcomodule of } M).\end{aligned}$$

Hence, $e' = \iota_{Q_1} \circ \pi_{Q_1}$ is a $C$-comodule endomorphism of $M$. In particular, any idempotent $e \in End^C(N)$ can be extended to the idempotent $C$-comodule morphism of $M$, that is, $e'$. □

## 4. Main Results

In module theory, it has been shown that all continuous modules are clean [7]. We investigate whether all continuous comodules are clean. We begin by proving some preliminary lemmas. In the context of clean modules, some researchers have investigated the cleanness of the endomorphisms of a vector space (over a field or a division ring), for example, [5]. Our first lemma shows that the endomorphisms of a $C$-comodule that is a direct sum of $n$ $C$-comodules are clean.

LEMMA 4.1. *Let $M$ be a comodule over a coalgebra $C$ and $L = \oplus_{n\geq 0} M_n$, where $M = M_n$ for each $n$. For any $m \in M$, we write $m_n$ for the element $m$ lying in $M_n = M$ and define the (forward) shift operator $f$ on $L$ by $f(m_n) = m_{n+1}$ for all $n$. Then $f$ is clean in $End^C(L)$.*

*Proof.* Suppose that $L = \oplus_{n\geq 0} M_n$ is a $C$-comodule, since the coproduct of $C$-comodules is a $C$-comodule [9]. Take any $f \in End^C(L) \subseteq End_R(L)$, that is,

$$f\colon L \to L, \text{ or equivalently, } f\colon \oplus_{n\geq 0} M_n \to \oplus_{n\geq 0} M_n,$$

where $f(\sum_{n\geq 0} m_n) = \sum_{n\geq 0} m_{n+1}$. In module theory, Ó Searcoid [5] has proven the corresponding lemma by choosing an $R$-module endomorphism of $\oplus_{n\geq 0} M_n$ such that $f = u + e$ for a unit $u$ and an idempotent $e$ in $End_R(L)$. By using a similar argument, we prove that the following $e$ is a $C$-comodule endomorphism:

$$e\colon \oplus_{n\geq 0} M_n \to \oplus_{n\geq 0} M_n$$

in $End_R(L)$, defined as $e(\sum_{n\geq 0} m_n) = \sum_{n\geq 0} e(m_n)$, where

$$e(m_{2n}) = m_{2n} \text{ and } e(m_{2n+1}) = m_{2n+2} - m_{2n}, \text{ for all } n \geq 0.$$

It can be seen that $e$ is an idempotent element in $End_R(L)$. Moreover, we want to prove that $e \in End^C(L)$. For any $\sum_{n\geq 0} m_n \in L$,

$$\begin{aligned}
\varrho^L \circ e(\sum_{n\geq 0} m_n) &= \varrho^L(m_0, m_4 - m_2, m_2, m_8 - m_6, m_4, \dots, m_{2n}, m_{2n+2} - m_{2n}, \dots) \\
&= (\varrho^M(m_0), \varrho^M(m_4 - m_2), \varrho^M(m_2), \varrho^M(m_8 - m_6), \varrho^M(m_4), \dots, \\
&\quad \varrho^M(m_{2n}), \varrho^M(m_{2n+2} - m_{2n}), \dots) \\
&= (m_{0\underline{0}} \otimes m_{0\underline{1}}, \dots, m_{2n\underline{0}} \otimes m_{2n\underline{1}}, m_{(2n+2)\underline{0}} - m_{2n\underline{0}} \otimes m_{(2n+2)\underline{1}} \\
&\quad - m_{2n\underline{1}}, \dots) \\
&= ((e \otimes I_C)(m_{0\underline{0}} \otimes m_{0\underline{1}}), \dots, (e \otimes I_C)(m_{2n\underline{0}} \otimes m_{2n\underline{1}}), (e \otimes I_C) \\
&\quad (m_{(2n+1)\underline{0}} \otimes m_{2n+1\underline{1}}, \dots)) \\
&= (e \otimes I_C)((m_{0\underline{0}} \otimes m_{0\underline{1}}), \dots, (m_{2n\underline{0}} \otimes m_{2n\underline{1}}), \\
&\quad (m_{2n+1\underline{0}} \otimes m_{2n+1\underline{1}}, \dots)) \\
&= (e \otimes I_C) \circ \varrho^M(\sum_{n\geq 0} m_n).
\end{aligned}$$

Therefore, $e$ is an idempotent in $End^C(L)$. Moreover, a $C$-comodule morphism satisfies $f: L \to L$, where $f(m_n) = m_{n+1}$ in $End^C(L)$ and $u := f - e$. Thus, we have

$$u(m_{2n}) = (f - e)(m_{2n}) = f(m_{2n}) - e(m_{2n}) = m_{2n+1} - m_{2n}$$

and

$$u(m_{2n+1}) = (f - e)(m_{2n+1}) = f(m_{2n+1}) - e(m_{2n+1}) = m_{2n+2} - (m_{2n+2} - m_{2n}) = m_{2n}.$$

That is, for all $n \geq 0$, we have

$$\begin{aligned}
u \circ u(m_{2n}) &= u(m_{2n+1} - m_{2n}) = u(m_{2n+1}) - u(m_{2n}) \\
&= m_{2n} - (m_{2n+1} - m_{2n}) = I_L(m_{2n}) - u(m_{2n}) = (I_L - u)(m_{2n})
\end{aligned}$$

and

$$u \circ u(m_{2n+1}) = u(m_{2n}) = m_{2n+1} - m_{2n} = I_L(m_{2n+1}) - u(m_{2n+1}) = I_L - u(m_{2n+1}).$$

Therefore, $u \circ u = I_L - u$ if and only if $u(u + I_L) = I_L$ for all $n \geq 0$. In particular, $u$ is a unit in $End^C(L)$, so $f = e + u$ is clean; thus, $L$ is a clean $C$-comodule. □

Using [7, Lemma 3.3 and Remark 3.4], we construct the set of all ordered pairs of $C$-subcomodules $M$ and define an $f$-invariant subcomodule. Let $f \in End_R(M)$. An $R$-submodule $N \subseteq M$ is said to be $f$-invariant if $f(N) \subseteq N$. Using this concept, we define the $f$-invariant subcomodules as follows.

Definition 4.2. Let $M$ be a $C$-comodule and $f \in End^C(M)$. A $C$-subcomodule $W \subseteq M$ is said to be $f$-invariant if $f(W) \subseteq W$.

Moreover, we define an essential monomorphism and a co-Hopfian comodule.

Definition 4.3. Let $M$ be a $C$-comodule.

1. A monomorphism $f \in End^C(M)$ is called an *essential monomorphism* if $Im(f) \subseteq^e M$.
2. $M$ is called a *co-Hopfian* comodule (respectively, an essential co-Hopfian comodule) if every monomorphism (respectively, essential monomorphism) in $End^C(M)$ is onto.

Consider the dual algebra $C^* = Hom_R(C, R)$, equipped with the convolution product (see Equation (1)). In [9], any $C$-comodule $M$ is shown to be a $C^*$-module. For any $0 \neq m$ in a right $C$-comodule $(M, \varrho^M)$, we construct the set of $C^* \rightharpoonup m$, that is,

$$C^* \rightharpoonup m = \{f \rightharpoonup m | f \in C^*\}, \tag{8}$$

where $f \rightharpoonup m = (I_m \otimes f) \circ \varrho^M(m)$. In [9], the category of $\mathbf{M}^C$ is shown to be a subcategory of ${}_{C^*}\mathbf{M}$, and it is a full subcategory if and only if $C$ satisfies the $\alpha$-condition. In this paper, we give a weaker condition than the $\alpha$-condition. For any $0 \neq m \in M$, define a map

$$\alpha_{M/C^* \rightharpoonup m} \colon M/C^*m \otimes_R C \to Hom_R(C^*, M/C^* \rightharpoonup m), x \otimes c \mapsto [f \mapsto f(c)x]. \tag{9}$$

An $R$-coalgebra $C$ satisfies the $\alpha^*$-condition if the map $\alpha_{M/C^* \rightharpoonup m}$ is injective. Hence, if $C^* \rightharpoonup m$ is a $C$-pure $R$-submodule of $M$ or $C$ is a flat $R$-module, then the set $C^* \rightharpoonup m$ will always to be a $C$-subcomodule of $M$.

Proposition 4.4. *Let $C$ be a flat $R$-module and $C$ an $R$-coalgebra. Let $M$ be a quasi-continuous $C$-comodule, $W \subseteq^e M$, and $f \in End^C(M)$ such that $f(W) \subseteq W$. If $f|_W - e$ is an essential monomorphism in $End^C(W)$ for some idempotent $e \in End^C(W)$, then there exists an idempotent $e' \in End^C(M)$ such that $e'|_W = e$ and $f - e'$ is an essential monomorphism in $End^C(M)$.*

*Proof.* By assumption, $Im(f|_W - e) \subseteq^e W$. Since $M$ is a quasi-continuous $C$-comodule, we may extend $e$ as an idempotent on $M$ (see Lemma 3.5). There is an $e' \in End^C(M)$ such that $e'|_W = e$ and $f - e \in End^C(M)$. Now, we show that $f - e'$ is an essential monomorphism in $End^C(M)$:

1. Since $C$ satisfies the $\alpha^*$-condition, for any $x \in M, C^* \rightharpoonup x$ is a $C$-subcomodule of $M$.
2. Suppose that $W \subseteq^e M$. Then, $C^* \rightharpoonup x \cap W \neq 0$. This means that there exists a $g \in C^*$ such that $0 \neq g \rightharpoonup x \in W$.
3. The monomorphism property of $f|_W - e$ implies that

$$\begin{aligned}(f - e')(g \rightharpoonup x) &= f(g \rightharpoonup x) - e'(g \rightharpoonup x)\\ &= f|_W(g \rightharpoonup x) - e'|_W(g \rightharpoonup x)\\ &= f|_W(g \rightharpoonup x) - e(g \rightharpoonup x)\\ &= (f|_W - e)(g \rightharpoonup x).\end{aligned}$$

   If $(f-e')(g \rightharpoonup x) = 0$, then $(f|_W - e)(g \rightharpoonup x) = 0$. By the injectivity of $(f|_W - e)$, $g \rightharpoonup x$ must be zero. Therefore, $f - e'$ is a monomorphism in $End^C(M)$.
4. Furthermore, we assumed that $Im(f|_W - e) \subseteq^e W$ and $W \subseteq^e M$. Thus, $Im(f|_W - e)$ is an essential $C$-subcomodule of $M$ (Lemma 3.2).
5. Since $Im(f|_W - e) \subseteq Im(f - e') \subseteq M$ and $Im(f|_W - e) \subseteq^e M$, $Im(f - e') \subseteq^e M$. Consequently, $Im(f - e')$ is an essential monomorphism in $End^C(M)$ (Lemma 3.2).

□

By Proposition 4.4, if $M$ is an essentially co-Hopfian comodule, then $f$ is onto. Thus, $f - e'$ is a unit in $End^C(M)$. In particular, $f$ is clean.

Remark 4.5. Let $M$ be a $C$-comodule and $f \in End^C(M)$. Let $\xi_f$ be the set of all ordered pairs $(W, e)$ for which $W$ is an $f$-invariant $C$-subcomodule of $M$ and $e \in$

$End^C(M)$ is an idempotent such that $f|_W - e$ is a unit in $End^C(W)$, or equivalently, $f|_W$ is clean. As $(0,0) \in \xi_f$, the set $\xi_f$ is not empty. Let $(W_1, e_1), (W_2, e_2) \in \xi_f$. We define a partial ordering by setting $(W_1, e_1) \leq (W_2, e_2)$ if and only if $W_1 \subseteq W_2$ and $e_2|_{W_1} = e_1$. Thus, any totally ordered set $\{(W_i, e_i)|i \in I\}$ is bounded above by $(N, e)$, where $N = \cup_{i \in I} W_i$ and $e(x) = e_i(x)$ for all $x \in W_i$. By Zorn's Lemma, any $(W_0, e_0) \in \xi_f$ is bounded by a maximal element of $\xi_f$. It can be seen that $f$ is clean in $End^C(M)$ if and only if there is an element $(W, e)$ in $\xi_f$ with $W = M$.

We will use this remark to prove the cleanness of continuous comodules in the main theorem. To understand the structure of $\xi_f$, we consider the characteristics of its maximal elements.

LEMMA 4.6. *Let $C$ be a flat $R$-module, and let $C$ be an $R$-coalgebra satisfying the $\alpha^*$-condition. If $f \in End^C(M), (W, e)$ is a maximal element in $\xi_f$, and $X$ is a $C$-subcomodule of $M$, where $X \cap W = 0$, then we have the following:*

(A): *For any $x \in X$, if $f(x) \in W$, then $x = 0$.*
(B): *For any $m \in W \oplus X$, if $f(m) \in W$, then $m \in W$.*

*Proof.*

(A): Suppose that $x \in X$ and $f(x) \in W$. Let $w = f(x) \in W$ and $X' = C^* \rightharpoonup x \subseteq X$. Since $C$ satisfies the $\alpha^*$-condition, $X = C^* \rightharpoonup x$ is a $C$-subcomodule of $M$. We prove that $W \oplus X'$ is an $f$-invariant as follows:

$$\begin{aligned}
f(W + X') &= f(W) + f(X') \\
&\subseteq W \oplus f(C^* \rightharpoonup x) && \text{(since } W \text{ is } f\text{-invariant)} \\
&= W \oplus C^* \rightharpoonup f(x) && \text{(by scalar multiplication of } C^*) \\
&= W \oplus C^* \rightharpoonup w \\
&\subseteq W && \text{(since } W \text{ is a } C^*\text{-module).}
\end{aligned}$$

Consequently, the $C$-comodule $W \oplus X'$ is $f$-invariant. Moreover, we have the following facts:

1. We can extend the idempotent endomorphism $e \in End^C(W)$, which is $(W, e) \in \xi_f$, to an endomorphism of $X'$ by defining $e(\alpha \rightharpoonup x) = \alpha \rightharpoonup e(x) = \alpha \rightharpoonup x$ for any $\alpha \rightharpoonup x \in X'$.
2. For any $w' + \alpha \rightharpoonup x \in W \oplus X'$, we have

$$\begin{aligned}
e \circ e(w' + \alpha \rightharpoonup x) &= e \circ e(w') + e \circ e(\alpha \rightharpoonup x) = e(w') + e(e(\alpha \rightharpoonup x)) \\
&= e(w') + e(\alpha \rightharpoonup e(x)) = e(w') + e(\alpha \rightharpoonup x) \\
&= e(w') + \alpha \rightharpoonup e(x) = e(w' + \alpha \rightharpoonup x).
\end{aligned}$$

   Hence, $e$ is an idempotent in $End^C(W \oplus X')$.
3. Let $f - e \colon W \oplus X' \to W \oplus X'$ be a $C$-comodule morphism. We need to check that $(f - e)W = W$. Since $(W, e) \in \xi_f$, $(f - e)|_W$ is a unit. That is, it is an automorphism and implies $(f - e)(W) \simeq W$. We prove that $f - e \in End^C(W \oplus X')$ is also an automorphism as follows:
   (a) We want to prove that $f - e \colon W \oplus X' \to W \oplus X'$ is onto. Because $(f - e)(W) \simeq W$, for $w = f(x) \in W$, there is a $w_1 \in W$ such that

$w = (f-e)(w_1)$. Then, we have

$$(f-e)(w_1+x) = (f-e)(w_1) - (f-e)(x)$$
$$= w - f(x) + e(x) = f(x) - f(x) + e(x) = x.$$

Therefore, for any $w' + \alpha \rightharpoonup x \in W \oplus X'$,

$$\begin{aligned} w' + \alpha \rightharpoonup x &= (f-e)(w_2) + \alpha((f-e)(w_1+x)) \qquad (\text{for some } w_2 \in W) \\ &= (f-e)(w_2) + (f-e)(\alpha \rightharpoonup w_1 + \alpha \rightharpoonup x) \\ &= (f-e)(w_2) + (f-e)(\alpha \rightharpoonup w_1) + (f-e)(\alpha \rightharpoonup x) \\ &= (f-e)(w_2 + \alpha \rightharpoonup w_1) + (f-e)(\alpha \rightharpoonup x). \\ &= (f-e)((w_2 + \alpha \rightharpoonup w_1) + \alpha \rightharpoonup x). \end{aligned}$$

This means that for any $w' + \alpha \rightharpoonup x \in W \oplus X'$, there exists a $(w_2 + \alpha \rightharpoonup w_1) + \alpha \rightharpoonup x \in W \oplus X'$ such that $w' + \alpha \rightharpoonup x = (f-e)((w_2 + \alpha \rightharpoonup w_1) + \alpha \rightharpoonup x)$. In particular, $W \oplus X' \in Im(f-e)$, so $f-e$ is onto.

(b) Next, suppose that $(f-e)(w_1 + \alpha \rightharpoonup x) = 0$ for some $w_1 + \alpha \rightharpoonup x \in W \oplus X'$; then

$$\begin{aligned} 0 = (f-e)(w_1 + \alpha \rightharpoonup x) &= (f-e)(w_1) + (f-e)(\alpha \rightharpoonup x) \\ &= w_2 + f(\alpha \rightharpoonup x) - e(\alpha \rightharpoonup x) \\ &\qquad (\text{for some } w_2 = (f-e)(w_1) \in W) \\ &= (w_2 + \alpha \rightharpoonup w) - \alpha \rightharpoonup x \qquad (\text{since } w = f(x)) \\ &= (w_2 + \alpha \rightharpoonup w) - \alpha \rightharpoonup x. \end{aligned}$$

Hence, we have $\alpha \rightharpoonup x = w_2 + \alpha \rightharpoonup w$, which implies that $\alpha \rightharpoonup x \in W \cap X'$. Since $W \cap X = W \cap X' = \{0\}$, $\alpha \rightharpoonup x = 0$. Then, $w_2 + \alpha \rightharpoonup x = w_1$. Thus, $(f-e)(w_2 + \alpha \rightharpoonup x) = (f-e)(w_1) = 0$. Moreover, since $(f-e)|_W$ is an automorphism, $w_1$ must be zero and $w_1 + \alpha \rightharpoonup x = 0$. In particular, $f - e \in End^C(W + X')$ is a monomorphism.

Thus, $f - e \in End^C(W + X')$ is an automorphism.

4. From the previous result, for a $C$-subcomodule $W \oplus X'$, we have the following:
   (a) $(f-e)|_{W \oplus X'}$ is an automorphism (a unit) and $f$-invariant.
   (b) If $e \in End^C(W)$, then $e$ is also an idempotent endomorphism of $W \oplus X'$. Therefore, $W \oplus X' \in \xi_f$. By the maximality of $(W, e)$, $X' = 0$ (since $W \subseteq W \oplus X'$.) Thus, when $X' = C^* \rightharpoonup x = 0$, $x$ must be zero.

(B)**:** Now we prove that for any $m \in W \oplus X$ and $f(m) \in W$, $m \in W$. Let $m = w + x \in W \oplus X$ such that $f(m) = f(w) + f(x)$. Since $W$ is $f$-invariant, $f(w) \in W$. Consequently, $f(x) = f(m) - f(w) \in W$. By using (A), we have $x = 0$ and $m = w \in W$.

□

We adapt [7, Theorem 3.7] to the comodule case.

THEOREM 4.7. *Let $C$ be a flat $R$-module and an $R$-coalgebra satisfying the $\alpha^*$-condition, and let $f \in End^C(M)$, where is $M$ either a semisimple $C$-comodule or a continuous $C$-comodule. No nonzero element of $M$ is annihilated by a left ideal of $C^*$. If $(W, e) \in \xi_f$, where $\xi_f$ is as in Remark 4.5, then we have the following:*

(A): *$(W, e)$ is a maximal element of $\xi_f$ if and only if $W = M$.*

(B): *Given any $(W_0, e_0) \in \xi_f$, there exists a clean decomposition $f = e + u$, where $e$ is an idempotent of $End^C(M)$ extending $e_0$, and $u$ is a unit of $End^C(M)$. In particular, $M$ is a clean $C$-comodule.*

(C): *Let $W_1$ and $W_2$ be $C$-subcomodules of $M$, with $W_1 \cap W_2 = 0$ such that $f|_{W_1}$ and $1 - f|_{W_2}$ are both automorphisms of $C$-comodules. Then there exists a clean decomposition $f = u + e$, where $e$ is an idempotent element of $End^C(M)$, and $u$ is a unit of $End^C(M)$ restricted to zero on $W_1$ and to identity on $W_2$.*

*Proof.*

(C): Let $W_1$ and $W_2$ be $C$-subcomodules of $M$, with $W_1 \cap W_2 = 0$ such that $f|_{W_1}$ and $1 - f|_{W_2}$ are both automorphisms of $C$-comodules. Take $W_0 = W_1 \oplus W_2$ and $e_0$ as the projection of $W_0$ onto $W_2$ with kernel $W_1$, that is,

$$e_0 \colon W_0 \to W_2, w_1 + w_2 \mapsto w_2.$$

We prove (C) by applying (B), Lemma 2.1, and Proposition 2.2. From the definition of $e_0$, $W_0 \in \xi_f$ since $f(W_0) \subseteq W_0$ and $f|_{W_0}$ is clean. Thus, from (B), there is an $f = e + u$ in $End^C(M)$, where $u$ is a unit and $e$ is an idempotent such that $e|_W = e_0$. We only need to check that $e_0|_{W_1}$ is a zero map and that $u|_{W_2}$ is the identity $I_{W_2}$. We use the following steps:

1. For any $w_0 = w_1 + w_2 \in W_0$, with $w_1 \in W_1$ and $w_2 \in W_2, e_0 \circ e_0(w_0) = e_0(w_0) = w_2$. Therefore, $e_0$ is an idempotent in $End^C(W_0)$. For any $w_1 \in W_1, w_1 = w_1 + 0 \in W_0$ and $e_0(w_1 + 0) = 0$. Thus, $e_0|_{W_1}$ is a zero map.
2. By [14], $W_0 = W_1 \oplus W_2 = Ker(e_0) \oplus Im(e_0)$, so $f(W_1) \subseteq Ker(e_0) = W_1$ and $(1 - f)(W_2) \subseteq Im(e_0) = W_2$. From Proposition 2.2, we have that $u|_{W_0} = f|_{W_0} - e_0 \in End^C(W_0)$ is a unit.
3. For any $w_2 = 0 + w_2 \in W_0$,

$$\begin{aligned} u|_{W_0}(w_2) &= f|_{W_0} - e_0(0 + w_2) = f|_{W_0}(w_2) + w_2 \\ &= f|_{W_0}(w_2) - I_{W_2})(w_2) = -(1 - f)_{W_0}(w_2) \\ &\subseteq Im(e_0) = W_2 = I_{W_2}(w_2). \end{aligned}$$

Thus, a unit $u$ in $End_R(M)$ is restricted to the identity in $W_2$.

(B): The proof follows from (A). From Remark 4.5, all $(W_0, e_0) \in \xi_f$ are bounded by a maximal element $(W, e)$ of $\xi_f$. From (A), if $(W, e)$ is the maximal element, then $W = M$. That is, there exists an idempotent $e \in End^C(M), e|_{W_0} = e_0$, so $f - e$ is clean in $End^C(M)$.

(A): If $W = M$, then $(W, e)$ is trivially a maximal element of $\xi_f$. Thus, we need only prove the nontrivial "only if" part of (A). Let $(W, e)$ be maximal. We want to prove that $W = M$ if $M$ is either a semisimple or continuous $C$-comodule and that no nonzero element of $M$ is annihilated by an essential left ideal of $C^*$. We divide the proof into three steps:

1. We shall first prove that $W$ is a summand of $M$:
   (a) If $M$ is a semisimple $C$-comodule, there is nothing to prove since every subcomodule of $M$ is a direct summand [9].
   (b) Let us assume that $M$ is a continuous comodule and that no nonzero element of $M$ is annihilated by an essential left ideal of $C^*$. Since $M$ is a continuous $C$-comodule, $M$ is $CS$ (satisfying $(M_1)$); that is, for every $C$-subcomodule $W$ of $M$, there exists a $C$-subcomodule $E \subseteq M$ where

$E$ is a summand $M$ such that $W \subseteq^e E$. We will prove that $W$ is a direct summand of $M$ by showing that $W$ is (essentially) closed in $M$ (i.e., it has no proper essential extension subcomodule in $M$ such that $W$ is a summand of $M$), that is, $W = E$:

(i) Let $E$ be a maximal essential extension $C$-subcomodule of $W$ in $M$ and $M = E \oplus X$ for some $C$-subcomodule $X$.

(ii) For $y \in E$, let
$$I := \{\alpha \in C^* | \alpha \rightharpoonup y \in W\} \subseteq C^*.$$
For any $\alpha_1, \alpha_2 \in I$, we have the following:
- $(\alpha_1 - \alpha_2) \rightharpoonup y = (\alpha_1 \rightharpoonup y) - (\alpha_2 \rightharpoonup y) \in W$ (by scalar multiplication $\rightharpoonup$), so $\alpha_1 - \alpha_2 \in I$.
- $(\alpha_1 * \alpha_2) \rightharpoonup y = \alpha_1 \rightharpoonup (\alpha_2 \rightharpoonup y) \in W$ (by $*$ as a scalar multiplication of the $C^*$-module $M$), so $(\alpha_1 * \alpha_2) \in I$.
- Since $W$ is a $C^*$-module and $\alpha \rightharpoonup y \in W$, then for any $\beta \in C^*$, $\beta \rightharpoonup (\alpha_1 \rightharpoonup y) \in W$. Hence, $C^* I \subseteq I$. That is, $I$ is a left ideal of $C^*$.

(iii) Now, we want to show that $f(E) \subseteq E$. Since $M = E \oplus X$, for $y \in E$, $f(y) = z + x \in M$ for some $z \in E$ and $x \in X$. For any $\alpha_1 \in I$,
$$\begin{aligned}
&\alpha_1 \rightharpoonup f(y) = \alpha_1 \rightharpoonup (z + x)\\
\Leftrightarrow\ &\alpha_1 \rightharpoonup f(y) = \alpha_1 \rightharpoonup z + \alpha_1 \rightharpoonup x\\
\Leftrightarrow\ &f(\alpha_1 \rightharpoonup y) = \alpha_1 \rightharpoonup z + \alpha_1 \rightharpoonup x\\
\Leftrightarrow\ &-(\alpha_1 \rightharpoonup x) = \alpha_1 \rightharpoonup z - (f(\alpha_1 \rightharpoonup y))\\
&\qquad\qquad \in E + W \quad \text{(since } W \text{ is } f\text{-invariant and } \alpha_1 \rightharpoonup y \in W)\\
&\qquad\qquad \subseteq E \qquad \text{(since } W \subseteq E).
\end{aligned}$$
This implies that $\alpha_1 \rightharpoonup x \in E \cap X = \{0\}$, and so $\alpha_1 \rightharpoonup x = 0$, for any $\alpha_1 \in I$, and $I \subseteq Ann_{C^*}(x) \subseteq C^*$. On the other hand, the $C$-comodule $M$ has no nonzero element of $M$ that is annihilated by a left ideal of $C^*$, which implies that if $\alpha_1 \rightharpoonup x = 0$, then $x = 0$. Thus, $f(y) = z + x = z$ for some $z \in E$, so $f(E) \subseteq E$.

(iv) If the $C$-comodule $M$ is continuous and $E$ is a summand of $M$, then $E$ is also continuous. Here, let us collect our results: $W \subseteq^e E, E$ is continuous, $W$ is $f$-invariant, and $(W, e) \in \xi_f$ such that $f|_W - e$ is a unit in $End^C(W)$ (essentially co-Hopfian). By Remark 4.5, there exists an idempotent $e' \in End^C(E)$ such that $e'|_W = e$ and $f|_E - e'$ is a unit in $End^C(E)$. That is, $(E, e') \in \xi_f$. By the maximality of $(W, e) \in \xi_f$, this implies that $W = E$ or $M = W \oplus X$.

From here on, we shall assume that $M$ is a continuous comodule; semisimplicity will suffice.

2. From Step 1, $M = W \oplus X$, and $W$ is $f$-invariant such that $f(M) = f(W) + f(X) \subseteq W + f(X) \subseteq M$. For any $x \in X, f(x + 0) = f(x)$. Thus, $f\colon X \to f(X)$ is an isomorphism and $W \cap f(X) = 0$. Now we consider two cases separately; in Step 3, we address the case $M \neq W \oplus f(X)$, but here we assume that $M = W \oplus f(X)$. For the idempotent endomorphism $e \in End^C(W)$, take $A = Ker(e)$ and $B = Im(e)$. We already know that $f|_W - e$ is a unit, since $(W, e) \in \xi_f$. By using Lemma 2.1,

$W = A \oplus B = C \oplus D$, where $f\colon A \to C$ and $1 - f\colon B \to D$ are isomorphisms. That is,

$$M = (A \oplus B) \oplus X \simeq (C \oplus D) \oplus f(X).$$

Because $f\colon A \to C$ and $f\colon X \to f(X)$ are isomorphic, $f\colon A \oplus X \to C \oplus f(X)$ is also an isomorphism. By Lemma 2.1, we can see that $f - e$ is a unit in $End^C(M)$, so $(M, e') \in \xi_f$, where $e'\colon M \to B$ is a projection with $Ker(e') = A \oplus X$. On the other hand, $(W, e) \leq (M, e')$. By using the maximality of $(W, e)$, we have $W = M$.

3. Now we assume that $W \oplus f(X) \neq M$. Since $M$ is continuous and $X \simeq f(X)$, from $(CM_1)$ and $(CM_2)$, we have that $W \oplus f(X)$ are summands of $M$. By using $(CM_1)$ for every $0 \neq v \in M$, the $C$-subcomodule $C^* \rightharpoonup v$ is essential inside a (direct) summand of $M$. Since $W \oplus f(X) \neq M$, there exists a $0 \neq v \in M$ such that

(10)
$$C^* \rightharpoonup v \cap (W \oplus f(X)) = 0.$$

However, $M = W \oplus X$, so

(11)
$$f(M) = f(W) + f(X) \subseteq W \oplus f(X) \qquad \text{(since } W \text{ is } f\text{-invariant).}$$

(a) **Claim 1**. The sum $W + \sum_{i=0}^{n\in\mathbb{N}} C^* \rightharpoonup f^i(v)$, where $v \in V$ is direct.

(i) Suppose that

(12)
$$0 = w + \sum_{i=0}^{n\in\mathbb{N}} \alpha_i \rightharpoonup f^i(v) \in W,$$

where $w \in W$, and $\alpha_i \in C^*$ for any $i \in \mathbb{N}$. From Equations (11) and (12), we have

$$\begin{aligned} \alpha_0 \rightharpoonup f^0(v) = \alpha_0 \rightharpoonup v &= -(w + \sum_{i=1}^{n\in\mathbb{N}} \alpha_i \rightharpoonup f^i(v)) \\ &\in W + f(M) \\ &\subseteq W \oplus f(X). \end{aligned}$$

Consequently, $\alpha_0 \rightharpoonup v \in (C^* \rightharpoonup v) \cap (W \oplus f(X))$, which implies that $\alpha_0 \rightharpoonup v = 0$ (see Equation (10)). This yields

$$\begin{aligned} & w + \sum_{i=1}^{n\in\mathbb{N}} f^i(\alpha_i \rightharpoonup v) = 0 \\ \Leftrightarrow & \sum_{i=1}^{n\in\mathbb{N}} f^i(\alpha_i \rightharpoonup v) = -w \qquad \text{(for some } w \in W) \\ \Leftrightarrow & f(\alpha_1 \rightharpoonup v + \cdots + f^{n-1}(\alpha_n \rightharpoonup v)) \in W. \end{aligned}$$

Take

$$\begin{aligned} m &= \alpha_1 \rightharpoonup v + \cdots + f^{n-1}(\alpha_n \rightharpoonup v) \\ &\in C^* \rightharpoonup v + f(M) \\ &\subseteq C^* \rightharpoonup v \oplus W \oplus f(X) \qquad \text{(since } C^* \rightharpoonup v \cap (W \oplus f(X)) = 0) \\ &= (C^* \rightharpoonup v \oplus f(X)) \oplus W. \end{aligned}$$

Consider $C^* \rightharpoonup v \oplus f(X)$ as a $C$-subcomodule of $M$ with

$$f(\alpha_1 \rightharpoonup v + \cdots + f^{n-1}(\alpha_n \rightharpoonup v)) \in W.$$

From Lemma 4.6, we have that $\alpha_1 \rightharpoonup v + \cdots + f^{n-1}(\alpha_n \rightharpoonup v) \in W$. Consequently,

$$\begin{aligned} \alpha_1 \rightharpoonup v + \cdots + f^{n-1}(\alpha_n \rightharpoonup v) = w_1 \qquad (\text{for some } w_1 \in W) \\ \alpha_1 \rightharpoonup v = w_1 - (f(\alpha_2 \rightharpoonup v) + \cdots + f^{n-1}(\alpha_n \rightharpoonup v)) \\ \subseteq W + f(M). \end{aligned}$$

Therefore, $\alpha_1 \rightharpoonup v \in C^* \rightharpoonup v \cap W \oplus f(X)$, and so $\alpha_1 \rightharpoonup v = 0$. Further repetition of this argument shows that $\alpha_i \rightharpoonup v = 0$ for all $i$. Consequently, we also have the result that $w = 0$. On the other hand, we find that

$$L := \oplus_{i=0}^{n\in\mathbb{N}} C^* \rightharpoonup f^i(v),$$

so $L$ is nonzero since $v \neq 0$. Thus, $W \cap L = \{0\}$.

(b) **Claim 2.** $f$ maps $f^i(C^* \rightharpoonup v)$ isomorphically onto $f^{i+1}(C^* \rightharpoonup v)$ for all $i \geq 0$. We break the next part of the argument into four steps:

(i) Let $f(f^i(g \rightharpoonup v)) = 0 \in W$, where $g \in C^*$. We consider $f^i(g \rightharpoonup v) \in f(M) \subseteq W \oplus f(X)$. Using Lemma 4.6(B) and Claim 1, we have $f^i(g \rightharpoonup v) \in W \cap f^i(C^* \rightharpoonup v) = 0$.

(ii) Hence, $f|_L$ is the (forward) shift operator on $L$. By Lemma 3.1, we can find an idempotent $e' \in End_R(L)$ such that $f|_L - e'$ is a unit, which implies that $e \oplus e'$ is an idempotent endomorphism in $End^C(W \oplus L)$.

(iii) The $C$-subcomodule $W \oplus L$ is $f$-invariant, since

$$\begin{aligned} f|_{W\oplus L} &= f(W + L) \\ &= f(W) + f(L) \\ &= f(W) + f(\oplus_{i=0}^{n\in\mathbb{N}} C^* \rightharpoonup f^i(v)) \\ &= f(W) + \oplus_{i=0}^{n\in\mathbb{N}} C^* \rightharpoonup f^{i+1}(v) \\ &\subseteq W \oplus L \qquad (\text{since W is } f\text{-invariant}). \end{aligned}$$

(iv) Moreover, $f|_{W\oplus L} - (e \oplus e') = (f|_W - e) \oplus (f|_L e')$ is a unit, since $f|_W - e$ and $f|_L - e'$ are units.

Thus, $(W \oplus L, e \oplus e') \in \xi_f$. This contradicts the maximality of $(W, e)$, which means that the case $W \oplus f(X) \neq M$ in Step 3 cannot arise.

□

From Theorem 4.7(A) and (B), we reach an important conclusion. Since any $(W_0, e_0) \in \xi_f$ is always bounded by an element maximal in $\xi_f$, if $M$ is either (1) a semisimple $C$-comodule or (2) a continuous $C$-comodule with no nonzero element of $M$ that is annihilated by an essential left ideal $C^*m$ of the dual algebra $C^*$, then $M$ is a clean $C$-comodule.

## References


[1] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278.

[2] R. B. Warfield Jr., *Exchange rings and decompositions of modules*, Math. Ann. **199** (1972), 31–36.

[3] P. Crawley and B. Jónsson, *Refinements for infinite direct decompositions of algebraic systems*, Pacific J. Math. **14** (1964), 797–855.
https://doi.org/10.2140/pjm.1964.14.797
[4] W. K. Nicholson and K. Varadarajan, *Countable linear transformations are clean*, Proc. Amer. Math. Soc. **126** (1998), 61–64.
[5] M. Ó. Searcoid, *Perturbation of linear operators by idempotents*, Irish Math. Soc. Bull. **39** (1997), 10–13.
[6] W. K. Nicholson, K. Varadarajan, and Y. Zhou, *Clean endomorphism rings*, Arch. Math. **83** (2004), 340–343.
[7] V. P. Camillo, D. Khurana, T. Y. Lam, W. K. Nicholson, and Y. Zhou, *Continuous modules are clean*, J. Algebra **304** (2006), 94–111.
[8] V. P. Camillo, D. Khurana, T. Y. Lam, W. K. Nicholson, and Y. Zhou, *A short proof that continuous modules are clean*, In: Contemporary Ring Theory 2011, World Sci. Publ., Singapore (2012), 165–169.
[9] T. Brzeziński and R. Wisbauer, *Corings and Comodules*, Cambridge Univ. Press, Cambridge (2003).
https://doi.org/10.1017/CBO9780511546495
[10] S. H. Mohamed and B. J. Muller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Note Ser. **147**, Cambridge Univ. Press, Cambridge (1990).
[11] N. P. Puspita, I. E. Wijayanti, and B. Surodjo, *Clean coalgebras and clean comodules from finitely generated projective modules*, Algebra Discrete Math. **31** (2021), 251–260.
[12] N. P. Puspita, I. E. Wijayanti, and B. Surodjo, *The necessary and sufficient condition of clean R-coalgebra* $e \rightharpoonup C \leftharpoonup e$, J. Indones. Math. Soc. **28** (2022), 215–220.
[13] N. P. Puspita and I. E. Wijayanti, *Bi-clean and clean Hopf modules*, AIMS Math. **7** (2022), 18784–18792.
[14] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Philadelphia (1991).
[15] A. Tuganbayev, *Rings Close to Regular*, Kluwer Acad. Publ., Dordrecht (2002).

**Nikken Prima Puspita**
Department of Mathematics, Universitas Diponegoro
*E-mail*: nikkenprima@lecturer.undip.ac.id

**Indah Emilia Wijayanti**
Department of Mathematics, Universitas Gadjah Mada
*E-mail*: ind_wijayanti@ugm.ac.id

**Budi Surodjo**
Department of Mathematics, Universitas Gadjah Mada
*E-mail*: surodjo_b@ugm.ac.id