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SINGULAR DYNAMICS ON MANIFOLDS: A CARTESIAN PRODUCT APPROACH TO HOMOTOPY GROUP

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Abstract. In this paper, we present the induced singular dynamics of the Cartesian product manifold and their homotopy groups. We also analyze the induced limit singular dynamics on the Cartesian product of manifolds and their associated homotopy group. The role played by the dynamical manifold in the wedge sum of manifolds and their homotopy group will be identified. We introduce a certain type of conditional singular dynamical manifold for free group elements and its homotopy group. Theorems concerning these relations are provided. The results we achieved provide new insights into the relationship between singular dynamics and topology by highlighting how a system's history reflects the algebraic structure of its core manifold.

1. Introduction and definitions

This paper explains singular dynamical manifolds from an algebra and topology viewpoint. We point out what the dynamical Cartesian product $N_1 \times N_2 \times ... \times N_n$ by the homotopy group $\pi_k(N_1 \times N_2 \times ... \times N_n)$. Manifolds are defined as spaces carrying algebraic and topological structure. If one manifold changed into another through a certain transformation, then the first has a different algebraic structure than the second.

A dynamical system is a mathematical model describing how a point changes in a space over time. The math rules used to describe the swinging of a pendulum, The way water flows in a pipe or the number of fish in a lake each spring are examples of dynamical systems. A dynamical system has a state determined by a collection of real numbers. Small changes in the state of the system correspond to small changes in the numbers. The numbers are also the coordinates of a geometrical space-a manifold. The evolution rule is a fixed rule describing how the system moves from now to the future. The rule is certain: each current state results in one future state for a time step. The idea of a dynamical system is based on Newtonian mechanics. In natural sciences and engineering, the evolution rule of a dynamical system is given by a relation that shows the system state only a short time ahead. The work in [6] studied topological methods involving concentric manifolds.

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By studying dynamical systems, we learn laws that let us predict future states from current ones. [15, 17]. Let (Z, τ) be a topological space, and let $T \subseteq \mathbb{R}$ be a time domain. A dynamic transformation on Y is a family of continuous maps $\Psi = \{\Psi_t : Z \to Z,\ t \in T\}$ satisfying the following conditions: (1) $\Psi : T \times Z \to Z$, defined by $\Psi(t, z) = \Psi_t(z)$ is continuous with respect to the product topology on $T \times Z$. (2) $\Psi_0 = id_x$ (3) $\Psi_{t+s}(z) = \Psi_t(\Psi_s(z)),\ \forall z \in Z,\ \forall t, s \in T$ [8]. A subset C of a space Z is a retract of Z if there exists a continuous map $r : Z \to C$, in which $r|C = id(C)$ [12]. For the chosen point $z_0 \in Z$, the k-th homotopy group of $(Z,\ z_0)$, denoted by $\pi_k(Z,\ z_0)$ is the set of all homotopy classes of continuous maps $f : (S^k,\ x_0) \to (Z,\ z_0)$ in which $(S^k,\ s_0)$ is the k-dimensional sphere with a chosen basepoint x_0 [9, 10]. Research on the homotopy groups of specific spaces has been carried out in [7, 11, 14, 16]. Consider spaces Z_1 and Z_2 with points $z_1 \in Z_1$ and $z_2 \in Z_2$, then the wedge sum $Z_1 \curlyvee Z_2$ is formed by joining $Z_1 \cup Z_2$ created by merging z_1 and z_2 into one point [9, 13]. In this paper, we write F_n for the free group of rank n. This work is devoted to the study of the k-th homotopy group $\pi_k(Z, z_0)$ for $k \geq 1$. Transformations on manifolds and graphs are discussed in [1–5].

2. The main results

Definition 2.1. *Let $\mathcal{N}$ be a topological manifold. Let $\Psi_t : D_t \subseteq \mathcal{N} \to \mathcal{N}$ and allow the domain to shrink over time as $t = 0,\ D_0 = y$, and $t \to t^*,\ D_t$ excludes some submanifold (e.g., a point or region), representing a hole.*

Example 2.2. *Consider $T = [0, 1]$ and let $D_t = S^2 - \{(0, 0, 1 - t)\} \subseteq S^2$, with $\Psi_t : D_t \to S^2$ defined by $\Psi_t(x) = x$. As $t = 1,\ D_1 = S^2 - \{(0, 0, 0)\}$, and as $t = 0,\ D_0 = S^2 - \{(0, 0, 1)\}$.*

Definition 2.3. *Let $\mathcal{N}$ be a topological manifold. A singular dynamic transformation on $\mathcal{N}$ is a family of maps $\Psi = \{\Psi_t : D_t \to \mathcal{N}_t,\ t \in T\}$ satisfying:*
the following conditions:
(1) Let $G = \{(t,\ z) \in T \times \mathcal{N} :\ z \in D_t\}$ with the map $\Psi : G \to \mathcal{N}$, defined by $\Psi(t,\ z) = \Psi_t(z)$, be continuous with respect to the subspace topology on $G \subseteq T \times \mathcal{N}$.
(2) $\Psi_0 = id_x$.
(3) $\Psi_{t+s}(z) = \Psi_t(\Psi_s(z)),\ \forall x \in \mathcal{N},\ \forall t,\ s \in T$ in which $D_t \subseteq \mathcal{N}_t \subseteq \mathcal{N}$. Holes are formed in which $z \notin D_t$ or $\Psi_t(z) \notin \mathcal{N}_t$. Some manifolds, like a sphere, can lose parts of themselves over time. This condition allows topology change; for example, S^2 and $S^2 - \{p\}$ are not topologically equivalent. For simplicity, we denote the singular dynamic transformation by singular dynamics. Such transformations appear in physics, Ricci flow, geometry, and black hole formation. Also, we define the fixed point as the point at which the singular dynamics remain unchanged over time.

Example 2.4. *Consider the manifold $\mathcal{N} = [0, 1]$ with the subspace topology from $\mathbb{R}$, and let $T = [0, \infty)$. Define $\Psi = \{\Psi_t : D_t \to \mathcal{N}_t,\ t \in T\}$ as follows $D_t = [0, 1], \forall t \in T$ (constant), $\mathcal{N}_t = [0, e^{-t}] \subseteq \mathcal{N}$ with $\Psi_t(z) = ze^{-t}, \forall z \in D_t$. Then, $G = T \times \mathcal{N} = [0, \infty) \times [0, 1]$. In this approach the map $\Psi : G \to \mathcal{N}$ given by $\Psi(t,\ z) = \Psi_t(z) = ze^{-t}$ is continuous on G since both z and e^{-t} are continuous. Also, at $t = 0,\ \Psi_0(z) = z,\ \forall z \in D_0 = [0, 1]$, and $\mathcal{N}_0 = [0, 1] = \mathcal{N}$. Therefore, $\Psi_0 = id_z$. Moreover, $\forall t,\ s \in T$ and $z \in D_s = [0, 1],\ \Psi_s(z) = ze^{-s}$. Now, $\Psi_s(z) \in D_t$ since $\Psi_s(z) \in [0, e^{-s}] \subseteq [0, 1] = D_t$. Thus, $\Psi_t(\Psi_s(z)) = \Psi_t(ze^{-s}) = ze^{-(s+t)} = \Psi_{t+s}(z)$. Hence, $\Psi_{t+s}(z) = \Psi_t(\Psi_s(z))$. The* *topological change when* $\mathcal{N}_t = [0, e^{-t}]$ *begins at* $\mathcal{N}_0 = [0,1]$ *at* $t = 0$ *and continuously collapses to* $\{0\}$ *as* $t \to \infty$*, introducing singularity and not holes.*

EXAMPLE 2.5. *Suppose that* $A \subseteq \mathcal{N}$ *is a closed subspace of a manifold* $\mathcal{N}$ *and let* $t \in (0,1]$ *with* $D_t = \mathcal{N}$, $\Psi_t(z) \in A$, $\forall z \in \mathcal{N}$ *and* $\Psi_t|_A = id_A$. *If* $r : \mathcal{N} \to A$ *is continuous, in which* $r(z) = \lim_{t \to 0^+} \Psi_t(z)$ *exists* $\forall z \in \mathcal{N}$. *Then* r *is a retraction.*

THEOREM 2.6. *The limit of the singular dynamical system is the fixed point.*

Proof. Suppose N^t is a manifold and let,
$\Psi^t : N^t \to N^{t_1}$, $N^{t_1} \subseteq N^{t_1}$, $\Psi^{t_1} : \Psi^t(N^t) \to N^{t_2}$, $N^{t_2} \subseteq N^{t_1}, \ldots,$ $\Psi^{t_m} : N^{t_{m-1}} \to N^{t_m}$, $N^{t_{m-1}} \subseteq N^{t_m}$ such that $\underset{m\to\infty}{lim}\ \Psi^t$ is the fixed point. □

LEMMA 2.7. *Let* N *be an arc-connected manifold, and consider a singular dynamic mapping* $\Psi : N \to N$. *Then there is an induced singular dynamic* $\overline{\Psi} : \pi_k(N) \to \pi_k(N)$ *such that* $\overline{\Psi}(\pi_k(N))$ *is isomorphic to* $\pi_k(\Psi(N))$.

Proof. The proof is obvious. □

THEOREM 2.8. *If* $\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_n$ *are arc-connected manifolds and* Ψ *is a singular dynamics from* $\overset{n}{\underset{i=1}{\curlyvee}} \mathcal{N}_i$ *to itself, then there is a resulting singular dynamics* $\overline{\Psi}$ *of* $\overset{n}{\underset{i=1}{*}} \pi_k(\mathcal{N}_i)$ *into itself that minimizes the rank of* $\overset{n}{\underset{i=1}{*}} \pi_k(\mathcal{N}_i)$.

Proof. Let $\Psi : \overset{n}{\underset{i=1}{\curlyvee}} \mathcal{N}_i \to \overset{n}{\underset{i=1}{\curlyvee}} \mathcal{N}_i$ be a singular dynamic of $\overset{n}{\underset{i=1}{\curlyvee}} \mathcal{N}_i$ into itself. Then $\Psi : \overset{n}{\underset{i=1}{\curlyvee}} \mathcal{N}_i \to \overset{n}{\underset{i=1}{\curlyvee}} \mathcal{N}_i$ has the following forms:
In a case of $\Psi(\overset{n}{\underset{i=1}{\curlyvee}} \mathcal{N}_i) = \mathcal{N}_1 \curlyvee \mathcal{N}_2 \curlyvee ... \curlyvee \Psi(\mathcal{N}_p) \curlyvee ... \curlyvee \mathcal{N}_n$ for $p = 1, 2, \ldots n$, we obtain $\overline{\Psi}(\overset{n}{\underset{i=1}{*}} \pi_k(\mathcal{N}_i)) = \pi_k(\Psi(\overset{n}{\underset{i=1}{\curlyvee}} \mathcal{N}_i)) \approx \pi_k(\mathcal{N}_1) * \pi_k(\mathcal{N}_2) * ... * \pi_k(\Psi(\mathcal{N}_p)) * ... * \pi_k(\mathcal{N}_n)$. Since $rank(\pi_k(\Psi(\mathcal{N}_p))) \leq rank(\pi_k(\mathcal{N}_p))$ it follows that $\overline{\Psi}$ reduces the rank of $\overset{n}{\underset{i=1}{*}} \pi_k(\mathcal{N}_i)$. Also, if $\Psi(\overset{n}{\underset{i=1}{\curlyvee}} \mathcal{N}_i) = \mathcal{N}_1 \curlyvee \mathcal{N}_2 \curlyvee .. \curlyvee \Psi(\mathcal{N}_p) \curlyvee ... \curlyvee \Psi(\mathcal{N}_q) \curlyvee ... \curlyvee \mathcal{N}_n$ for $q = 1, 2, ..., n$, $p < q$ then $\overline{\Psi}(\overset{n}{\underset{i=1}{*}} \pi_k(\mathcal{N}_i)) = \pi_k(\Psi(\overset{n}{\underset{i=1}{\curlyvee}} \mathcal{N}_i)) \approx \pi_k(\mathcal{N}_1) * \pi_k(\mathcal{N}_2) * ... * \pi_k(\Psi(\mathcal{N}_p)) * ... * \pi_k(\Psi(\mathcal{N}_q)) * ... * \pi_k(\mathcal{N}_n)$ and so $\overline{\Psi}$ reduces the rank of $\overset{n}{\underset{i=1}{*}} \pi_k(\mathcal{N}_i)$. Similarly, if we extend this process $\Psi(\overset{n}{\underset{i=1}{\curlyvee}} \mathcal{N}_i) = \overset{n}{\underset{i=1}{\curlyvee}} \Psi(\mathcal{N}_i)$, we find that $\overline{\Psi}(\overset{n}{\underset{i=1}{*}} \pi_k(\mathcal{N}_i)) = \pi_k(\Psi(\overset{n}{\underset{i=1}{\curlyvee}} \mathcal{N}_i)) = \pi_k(\overset{n}{\underset{i=1}{\curlyvee}} \Psi(\mathcal{N}_i)) \approx \overset{n}{\underset{i=1}{*}} \pi_k(\Psi(\mathcal{N}_i))$. Hence, $\overline{\Psi}$ reduces the rank of $\overset{n}{\underset{i=1}{*}} \pi_k(\mathcal{N}_i)$. □

THEOREM 2.9. *There exists a sequence of singular dynamic* $\Psi_q : \overset{n}{\underset{i=1}{\curlyvee}} S_i^1 \to \overset{n}{\underset{i=1}{\curlyvee}} S_i^1$, $q = 1,\ 2, \ldots n$, *which induces a sequence of singular dynamic*
$\overline{\Psi_q} : \overset{n}{\underset{i=1}{*}} \pi_k(S_i^1 \to \overset{n}{\underset{i=1}{*}} \pi_k(S_i^1$ *of* $\overset{n}{\underset{i=1}{*}} \pi_k(S_i^1)$ *into itself such that* $\overline{\Psi_q}(\overset{n}{\underset{i=1}{*}} \pi_k(S_i^1)) = F_{n-q}$.

Proof. Let $\Psi_1 : \overset{n}{\underset{i=1}{\curlyvee}} S_i^1 \to \overset{n}{\underset{i=1}{\curlyvee}} S_i^1$ be a singular dynamic such that $\Psi_1(\overset{n}{\underset{i=1}{\curlyvee}} S_i^1) = S_1^1 \curlyvee S_2^1 \curlyvee ... \curlyvee \Psi_1(S_t^1) \curlyvee ... \curlyvee S_n^1$ for $p = 1, 2, ..., n$ and $\Psi_1(S_t^1) \neq S_t^1$ is a singular dynamic with singularity as in FIGURE 1, then there is an induced singular dynamic $\overline{\Psi_1} : \overset{n}{\underset{i=1}{*}} \pi_k(S_i^1) \to \overset{n}{\underset{i=1}{*}} \pi_k(S_i^1)$ such that $\overline{\Psi_1}(\overset{n}{\underset{i=1}{*}} \pi_k(S_i^1)) = \pi_k(\Psi_1(\overset{n}{\underset{i=1}{\curlyvee}} S_i^1))$ and so

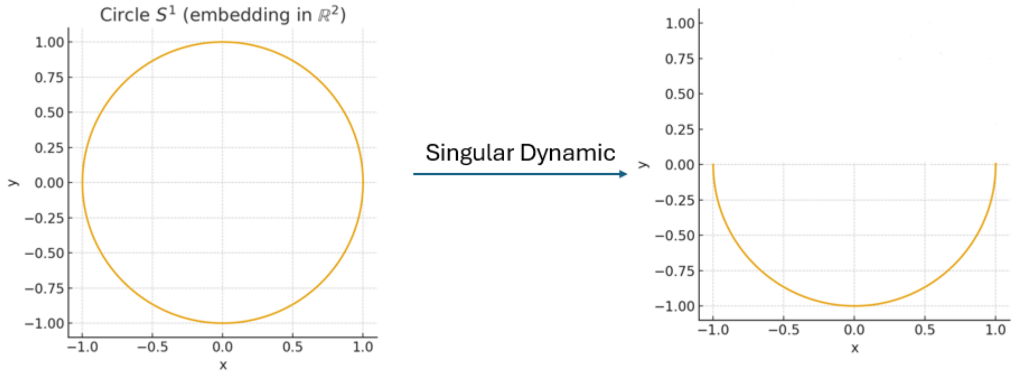

FIGURE 1. Singular dynamic with singularity on S^1

$\overline{\Psi_1}(\mathop{\ast}\limits_{i=1}^{n}\pi_k(S_i^1)) \approx \pi_k(S_1^1) \ast \pi_k(S_2^1) \ast \ldots \ast \pi_k(\Psi_1(S_p^1)) \ast \ldots \ast \pi_k(S_n^1)$. Since $\pi_k(\Psi_1(S_t^1)) = 0$ and $\pi_k(S_i^1) \approx \mathbb{Z}$, it follows that $\overline{\Psi_1}(\mathop{\ast}\limits_{i=1}^{n}\pi_k(S_i^1)) = \digamma_{n-1}$. Also, let $\Psi_2 : \mathop{\curlyvee}\limits_{i=1}^{n} S_i^1 \to \mathop{\curlyvee}\limits_{i=1}^{n} S_i^1$ be a singular dynamic such that $\Psi_2(\mathop{\curlyvee}\limits_{i=1}^{n} S_i^1) = S_1^1 \curlyvee S_2^1 \curlyvee \ldots \curlyvee \Psi_2(S_p^1) \curlyvee \ldots \curlyvee \Psi_2(S_q^1) \ldots \curlyvee S_n^1$ for $p,\ q = 1,\ 2, \ldots, n,\ p < q$ and $\Psi_2(S_p^1) \neq S_p^1, \Psi_2(S_t^1) \neq S_t^1$ then we can get the induced singular dynamic $\overline{\Psi_2} : \mathop{\ast}\limits_{i=1}^{n}\pi_k(S_i^1) \to \mathop{\ast}\limits_{i=1}^{n}\pi_k(S_i^1)$ such that $\overline{\Psi_2}(\mathop{\ast}\limits_{i=1}^{n}\pi_k(S_i^1)) = \digamma_{n-2}$.

Likewise, by extending this process we obtain the singular dynamic $\Psi_n : \mathop{\curlyvee}\limits_{i=1}^{n} S_i^1 \to \mathop{\curlyvee}\limits_{i=1}^{n} S_i^1$such that $\Psi_n(\mathop{\curlyvee}\limits_{i=1}^{n} S_i^1) = \mathop{\curlyvee}\limits_{i=1}^{n} \Psi_n(S_i^1)$ and $\Psi_n(S_i^1) \neq S_i^1$, which induces a singular dynamic $\overline{\Psi_n} : \mathop{\ast}\limits_{i=1}^{n}\pi_k(S_i^1) \to \mathop{\ast}\limits_{i=1}^{n}\pi_k(S_i^1)$ such that $\overline{\Psi_n}(\mathop{\ast}\limits_{i=1}^{n}\pi_k(S_i^1)) = \digamma_0$. □

THEOREM 2.10. *Let $Q_n,\ n \geq 2$ represent the disjoint union of n discs on the sphere, and let $\{\Psi_m,\ m \in N\}$ be a singular dynamic $S^2 - Q_n$ into itself, leading to an induced singular dynamic $\overline{\Psi}_m : \pi_k(S^2 - Q_n) = \pi_k(S^2 - Q_n)$ such that $\pi_k(\lim\limits_{m\to\infty}(\Psi_m(S^2 - Q_n)) =$*

$$\begin{cases} \digamma_{n-2} & if \quad k = 1 \\ \digamma_2 & if \quad k = 2 \\ \pi_k(S^2) \ast \pi_k(S^2) & if \quad k \geq 3. \end{cases}$$

Proof. Let $Q_n,\ n \geq 2$, be the disjoint union of n-discs on the sphere S^2. Then we can define a sequence of singular dynamics $\Psi_m : S^2 - Q_n \to S^2 - Q_n,\ m = 1, 2, \ldots$ such that $\lim\limits_{m\to\infty}(\Psi_m(S^2 - Q_n) = (S^2 - Q_P) \curlyvee (S^2 - Q_q)$ with $p + q = n$, see FIGURE 2, thus $\pi_k(\lim\limits_{m\to\infty}(\Psi_m(S^2 - Q_n)) = \pi_k(S^2 - Q_p) \ast \pi_k(S^2 - Q_q)$.

Since $\pi_k(S^2 - Q_p) = \begin{cases} \digamma_{p-1} & if \quad k = 1 \\ \digamma_1 & if \quad k = 2 \\ \pi_k(S^2) & if \quad k \geq 3, \end{cases}$

and $\pi_k(S^2 - Q_q) = \begin{cases} \digamma_{q-1} & if \quad k = 1 \\ \digamma_1 & if \quad k = 2 \\ \pi_k(S^2) & if \quad k \geq 3, \end{cases}$

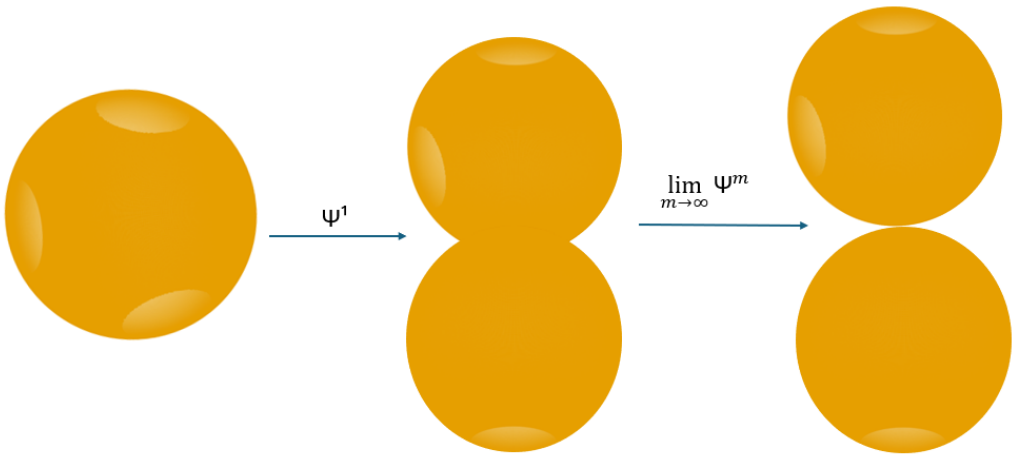

FIGURE 2. The sequence of singular dynamics on $S^2 - Q_n$

we obtain, $\pi_k(\lim_{m\to\infty}(\Psi_m(S^2 - Q_n)) = \begin{cases} F_{p+q-2} & if \quad k = 1 \\ F_2 & if \quad k = 2 \\ \pi_k(S^2) * \pi_k(S^2) & if \quad k \geq 3. \end{cases}$

Therefore, $\pi_k(\lim_{m\to\infty}(\Psi_m(S^2 - \Psi_n)) = \begin{cases} F_{n-2} & if \quad k = 1 \\ F_2 & if \quad k = 2 \\ \pi_k(S^2) * \pi_k(S^2) & if \quad k \geq 3. \end{cases}$ □

THEOREM 2.11. *If $\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_n$ are arc-connected manifolds and Ψ is a singular dynamic from $\prod_{i=1}^{n} \mathcal{N}_i$ to itself, then there exists an induced singular dynamic $\overline{\Psi}$ of $\pi_k(\prod_{i=1}^{n} \mathcal{N}_i)$ into itself in which $(\pi_k(\prod_{i=1}^{n} \mathcal{N}_i)) \approx \pi_k(\mathcal{N}_1) \times \pi_k(\mathcal{N}_2) \times ... \times \pi_k(\Psi(\mathcal{N}_p)) \times ... \times \pi_k(\mathcal{N}_n)$ with $p = 1, 2, ..., n$ or $\approx \pi_k(\mathcal{N}_1) \times \pi_k(\mathcal{N}_2) \times ... \times \pi_k(\Psi(\mathcal{N}_p)) \times ... \times \pi_k(\Psi(\mathcal{N}_q)) \times ... \times \pi_k(\mathcal{N}_n)$ with $p,\ q = 1,\ 2, ..., n,\ p < q, ...,$ or $\approx \pi_k(\prod_{i=1}^{n} \Psi(\mathcal{N}_i))$.*

Proof. $\overline{\Psi}(\pi_k(\prod_{i=1}^{n} \mathcal{N}_i)) = \pi_k(\Psi(\prod_{i=1}^{n} \mathcal{N}_i)) \approx \pi_k(\Psi(\mathcal{N}_1)) \times ... \times \pi_k(\Psi(\mathcal{N}_n)) = \pi_k(\prod_{i=1}^{n} \Psi(\mathcal{N}_i)).\Psi :$ $\prod_{i=1}^{n} \mathcal{N}_i \to \prod_{i=1}^{n} \mathcal{N}_i$ be a singular dynamic from $\prod_{i=1}^{n} \mathcal{N}_i$ into itself; then Ψ is a continuous map. Thus, we get the coordinate system of $\prod_{i=1}^{n} \mathcal{N}_i$ included on the form $\{(U_{\gamma_1} \times U_{\gamma_2} \times ... \times U_{\gamma_n}), (\mathcal{N}_{\gamma_1} \times \mathcal{N}_{\gamma_2} \times ... \times \mathcal{N}_{\gamma_n})\}$, where $\mathcal{N}_{\gamma_i}$ is a one-to-one, bicontinuous function from an open set $U_{\gamma_i} \subseteq R^{n_i} \to \mathcal{N}_i$ for $i = 1, 2, ..., n$ and $\{(U_{\gamma_i}, \mathcal{N}_{\gamma_i})\}$ is the atlas of $\mathcal{N}_i$, $\forall i$, $\Psi : \prod_{i=1}^{n} \mathcal{N}_i \to \prod_{i=1}^{n} \mathcal{N}_i$ takes these forms: If $\Psi(\prod_{i=1}^{n} \mathcal{N}_i) = \Psi(U_{\gamma_1} \times U_{\gamma_2} \times ... \times U_{\gamma_n}, \mathcal{N}_{\gamma_1} \times \mathcal{N}_{\gamma_2} \times ... \times \mathcal{N}_{\gamma_n}) = (U_{\gamma_1} \times U_{\gamma_2} \times ... \times U_{\gamma_n}, \Psi(U_{\gamma_p}, \mathcal{N}_{\gamma_p}), \mathcal{N}_{\gamma_1} \times \mathcal{N}_{\gamma_2} \times ... \times \mathcal{N}_{\gamma_n}) = \mathcal{N}_1 \times \mathcal{N}_2 \times ... \times \Psi(\mathcal{N}_p) \times ... \times \mathcal{N}_n$ for $p = 1, 2, ..., n$, then $\overline{\Psi}(\pi_k(\prod_{i=1}^{n} \mathcal{N}_i)) = \pi_k(\Psi(\prod_{i=1}^{n} \mathcal{N}_i)) \approx \pi_k(\mathcal{N}_1) \times \pi_k(\mathcal{N}_2) \times ... \times \pi_k(\Psi(\mathcal{N}_p)) \times ... \times \pi_k(\mathcal{N}_n)$. Also, if $\Psi(\prod_{i=1}^{n} \mathcal{N}_i) = \Psi(U_{\gamma_1} \times U_{\gamma_2} \times ... \times U_{\gamma_n}, \mathcal{N}_{\gamma_1} \times \mathcal{N}_{\gamma_2} \times ... \times \mathcal{N}_{\gamma_n}) = (U_{\gamma_1} \times U_{\gamma_2} \times ... \times U_{\gamma_n}, \Psi(U_{\gamma_p}, \mathcal{N}_{\gamma_p}), \Psi(U_{\gamma_q}, \mathcal{N}_{\gamma_q}), \mathcal{N}_{\gamma_1} \times \mathcal{N}_{\gamma_2} \times ... \times \mathcal{N}_{\gamma_n}) =$
$\mathcal{N}_1 \times \mathcal{N}_2 \times ... \times \Psi(\mathcal{N}_p) \times \Psi(\mathcal{N}_q) \times ... \times M_n$ for $p,\ q = 1,\ 2, ..., n,\ p < q$, then $\overline{\Psi}(\pi_k(\prod_{i=1}^{n} \mathcal{N}_i)) =$

$\pi_k(\Psi(\prod_{i=1}^{n}\mathcal{N}_i)) \approx \pi_k(\mathcal{N}_1)\times\pi_k(\mathcal{N}_2)\times...\times\pi_k(\Psi(\mathcal{N}_p))\times\pi_k(\Psi(\mathcal{N}_q))\times...\times\pi_k(\mathcal{N}_n)$. Moreover, by continuing this process if $\Psi(\prod_{i=1}^{n}\mathcal{N}_i) = \Psi(U_{\gamma_1}\times U_{\gamma_2}\times...\times U_{\gamma_n}, \mathcal{N}_{\gamma_1}\times\mathcal{N}_{\gamma_2}\times...\times\mathcal{N}_{\gamma_n}) = (\Psi(U_{\gamma_1},\mathcal{N}_{\gamma_1}), \Psi(U_{\gamma_2},\mathcal{N}_{\gamma_2}), ..., \Psi(U_{\gamma_n},\mathcal{N}_{\gamma_n})) = \Psi(\mathcal{N}_1)\times\Psi(\mathcal{N}_2)\times ... \times\Psi(\mathcal{N}_n)$, then, $\overline{\Psi}(\pi_k(\prod_{i=1}^{n}N_i)) = \pi_k(\Psi(\prod_{i=1}^{n}N_i)) \approx \pi_k(\Psi(N_1))\times ... \times\pi_k(\Psi(N_n)) = \pi_k(\prod_{i=1}^{n}\Psi(N_i))$. □

THEOREM 2.12. *Let $\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_n$ be arc-connected manifolds, and Ψ is a singular dynamic in which $\Psi(\prod_{i=1}^{n}\mathcal{N}_i) \neq \prod_{i=1}^{n}\Psi(\mathcal{N}_i)$.*
Then $\pi_k(\lim_{m\to\infty}(\Psi_m(\prod_{i=1}^{n}\mathcal{N}_i)) \neq \prod_{i=1}^{n}(\pi_k(\lim_{m\to\infty}(\Psi_m(\mathcal{N}_i)))$.

Proof. Let $T^1 = \mathcal{S}^1\times\mathcal{S}^1$ represent a torus. Set $\mathcal{N}_1 = \mathcal{S}^1$, $\mathcal{N}_2 = \mathcal{S}^1$, and $k = 1$ and consider the case whenever $\Psi(T^1) = \Psi(\mathcal{S}^1)\times\Psi(\mathcal{S}^1)$. Since $\lim_{m\to\infty}(\Psi_m(\mathcal{S}^1)$ is a single point, we obtain $\lim_{m\to\infty}(\Psi_m(\mathcal{S}^1)\times\lim_{m\to\infty}(\Psi_m(\mathcal{S}^1)$ as a single point, which implies that $\pi_k(\lim_{m\to\infty}(\Psi_m(\mathcal{S}^1)\times\lim_{m\to\infty}(\Psi_m(\mathcal{S}^1)) = 0$. For other cases, if $\Psi(T^1)\neq\Psi(\mathcal{S}^1)\times\Psi\mathcal{S}^1)$, we also have $\Psi(T^1) = \Psi(\mathcal{S}^1)\times S^1$ or $\Psi(T^1) = \mathcal{S}^1\times\Psi(\mathcal{S}^1)$, and so, $\lim_{m\to\infty}\Psi_m(T^1) = \mathcal{S}^1$. Thus, $\pi_k(\lim_{m\to\infty}\Psi_m(T^1)) = \pi_k(\mathcal{S}^1) = F_1$. Therefore, $\pi_k(\lim_{m\to\infty}(\Psi_m(T^1))\neq\pi_k(\lim_{m\to\infty}(\Psi_m(\mathcal{S}^1))\times\pi_k(\lim_{m\to\infty}(\Psi_m(\mathcal{S}^1))$. □

Author Contributions
The sole author was responsible for all aspects of the work, including conceptualization, methodology, investigation, formal analysis, writing (original draft and review/editing), and visualization.

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The data that support the findings.

3. Conclusion

In this article, the concept of singular dynamics on manifolds that provide geometric and topological features is introduced. The homotopy group is affected by the induced singular dynamics on a manifold or Cartesian product of a finite number of manifolds. The homotopy group operates as a convincing algebraic and topological tool that takes essential information about the connectivity and structure of a space, linking algebra with topology and geometry in a profound way. In sum, these concepts highlight the complexity and profundity of modern topology, where local and global viewpoints are interconnected and abstract structures provide an actual understanding of the nature of science.

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