Korean J. Math. **34** (2026), No. 1, pp. 65–74
https://dx.doi.org/10.11568/kjm.2026.34.1.65

ESTIMATES FOR A SUBCLASS OF STARLIKE FUNCTIONS INVOLVING A EXPONENTIAL FUNCTION

Kadhavoor R. Karthikeyan* and Dharmaraj Mohankumar

Abstract. In this paper, a new class of analytic function is defined by using an analytic characterization which is influenced by the multiplicative derivative. Multiplicative derivative is defined in a domain which excludes zero, so here the defined subclass did not involve swapping the ordinary derivative with a multiplicative derivative. But we have just used the motivation behind the purpose of such a restrictive calculus, given the circumstances that we have a more versatile calculus of Newton and Euler. Estimates involving the initial coefficients, inclusion and closure properties, which belong to the defined function class are our main results.

1. Introduction and Definition

We let $\mathcal{S}$ to denote the subclass of $\mathcal{A}$ which are analytic in $\mathbb{U} = \{\nu : |\nu| < 1\}$ with the normalization $\varphi(0) = \varphi'(0) - 1 = 0$. Let $\mathcal{P}(\rho)$ $(0 \leq \rho \leq 1)$ denote the class of Carathéodory's functions (see [13]). Famous subclasses of $\mathcal{S}$ are the so-called classes of starlike and convex function which we will denote it by $\mathcal{S}^*$ and $\mathcal{C}$ respectively. Refer to [23], for formal definitions of various subclasses of $\mathcal{S}$. Lewandowski et al. [18, Theorem 1, pg. 54] established the following inclusion

$$\operatorname{Re}\left[\left(\frac{\nu\varphi'(\nu)}{\varphi(\nu)}\right)^{\gamma}\left(\frac{(\nu\varphi'(\nu))'}{\varphi'(\nu)}\right)^{1-\gamma}\right] > 0 \quad \Rightarrow \quad \varphi \in \mathcal{S}^*, \quad (\forall\gamma \in \mathbb{R};\, \varphi \in \mathcal{A})\,. \tag{1}$$

The class of functions $\varphi \in \mathcal{A}$ satisfying the differential inequality in (1) is known as *gamma starlike function* and here we will denote the class of gamma starlike functions as L_γ. In [5], Altinkaya and Yalçin studied a class of analytic functions satisfying the differential characterization

$$\operatorname{Re}\left\{\frac{1}{2}\left(\frac{\nu\varphi'(\nu)}{\varphi(\nu)} + \left(\frac{\nu\varphi'(\nu)}{\varphi(\nu)}\right)^{\frac{1}{\gamma}}\right)\right\} > 0, \quad (0 < \gamma \leq 1)\,, \tag{2}$$

where $\varphi(\nu) = \nu + \sum_{k=1}^{\infty} a_{mk+1}\nu^{mk+1}$ and $\nu \in \mathbb{U}$. Altinkaya and Yalçin studied the class subject to satisfying a additional stringent condition.

Received October 4, 2025. Revised January 2, 2026. Accepted January 14, 2026.
2010 Mathematics Subject Classification: 30C45.
Key words and phrases: analytic function, univalent function, Mittag-Leffler function, subordination, coefficient inequalities, Fekete-Szegő inequality.
∗ Corresponding author.

1.1. Multiplicative Calculus. Bashirov et al. [7] (also see [8, 9, 24]) studied the properties of *Multiplicative calculus* which is not adaptable to many application problems when compared with the calculus due to Newton and Euler. The multiplicative derivative denoted by φ^* whose definition is given explicitly by

$$\varphi^*(x) = \lim_{h\to 0}\left(\frac{\varphi(x+h)}{\varphi(x)}\right)^{\frac{1}{h}} = e^{\frac{\varphi'(x)}{\varphi(x)}} = e^{[\ln\varphi(x)]'} \quad (x \in \mathbb{R}),$$

where $\varphi'(x)$ is the ordinary derivative. The $*$-derivative of φ at ν satisfies the same as for the real variable but in neighborhood where $\varphi(\nu) \neq 0$. Influenced by multiplicative derivative, Karthikeyan and Murugusundaramoorthy in [17] (also see [11]) introduced a class of analytic functions $\mathcal{R}(\psi)$ satisfying the subordination condition

$$\frac{\nu e^{\frac{\nu^2\varphi'(\nu)}{\varphi(\nu)}}}{\varphi(\nu)} \prec \psi(\nu), \tag{3}$$

where $\psi \in \mathcal{P}$ is starlike symmetric with respect to horizontal axis and is of the form

$$\psi(\nu) = 1 + L_1\nu + L_2\nu^2 + L_3\nu^3 + \cdots, \qquad (L_1 \neq 0;\ \nu \in \mathbb{U}). \tag{4}$$

For the detailed analysis and closure properties of the class $\mathcal{R}(\psi)$, refer to [17]. For details of other prominent studies in univalent function theory involving exponential and logarithm, refer to [6, 12, 33] and references provided therein.

1.2. Objectives: A very limited number of studies on *Geometric Function Theory* involve expressions containing exponentials of some analytic characterization, although the use of other special functions is not uncommon. The main purpose of this study is to fill this gap. We intend to study a new family of analytic functions involving a differential operator, which was defined using Mittag-Leffler function. The main result are the coefficient inequalities, which would help us understand the geometric and algebraic properties.

1.3. A family of differential operator. Now we will give a short introduction to the differential operator defined by Murat et al. [22]. *Mittag-Leffler function*, is a special transcendental function and it reduces to a well-known elementary functions. Refer to Srivastava [25–27] and Srivastava et al. [28–32] for details on Mittag-Leffler function. The generalized Mittag-Leffler three parameter function $E^{\sigma}_{u,\varkappa}(\omega)$ is popularly known as Prabhakar function. Explicitly,

$$E^{\sigma}_{u,\varkappa}(\nu) = \sum_{n=0}^{\infty}\frac{(\sigma)_n\nu^n}{\Gamma(un+\varkappa)\,n!}, \quad (\nu,\ u,\ \varkappa,\ \sigma \in \mathbb{C}, Re(u) > 0), \tag{5}$$

where $(x)_n$ denotes the usual Pochhammer symbol. Motivated by Ibrahim and Darus [15, 16] and using (5), Murat et al. [22] ([10, 34]) defined an operator $D^{\varpi}_r(u,\ \varkappa,\ \sigma)\varphi : \mathbb{U} \longrightarrow \mathbb{U}$ by

(6)

$$D^{\varpi}_r(u,\ \varkappa,\ \sigma)\varphi(\nu) = \nu + \sum_{n=2}^{\infty}\left[n + \frac{r}{2}(1+(-1)^{n+1})\right]^{\varpi}\frac{\Gamma(\varkappa)(\sigma)_{n-1}}{\Gamma(\varkappa+u(n-1))\,(n-1)!}a_n\nu^n,$$

$$(\varpi,\ r \in \mathbb{N}_0 = \{0,\ 1,\ 2, \ldots\},\ \nu,\ u,\ \varkappa,\ \sigma \in \mathbb{C}, Re(u) > 0).$$

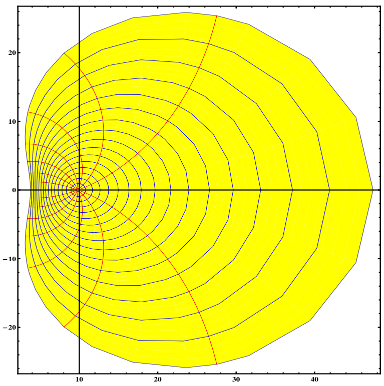

FIGURE 1. Mapping of $G(\nu)$ if $|\nu| < 1$

1.4. A new family of analytic functions. Motivated by class $\mathcal{L}_\gamma$ and the equation (2), we introduce a new class influenced by the definition of multiplicative derivative.

DEFINITION 1.1. For some fixed values of $0 < \gamma \leq 1$, ϖ, $r \in \mathbb{N}_0$ and ω, u, $\varkappa$, $\sigma \in \mathbb{C}$ such that $\Re(u) > 0$, a function φ belongs to the class $\mathcal{K}_r^\varpi(\gamma;\, \omega,\, u,\, \varkappa,\, \sigma, \psi)$ if it satisfies

$$\frac{1}{2}\left(\frac{\nu e^{\frac{\nu^2 D_r^\varpi(u,\,\varkappa,\,\sigma)\varphi'(\nu)}{D_r^\varpi(u,\,\varkappa,\,\sigma)\varphi(\nu)}}}{D_r^\varpi(u,\,\varkappa,\,\sigma)\varphi(\nu)} + \left(\frac{\nu e^{\frac{\nu^2 D_r^\varpi(u,\,\varkappa,\,\sigma)\varphi'(\nu)}{D_r^\varpi(u,\,\varkappa,\,\sigma)\varphi(\nu)}}}{D_r^\varpi(u,\,\varkappa,\,\sigma)\varphi(\nu)}\right)^{\frac{1}{\gamma}}\right) \prec \psi(\nu), \tag{7}$$

where $\psi(\nu) \in \mathcal{P}$ is of the form (4).

REMARK 1.2. Letting $u \to 0$, $\sigma = \gamma = 1$ and $\varpi = 0$ in Definition (1.1), the class $\mathcal{K}_r^\varpi(\gamma;\, \omega,\, u,\, \varkappa,\, \sigma, \psi)$ reduces to the class $\mathcal{R}(\psi)$ recently studied and introduced by Karthikeyan and Murugusundaramoorthy in [17].

Letting $u \to 0$, $\sigma = 1$ and $\varpi = 0$ in Definition (1.1), the class $\mathcal{K}_r^\varpi(\gamma;\, \omega,\, u,\, \varkappa,\, \sigma, \psi)$ reduces to the class

$$\mathcal{K}(\gamma;\, \psi) = \left\{\varphi \in \mathcal{A};\ \frac{1}{2}\left(\frac{\nu e^{\frac{\nu^2\varphi'(\nu)}{\varphi(\nu)}}}{\varphi(\nu)} + \left(\frac{\nu e^{\frac{\nu^2\varphi'(\nu)}{\varphi(\nu)}}}{\varphi(\nu)}\right)^{\frac{1}{\gamma}}\right) \prec \psi(\nu)\right\}.$$

The class $\mathcal{K}(\gamma;\, \psi)$ is the class defined analogous to class recently studied by Altinkaya and Yalçin [5].

EXAMPLE 1.3. Let $\varphi(\nu) = \frac{\nu}{(5-\nu)}$. The function $\varphi(\nu) = \frac{\nu}{(5-\nu)}$ is convex univalent and maps $\mathbb{U}$ onto a circular shaped region in w-plane. Now

$$G(\nu) = \frac{1}{2}\left(\frac{\nu e^{\frac{\nu^2\varphi'(\nu)}{\varphi(\nu)}}}{\varphi(\nu)} + \left(\frac{\nu e^{\frac{\nu^2\varphi'(\nu)}{\varphi(\nu)}}}{\varphi(\nu)}\right)^{\frac{5}{3}}\right) = \frac{1}{2}\left(\frac{\nu}{\left(\frac{\nu}{(5-\nu)}\right)}e^{\frac{5\nu}{5-\nu}} + \left(\frac{\nu}{\left(\frac{\nu}{(5-\nu)}\right)}e^{\frac{5\nu}{5-\nu}}\right)^{\frac{5}{3}}\right).$$

We can see that the function $G(\nu) = \frac{1}{2}\left(\frac{\nu}{\left(\frac{\nu}{(5-\nu)}\right)}e^{\frac{5\nu}{5-\nu}} + \left(\frac{\nu}{\left(\frac{\nu}{(5-\nu)}\right)}e^{\frac{5\nu}{5-\nu}}\right)^{5/3}\right)$ maps the unit disc onto a cardioid region in the right half-plane (see Figure 1). Further, Figure 1 illustrates that the class $\mathcal{K}(\gamma;\, \psi)$ is non-empty, for the fixed value of $\gamma = \frac{3}{5}$. Note that function $\varphi(\nu) = \frac{\nu}{5-\nu}$ does not satisfy the condition (7) for all values of $\gamma(0 < \gamma \leq 1)$.

2. Coefficient Inequalities And Fekete-Szegő Inequality Of $\mathcal{K}_r^{\varpi}(\gamma;\,\omega,\,u,\,\varkappa,\,\sigma,\psi)$

We will need the following inequality obtained by Ma and Minda [20] for the class $\mathcal{P}$. We will need the following lemmas to establish our main results.

Lemma 2.1. *([14, Theorem 1])* If $L(\xi) = 1 + \sum\limits_{r=1}^{\infty} \ell_r \xi^r \in \mathcal{F}$, and $\rho \in \mathbb{C}$, then

$$|\ell_\varepsilon - \rho \ell_r \ell_{\varepsilon - r}| \leq 2 \max \{1; |2\rho - 1|\},$$

for all $1 \leq r \leq \varepsilon - 1$.

Note that the above results are generalization of the well-known results of Ma-Minda [20, p. 162] and Livingston [19, Lemma 1].

Throughout this paper, we denote

$$\Gamma_n = \left[n + \frac{r}{2}(1 + (-1)^{n+1})\right]^{\varpi} \frac{\Gamma(\varkappa)(\sigma)_{n-1}}{\Gamma\left(\varkappa + u(n-1)\right)(n-1)!},$$

which may be real or complex parameters.

Theorem 2.2. *If $\varphi(\nu) \in \mathcal{K}_r^{\varpi}(\gamma;\,\omega,\,u,\,\varkappa,\,\sigma,\psi)$, then we have*

$$|a_2| \leq \frac{1}{|\Gamma_2|}\left[\frac{2\gamma\,|L_1|}{(1+\gamma)} + 1\right], \tag{8}$$

$$|a_3| \leq \frac{2\gamma |L_1|}{(1+\gamma)\,|\Gamma_3|} \max\left\{1; \left|\frac{L_2}{L_1} - \frac{L_1\left[2\gamma^2 + \gamma + 1\right]}{(1+\gamma)^2}\right|\right\} + \frac{4|L_1|}{(1+\gamma)|\Gamma_3|} + \frac{3}{2|\Gamma_3|}, \tag{9}$$

and for all $\rho \in \mathbb{C}$

$$\begin{aligned} \left|a_3 - \rho a_2^2\right| \leq {} & \frac{2\gamma |L_1|}{(1+\gamma)\,|\Gamma_3|} \max\left\{1; \left|\frac{L_2}{L_1} - \frac{L_1\left[2\gamma^2 + \gamma + 1\right]}{(1+\gamma)^2} + \frac{2\rho\gamma L_1 \Gamma_3}{(\gamma+1)\Gamma_2^2}\right|\right\} \\ & + \frac{4\gamma|L_1|}{(1+\gamma)}\left|\frac{\rho}{\Gamma_2^2} - \frac{\gamma}{\Gamma_3}\right| + \left|\frac{3}{2\Gamma_3} - \frac{\rho}{\Gamma_2^2}\right|. \end{aligned} \tag{10}$$

Proof. As $\varphi \in \mathcal{K}_r^{\varpi}(\gamma;\,\omega,\,u,\,\varkappa,\,\sigma,\psi)$, by (7) we have

$$\frac{1}{2}\left(\frac{\nu e^{\frac{\nu^2 D_r^{\varpi}(u,\varkappa,\sigma)\varphi'(\nu)}{D_r^{\varpi}(u,\varkappa,\sigma)\varphi(\nu)}}}{D_r^{\varpi}(u,\,\varkappa,\,\sigma)\varphi(\nu)} + \left(\frac{\nu e^{\frac{\nu^2 D_r^{\varpi}(u,\varkappa,\sigma)\varphi'(\nu)}{D_r^{\varpi}(u,\varkappa,\sigma)\varphi(\nu)}}}{D_r^{\varpi}(u,\,\varkappa,\,\sigma)\varphi(\nu)}\right)^{\frac{1}{\gamma}}\right) = \psi\left[w(\nu)\right]. \tag{11}$$

Thus, let $\vartheta \in \mathcal{P}$ be of the form $\vartheta(\nu) = 1 + \sum_{k=1}^{\infty} \vartheta_n \nu^n$ and defined by

$$\vartheta(\nu) = \frac{1 + w(\nu)}{1 - w(\nu)}, \quad \nu \in \mathbb{U}.$$

On computation, the right hand side of (11)

$$\psi\left[w(\nu)\right] = 1 + \frac{\vartheta_1 L_1}{2}\nu + \frac{L_1}{2}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{L_2}{L_1}\right)\right]\nu^2 + \cdots. \tag{12}$$

Expanding (11), we have

$$\frac{1}{2}\left(\frac{\nu e^{\frac{\nu^2 D_r^{\varpi}(u,\varkappa,\sigma)\varphi'(\nu)}{D_r^{\varpi}(u,\varkappa,\sigma)\varphi(\nu)}}}{D_r^{\varpi}(u,\,\varkappa,\,\sigma)\varphi(\nu)}+\left(\frac{\nu e^{\frac{\nu^2 D_r^{\varpi}(u,\varkappa,\sigma)\varphi'(\nu)}{D_r^{\varpi}(u,\varkappa,\sigma)\varphi(\nu)}}}{D_r^{\varpi}(u,\,\varkappa,\,\sigma)\varphi(\nu)}\right)^{\frac{1}{\gamma}}\right)=1+(1-a_2\Gamma_2)\left(1+\frac{1}{\gamma}\right)\nu+$$

$$\frac{1}{4\gamma^2}\left[(1+\gamma^2)+2a_2\Gamma_2(\gamma-1)+a_2^2\Gamma_2^2[2\gamma^2+\gamma+1]-2a_3\Gamma_3\gamma(\gamma+1)\right]\nu^2+\cdots. \tag{13}$$

From (13) and (12), we obtain

$$a_2=-\frac{1}{\Gamma_2}\left[\frac{\gamma\vartheta_1 L_1}{(1+\gamma)}-1\right] \tag{14}$$

and

$$a_3=\frac{-\gamma L_1}{(1+\gamma)\Gamma_3}\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{L_2}{L_1}+\frac{L_1\left[2\gamma^2+\gamma+1\right]}{(1+\gamma)^2}\right)\right]-\frac{2\gamma^2\vartheta_1 L_1}{(1+\gamma)\Gamma_3}+\frac{3}{2\Gamma_3}. \tag{15}$$

It is well-known that $|\vartheta_1|\le 2$. Equations (8) can be obtained by applying the bounds of ϑ_1 in (14). From (15), we have

$$|a_3|\le\frac{\gamma|L_1|}{(1+\gamma)|\Gamma_3|}\left|\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{L_2}{L_1}+\frac{L_1\left[2\gamma^2+\gamma+1\right]}{(1+\gamma)^2}\right)\right|+\frac{2\gamma^2\left|\vartheta_1 L_1\right|}{(1+\gamma)\left|\Gamma_3\right|}+\frac{3}{2\left|\Gamma_3\right|}. \tag{16}$$

Applying Lemma 2.1 together with inequalities $|\vartheta_1|\le 2$ in (16), we get (9).

To establish (10), we consider

$$\begin{aligned}\left|a_3-\rho a_2^2\right|=&\left|\frac{-\gamma L_1}{(1+\gamma)\Gamma_3}\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{L_2}{L_1}+\frac{L_1\left[2\gamma^2+\gamma+1\right]}{(1+\gamma)^2}\right)\right]-\frac{2\gamma^2\vartheta_1 L_1}{(1+\gamma)\Gamma_3}+\frac{3}{2\Gamma_3}\right.\\&\left.-\frac{\rho}{\Gamma_2^2}\left(\frac{\gamma^2\vartheta_1^2L_1^2}{(\gamma+1)^2}-\frac{2\gamma\vartheta_1 L_1}{(\gamma+1)}+1\right)\right|\\=&\left|\frac{-\gamma L_1}{(1+\gamma)\Gamma_3}\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{L_2}{L_1}+\frac{L_1\left[2\gamma^2+\gamma+1\right]}{(1+\gamma)^2}-\frac{2\rho\gamma L_1\Gamma_3}{(\gamma+1)\Gamma_2^2}\right)\right]\right.\\&\left.+\frac{2\gamma\vartheta_1 L_1}{(1+\gamma)}\left(\frac{\rho}{\Gamma_2^2}-\frac{\gamma}{\Gamma_3}\right)+\left(\frac{3}{2\Gamma_3}-\frac{\rho}{\Gamma_2^2}\right)\right|.\end{aligned}$$

Using Lemma 2.1, we obtain the assertion (10). □

Letting $u\to 0$, $\sigma=\gamma=1$ and $\varpi=0$ in Theorem 2.2

Corollary 2.3. ([17, Corollary 1]) If $\varphi(\nu)\in\mathcal{R}(\psi)$, then we have

$$|a_2|\le 1+|L_1|,$$

$$|a_3|\le|L_1|\left\{\max\left\{1,\left|\frac{L_2}{L_1}-L_1\right|\right\}+\frac{3}{2\left|L_1\right|}+2\right\},$$

and for all $\rho\in\mathbb{C}$

$$\left|a_3 - \rho a_2^2\right| \leq |L_1| \left[\max\left\{1, \left|\frac{L_2}{L_1} - L_1(1-\rho)\right|\right\} + 2|1-\rho| + \frac{1}{2|L_1|}|3-2\rho| \right].$$

Letting $\psi(\nu) = (1+\nu)/(1-\nu)$ in Corollary 2.3, we get

COROLLARY 2.4. *[17, Corollary 2] Let $\varphi \in \mathcal{A}$ satisfy the condition*

$$\operatorname{Re}\left(\frac{ze^{\frac{\nu^2\varphi'(\nu)}{\varphi(\nu)}}}{\varphi(\nu)}\right) > 0.$$

Then,

$$|a_2| \leq 3, \qquad |a_3| \leq \frac{15}{2},$$

and for a complex number ρ,

$$\left|a_3 - \rho a_2^2\right| \leq 2\left[\max\{1;\ |2\rho - 1|\} + 2|1-\rho| + \frac{1}{4}|3-2\rho|\right].$$

3. Inverse Function Coefficient Estimates

Every function $\varphi(\nu) = \nu + \sum_{n=2}^{\infty} a_n \nu^n$ in $\mathcal{S}$ has an inverse φ^{-1}, defined by $\varphi^{-1}(\varphi(\nu)) = \nu$, $\nu \in \mathbb{U}$ and $\varphi(\varphi^{-1}(w)) = w$, $(|w| < r;\ r \geq 1/4)$ where

$$\chi(w) = \varphi^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^2 - \left(5a_2^2 - 5a_2a_3 + a_4\right)w^4 + \cdots. \tag{17}$$

Now we will obtain the inverse function coefficient bounds for functions in $\mathcal{K}_r^{\varpi}(\gamma;\ \omega,\ u,\ \varkappa,\ \sigma, \psi)$.

THEOREM 3.1. *Let $\varphi \in \mathcal{K}_r^{\varpi}(\gamma;\ \omega,\ u,\ \varkappa,\ \sigma, \psi)$ and let φ^{-1} be the inverse of φ defined by*

$$\varphi^{-1}(w) = w + \sum_{k=2}^{\infty} d_k w^k, \quad (|w| < r;\ r \geq 1/4),$$

then

$$|d_2| \leq \frac{1}{|\Gamma_2|}\left[\frac{2\gamma|L_1|}{(1+\gamma)} + 1\right]$$

and

$$|d_3| \leq \frac{2\gamma|L_1|}{(1+\gamma)|\Gamma_3|}\max\left\{1;\ \left|\frac{L_2}{L_1} - \frac{L_1[2\gamma^2+\gamma+1]}{(1+\gamma)^2} + \frac{4\gamma L_1\Gamma_3}{(\gamma+1)\Gamma_2^2}\right|\right\} + \frac{4\gamma|L_1|}{(1+\gamma)}\left|\frac{2}{\Gamma_2^2} - \frac{\gamma}{\Gamma_3}\right| + \left|\frac{3}{2\Gamma_3} - \frac{2}{\Gamma_2^2}\right|.$$

Also, for all $\tau \in \mathbb{C}$, we have

$$\left|d_3 - \tau d_2^2\right| \leq \frac{2\gamma|L_1|}{(1+\gamma)|\Gamma_3|}\max\left\{1;\ \left|\frac{L_2}{L_1} - \frac{L_1[2\gamma^2+\gamma+1]}{(1+\gamma)^2} + \frac{2(\tau-2)\gamma L_1\Gamma_3}{(\gamma+1)\Gamma_2^2}\right|\right\} + \frac{4\gamma|L_1|}{(1+\gamma)}\left|\frac{(\tau-2)}{\Gamma_2^2} - \frac{\gamma}{\Gamma_3}\right| + \left|\frac{3}{2\Gamma_3} - \frac{(\tau-2)}{\Gamma_2^2}\right|.$$

Proof. From $\varphi(\nu) = \nu + \sum_{n=2}^{\infty} a_n \nu^n$ and (17), we have

$$d_2 = -a_2 \quad \text{and} \quad d_3 = 2a_2^2 - a_3.$$

The estimate for $|d_2| = |a_2|$ follows immediately from (14). Letting $\rho = 2$ in (10), we get the estimate $|d_3|$. Now, consider

$$\left|d_3 - \tau d_2^2\right| = \left|2a_2^2 - a_3 - \tau a_2^2\right| = \left|a_3 - (\tau - 2)a_2^2\right|.$$

Changing $\rho = (\tau - 2)$ in the (10), we get the desired result. □

Letting $u \to 0$, $\sigma = \gamma = 1$ and $\varpi = 0$ in Theorem 3.1, we get the following result.

Corollary 3.2. *If $\varphi(\nu) \in \mathcal{R}(\psi)$ and let φ^{-1} be the inverse of φ defined by*

$$\varphi^{-1}(w) = w + \sum_{k=2}^{\infty} d_k w^k, \quad (|w| < r;\ r \geq 1/4),$$

then we have $|d_2| \leq 1 + |L_1|$,

$$|d_3| \leq |L_1| \max\left\{1;\ \left|\frac{L_2}{L_1} + L_1\right|\right\} + 2|L_1| + \frac{1}{2},$$

and for all $\rho \in \mathbb{C}$

$$\left|d_3 - \tau d_2^2\right| \leq |L_1| \max\left\{1;\ \left|\frac{L_2}{L_1} - L_1(3 - \tau)\right|\right\} + 2|L_1(\tau - 3)| + \left|\frac{7}{2} - \tau\right|.$$

Letting $\psi(\nu) = \frac{1+\nu}{1-\nu}$ in Corollary 3.2, we get

Corollary 3.3. *Let $\varphi \in \mathcal{A}$ satisfy the condition*

$$\operatorname{Re}\left(\frac{ze^{\frac{\nu^2\varphi'(\nu)}{\varphi(\nu)}}}{\varphi(\nu)}\right) > 0,$$

and let φ^{-1} be the inverse of φ defined by

$$\varphi^{-1}(w) = w + \sum_{k=2}^{\infty} d_k w^k, \quad (|w| < r;\ r \geq 1/4),$$

then we have $|d_2| \leq 3$, $\quad |d_3| \leq \frac{17}{2}$, and for all $\rho \in \mathbb{C}$

$$\left|d_3 - \tau d_2^2\right| \leq 2\max\{1;\ |(5 - 2\tau)|\} + 4|\tau - 3| + \left|\frac{7}{2} - \tau\right|.$$

4. Logarithmic Coefficients

Milin in [21] highlighted the use of the logarithmic coefficients to resolve the bounds of the Taylor coefficients of univalent functions. Refer to [1–4] for the detailed study on the significance properties of the logarithmic coefficients.

The logarithmic coefficients ℓ_n of a function $\varphi \in \mathcal{A}$ such that $\dfrac{\varphi(\nu)}{\nu} \neq 0$ for all $\nu \in \mathbb{U}$ is defined by

$$\log \varphi(\nu) = 2\sum_{n=1}^{\infty} \ell_n \nu^n. \tag{18}$$

Now we will add additional criterion to the class $\mathcal{K}_r^{\varpi}(\gamma;\, \omega,\, u,\, \varkappa,\, \sigma, \psi)$, so that logarithmic coefficients of $\mathcal{K}_r^{\varpi}(\gamma;\, \omega,\, u,\, \varkappa,\, \sigma, \psi)$ is well-defined. That is, we let $\mathcal{LK}_r^{\varpi}(\gamma;\, \omega,\, u,\, \varkappa,\, \sigma, \psi) = \mathcal{K}_r^{\varpi}(\gamma;\, \omega,\, u,\, \varkappa,\, \sigma, \psi) \cap \left\{ \varphi \text{ is analytic in } \mathcal{U} : \frac{\varphi(\nu)}{\nu} \neq 0,\, \nu \in \mathcal{U} \right\}$. Note that for all functions $\mathcal{LK}_r^{\varpi}(\gamma;\, \omega,\, u,\, \varkappa,\, \sigma, \psi)$, the relation (18) is well-defined.

Theorem 4.1. *If $\varphi(\nu) \in \mathcal{LK}_r^{\varpi}(\gamma;\, \omega,\, u,\, \varkappa,\, \sigma, \psi)$, with the logarithmic coefficients given by* (18)*, then,*

$$|\ell_1| \leq \frac{1}{2\,|\Gamma_2|}\left[\frac{2\gamma\,|L_1|}{(1+\gamma)} + 1\right],$$

$$|\ell_2| \leq \frac{\gamma|L_1|}{(1+\gamma)\,|\Gamma_3|} \max\left\{1;\, \left|\frac{L_2}{L_1} - \frac{L_1\,[2\gamma^2+\gamma+1]}{(1+\gamma)^2} + \frac{\gamma L_1 \Gamma_3}{(\gamma+1)\Gamma_2^2}\right|\right\} + \frac{4\gamma|L_1|}{(1+\gamma)}\left|\frac{1}{2\Gamma_2^2} - \frac{\gamma}{\Gamma_3}\right| + \left|\frac{3}{2\Gamma_3} - \frac{1}{2\Gamma_2^2}\right|.$$

For $\mu \in \mathbb{C}$ we have

$$|\ell_2 - \mu\ell_1^2| \leq \frac{\gamma|L_1|}{(1+\gamma)\,|\Gamma_3|} \max\left\{1;\, \left|\frac{L_2}{L_1} - \frac{L_1\,[2\gamma^2+\gamma+1]}{(1+\gamma)^2} + \frac{(1+\mu)\gamma L_1 \Gamma_3}{(\gamma+1)\Gamma_2^2}\right|\right\} + \frac{4\gamma|L_1|}{(1+\gamma)}\left|\frac{(1+\mu)}{\Gamma_2^2} - \frac{\gamma}{\Gamma_3}\right| + \left|\frac{3}{2\Gamma_3} - \frac{(1+\mu)}{\Gamma_2^2}\right|. \tag{19}$$

Proof. From $\varphi(\nu) = \nu + \sum_{n=2}^{\infty} a_n \nu^n$ and equating the first two coefficients of the relation (18) we get

$$\ell_1 = \frac{a_2}{2}, \quad \ell_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right).$$

Using (8) and (10) in the above expression, we can find the estimates for ℓ_1 and ℓ_2. To find the estimate (4.1), consider

$$\left|\ell_2 - \mu\ell_1^2\right| = \frac{1}{2}\left[a_3 - \frac{(1+\mu)}{2}a_2^2\right].$$

Changing $\rho = \frac{1+\mu}{2}$ in (10), we get the desired result. □

Remark 4.2. For different choices of ψ and parameters involved in the Definition 1.1, the class $\mathcal{K}_r^{\varpi}(\gamma;\, \omega,\, u,\, \varkappa,\, \sigma, \psi)$ reduces to classes impacted by conic regions but classes like starlike, convex, and spirallike are not special case of $\mathcal{K}_r^{\varpi}(\gamma;\, \omega,\, u,\, \varkappa,\, \sigma, \psi)$. Hence main results have lots of applications, here we pointed out only few of them.

References

[1] E. A. Adegani, N. E. Cho, and M. Jafari, *Logarithmic coefficients for univalent functions defined by subordination*, Mathematics **7** (5) (2019), 408.
https://doi.org/10.3390/math7050408

[2] E. A. Adegani, T. Bulboacă, N. H. Mohammed, and P. Zaprawa, *Solution of logarithmic coefficients conjectures for some classes of convex functions*, Mathematica Slovaca **73** (1) (2023), 79–88.
https://doi.org/10.1515/ms-2023-0009

[3] D. Alimohammadi, N. E. Cho, E. A. Adegani, and A. Motamednezhad, *Argument and coefficient estimates for certain analytic functions*, Mathematics **8** (1) (2020), 88.
https://doi.org/10.3390/math8010088

[4] D. Alimohammadi, E. A. Adegani, T. Bulboacă, and N. E. Cho, *Logarithmic coefficients for classes related to convex functions*, Bull. Malays. Math. Sci. Soc. **44** (2021), 2659–2673.
https://doi.org/10.1007/s40840-021-01085-z

[5] S. Altinkaya and S. Yalçin, *On some subclasses of M-fold symmetric bi-univalent functions*, Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **67** (1) (2018), 29–36.
https://doi.org/10.1501/Commua1_0000000827

[6] J. H. Arango, D. Mejía, and S. Ruscheweyh, *Exponentially convex univalent functions*, Complex Variables Theory Appl. **33** (1-4) (1997), 33–50.
https://doi.org/10.1080/17476939708815010

[7] A. E. Bashirov, E. M. Kurpinar, and A. Özyapıcı, *Multiplicative calculus and its applications*, J. Math. Anal. Appl. **337** (1) (2008), 36–48.
https://doi.org/10.1016/j.jmaa.2007.03.081

[8] A. E. Bashirov, E. Mısırlı, Y. Tandoğdu, and A. Özyapıcı, *On modeling with multiplicative differential equations*, Appl. Math. J. Chin. Univ. **26** (2011), 425–438.
https://doi.org/10.1007/s11766-011-2767-6

[9] A. E. Bashirov and M. Riza, *On complex multiplicative differentiation*, TWMS J. Appl. Eng. Math. **1** (1) (2011), 75–85.

[10] D. Breaz, K. R. Karthikeyan, and E. Umadevi, *Non-Carathéodory analytic functions with respect to symmetric points*, Math. Comput. Model. Dyn. Syst. **30** (1) (2024), 266–283.
https://doi.org/10.1080/13873954.2024.2341691

[11] D. Breaz, K. R. Karthikeyan, and G. Murugusundaramoorthy, *Applications of Mittag–Leffler functions on a subclass of meromorphic functions influenced by the definition of a non-Newtonian derivative*, Fractal Fract. **8** (2024), 509.
https://doi.org/10.3390/fractalfract8090509

[12] D. Breaz, K. R. Karthikeyan, D. Mohankumar, and A. Senguttuvan, *Properties of a subclass of starlike functions involving the quantum derivative operator*, MethodsX **15** (2025), 103740.
https://doi.org/10.1016/j.mex.2025.103740

[13] C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen*, Math. Ann. **64** (1) (1907), 95–115.
https://doi.org/10.1007/BF01449883

[14] I. Efraimidis, *A generalization of Livingston's coefficient inequalities for functions with positive real part*, J. Math. Anal. Appl. **435** (1) (2016), 369–379.
https://doi.org/10.1016/j.jmaa.2015.10.050

[15] R. W. Ibrahim and M. Darus, *Subordination inequalities of a new Salagean-difference operator*, Int. J. Math. Comput. Sci. **14** (3) (2019), 573–582.
https://future-in-tech.net/14.3/R-RabhaIbrahim.pdf

[16] R. W. Ibrahim, *On a Janowski formula based on a generalized differential operator*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **69** (2) (2020), 1320–1328.

[17] K. R. Karthikeyan and G. Murugusundaramoorthy, *Properties of a class of analytic functions influenced by multiplicative calculus*, Fractal Fract. **8** (3) (2024), 131.
https://doi.org/10.3390/fractalfract8030131

[18] Z. Lewandowski, S. S. Miller, and E. Złotkiewicz, *Gamma-starlike functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **28** (1974), 53–58.

[19] A. E. Livingston, *The coefficients of multivalent close-to-convex functions*, Proc. Amer. Math. Soc. **21** (1969), 545–552.
https://doi.org/10.2307/2036417

[20] W. C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, Proc. Conf. Complex Analysis (Tianjin, 1992), Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA (1992), 157–169.

[21] I. M. Milin, *Univalent functions and orthonormal systems*, Nauka, Moscow (1971); English translation: American Mathematical Society, Providence (1977).

[22] C. Murat, K. R. Karthikeyan, and G. Murugusundaramoorthy, *Inequalities on a class of analytic functions defined by generalized Mittag-Leffler function*, Filomat **37** (19) (2023), 6277–6288.
https://doi.org/10.2298/FIL2319277C

[23] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen (1975).

[24] M. Riza, A. Özyapici, and E. Mısırlı, *Multiplicative finite difference methods*, Quart. Appl. Math. **67** (4) (2009), 745–754.
https://doi.org/10.1090/S0033-569X-09-01158-2

[25] H. M. Srivastava, *A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics*, Symmetry **13** (2021), 2294.
https://doi.org/10.3390/sym13122294

[26] H. M. Srivastava, *An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions*, J. Adv. Engrg. Comput. **5** (3) (2021), 135–166.
http://dx.doi.org/10.55579/jaec.202153.340

[27] H. M. Srivastava, *Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations*, J. Nonlinear Convex Anal. **22** (8) (2021), 1501–1520.

[28] H. M. Srivastava, A. Kumar, S. Das, and K. Mehrez, *Geometric properties of a certain class of Mittag–Leffler-type functions*, Fractal Fract. **6** (2022), 54.
https://doi.org/10.3390/fractalfract6020054

[29] H. M. Srivastava, A. Fernandez, and D. Baleanu, *Some new fractional-calculus connections between Mittag–Leffler functions*, Mathematics **7** (2019), 485.
https://doi.org/10.3390/math7060485

[30] H. M. Srivastava, M. Bansal, and P. Harjule, *A study of fractional integral operators involving a certain generalized multi-index Mittag-Leffler function*, Math. Methods Appl. Sci. **41** (16) (2018), 6108–6121.
https://doi.org/10.1002/mma.5122

[31] H. M. Srivastava and Z. Tomovski, *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel*, Appl. Math. Comput. **211** (1) (2009), 198–210.
https://doi.org/10.1016/j.amc.2009.01.055

[32] H. M. Srivastava, *On an extension of the Mittag-Leffler function*, Yokohama Math. J. **16** (1968), 77–88.

[33] S. S. Varma, T. Rosy, and U. Vadivelan, *Radius of exponential convexity of certain subclass of analytic functions*, Creat. Math. Inform. **29** (1) (2020), 109–112.
https://doi.org/10.37193/CMI.2020.01.13

[34] E. Umadevi and K. R. Karthikeyan, *A Subclass of Close-to-Convex Function Involving Srivastava-Tomovski Operator*, Recent Developments in Algebra and Analysis, ICRDM 2022, Trends in Mathematics, Birkhäuser, Cham (2024).
https://doi.org/10.1007/978-3-031-37538-5_25

Kadhavoor R. Karthikeyan
Department of Applied Mathematics and Science,
National University of Science & Technology, Muscat P.O. Box 620, Oman.
E-mail: karthikeyan@nu.edu.om

Dharmaraj Mohankumar
Department of Mathematics for Innovation, Saveetha School of Engineering,
Saveetha Institute of Medical and Technical Sciences(SIMATS),
Thandalam, Chennai, 602 105, India.
E-mail: dmohankumarmaths@gmail.com