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NONLINEAR n-TUPLE λ-JORDAN $*$-DERIVATIONS ON VON NEUMANN ALGEBRAS

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Abstract. Let $\mathfrak{A}$ be a factor von Neumann algebra and Ω preserves n-tuple λ-Jordan $*$-derivations on $\mathfrak{A}$, that is, for every $A_1, A_2, \ldots, A_n \in \mathfrak{A}$,

$$\begin{aligned}\Omega(A_1 \triangleright A_2 \triangleright \ldots \triangleright A_n) &= \Omega(A_1) \triangleright A_2 \triangleright \ldots \triangleright A_n + A_1 \triangleright \Omega(A_2) \triangleright \ldots \triangleright A_n \\ &\quad + \ldots + A_1 \triangleright A_2 \triangleright \ldots \triangleright \Omega(A_n)\end{aligned}$$

where $A_i \triangleright A_j = A_iA_j + \lambda A_jA_i^*$ for $\lambda \in \mathbb{R}$, $\lambda \notin \{0, \pm 1\}$, then Ω is additive $*$-derivation.

1. Introduction

In recent years, numerous researchers have explored the preserver problems, which involve characterizing maps on operator algebras that maintain certain functions, subsets, relations, and so on. The central question is to identify intrinsic isomorphism invariants between operator algebras. For instance, one area of focus is investigating maps that preserve specific products, such as the Jordan product, Lie product, skew Lie product, and others [3, 7, 11].

Let $\mathcal{R}$ and $\mathcal{R}'$ be rings. We say that a map $\Omega : \mathcal{R} \to \mathcal{R}'$ preserves the product or is multiplicative if $\Omega(AB) = \Omega(A)\Omega(B)$ for all $A, B \in \mathcal{R}$. The question of when a product-preserving or multiplicative map is also additive has been discussed by several authors, as seen in [13] and related references.

In [13], Martindale proved the following theorem.

Theorem 1.1. *Let $\mathcal{R}$ be a ring containing a family of $\{e_\alpha : \alpha \in \Lambda\}$ of idempotent which satisfies:*

1. *$x\mathcal{R} = 0$ implies $x = 0$.*
2. *If $e_\alpha \mathcal{R} x = 0$ for each $\alpha \in \Lambda$, then $x = 0$.*
3. *For each $\alpha \in \Lambda$, $e_\alpha x e_\alpha \mathcal{R}(1 - e_\alpha) = 0$ implies $e_\alpha x e_\alpha = 0$.*

Then any multiplicative isomorphism of $\mathcal{R}$ onto an arbitrary ring is additive.

Inspired by this, numerous authors have focused on maps between rings (and algebras) that preserve the Lie product $[A, B] = AB - BA$ or the Jordan product

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$A \circ B = AB + BA$ (for more details, see [1, 4–6, 10–12, 15]). These findings demonstrate that, in a certain sense, the structure of the Jordan or Lie product alone is sufficient to characterize the ring or algebraic structure. Over time, many mathematicians have concentrated on the study of additive or linear Jordan or Lie product preservers between rings or operator algebras. Such maps are commonly referred to as Jordan homomorphisms or Lie homomorphisms. Here, we highlight several results [1, 13, 16, 18, 21].

Let $\mathcal{R}$ be a $*$-ring. For $A, B \in \mathcal{R}$, we define the $*$-Jordan product $A \triangleright B = AB + BA^*$ and the $*$-Lie product $[A, B]_* = AB - BA^*$. These products have proven to be significant in various research areas, and their study has recently gained considerable attention from many authors (for instance, see [2, 5, 15, 16, 20]).

Next, we define the λ-Jordan $*$-product by $A \triangleright_\lambda B = AB + \lambda BA^*$. We say that a map Ω is a λ-Jordan $*$-derivation if it satisfies the property $\Omega(A \triangleright_\lambda B) = \Omega(A) \triangleright_\lambda B + A \triangleright_\lambda \Omega(B)$. It is clear that when $\lambda = -1$ and $\lambda = 1$, the λ-Jordan $*$-derivation map corresponds to a $*$-Lie derivation and a $*$-Jordan derivation, respectively [1]. Here, we emphasize that whenever we say Ω preserves derivation, it means that $\Omega(AB) = \Omega(A)B + A\Omega(B)$.

A von Neumann algebra $\mathfrak{A}$ is a self-adjoint subalgebra of some $B(H)$, the algebra of bounded linear operators acting on a complex Hilbert space, which satisfies the double commutant property: $\mathfrak{A}'' = \mathfrak{A}$, where $\mathfrak{A}' = \{T \in B(H) : TA = AT, \forall A \in \mathfrak{A}\}$ and $\mathfrak{A}'' = \{\mathfrak{A}'\}'$. The center of $\mathfrak{A}$ is denoted by $\mathcal{Z}(\mathfrak{A}) = \mathfrak{A}' \cap \mathfrak{A}$. A von Neumann algebra $\mathfrak{A}$ is called a factor if its center is trivial, i.e., $\mathcal{Z}(\mathfrak{A}) = \mathbb{C}I$. Clearly, if $\mathfrak{A}$ is a factor von Neumann algebra, it is prime, meaning that for $A, B \in \mathfrak{A}$, if $A\mathfrak{A}B = \{0\}$, then $A = 0$ or $B = 0$. For $A \in \mathfrak{A}$, the central carrier of A, denoted by $\overline{A}$, is the smallest central projection P such that $PA = A$. It is straightforward to observe that $\overline{A}$ is the projection onto the closed subspace spanned by $\{BAx : B \in \mathfrak{A}, x \in H\}$. If A is self-adjoint, the core of A, denoted by $\underline{A}$, is given by $\sup\{S \in \mathcal{Z}(\mathfrak{A}) : S = S^*, S \leq A\}$. If $A = P$ is a projection, then $\underline{P}$ is the largest central projection Q satisfying $Q \leq P$. A projection P is said to be core-free if $\underline{P} = 0$ (see [14]). It is evident that $\underline{P} = 0$ if and only if $\overline{I - P} = I$, as shown in [8, 9].

Recently, Yu and Zhang [22] proved that every non-linear $*$-Lie derivation from a factor von Neumann algebra into itself is an additive $*$-derivation. Additionally, Li, Lu, and Fang [4] investigated non-linear λ-Jordan $*$-derivations. They demonstrated that if $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra without central abelian projections and λ is a non-zero scalar, then a map $\Omega : \mathfrak{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a non-linear λ-Jordan $*$-derivation if and only if Ω is an additive $*$-derivation.

In [21], we showed that a $*$-Jordan derivation map (i.e., $\Omega(A \triangleright B) = \Omega(A) \triangleright B + A \triangleright \Omega(B)$) on every factor von Neumann algebra $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ is an additive $*$-derivation.

The authors in [10] introduced the concept of Lie triple derivations. A map $\Omega : \mathfrak{A} \to \mathfrak{A}$ is a nonlinear skew Lie triple derivation if it satisfies the equation $\Omega([[A, B]_*, C]_*) = [[\Omega(A), B]_*, C]_* + [[A, \Omega(B)]_*, C]_* + [[A, B]_*, \Omega(C)]_*$ for all $A, B, C \in \mathfrak{A}$, where $[A, B]_* = AB - BA^*$. They showed that if Ω preserves the above characterizations on factor von Neumann algebras, then Ω is an additive $*$-derivation. In [17], we proved the above result for prime $*$-algebras.

In [19], we considered a map Ω on a prime $*$-algebra $\mathfrak{A}$ that satisfies the following conditions:

$$\Omega(A \triangleright_\lambda B \triangleright_\lambda C) = \Omega(A) \triangleright_\lambda B \triangleright_\lambda C + A \triangleright_\lambda \Omega(B) \triangleright_\lambda C + A \triangleright_\lambda B \triangleright_\lambda \Omega(C)$$

where $A \triangleright_\lambda B = AB + \lambda BA^*$ for $i, j \in \mathbb{N}, \lambda \in \mathbb{R}$, and $|\lambda| \neq 0, 1$, then Ω is additive. Also, if $\Omega(I)$ is self-adjoint, then Ω is a $*$-derivation.

In [23], the authors considered nonlinear $*$-Lie higher derivations on factor von Neumann algebras acting on a complex Hilbert space H with $\dim H \geq 2$.

In this paper, motivated by the above results, we consider a map Ω on a factor von Neumann algebra $\mathfrak{A}$ that satisfies the following conditions:

$$\begin{aligned}\Omega(A_1 \triangleright A_2 \triangleright \ldots \triangleright A_n) &= \Omega(A_1) \triangleright A_2 \triangleright \ldots \triangleright A_n + A_1 \triangleright \Omega(A_2) \triangleright \ldots \triangleright A_n \\ &\quad + \ldots + A_1 \triangleright A_2 \triangleright \ldots \triangleright \Omega(A_n)\end{aligned}$$

where $A_i \triangleright A_j = A_iA_j + \lambda A_jA_i^*$ for $\lambda \in \mathbb{R}$, $\lambda \notin \{0, \pm 1\}$, then Ω is additive $*$-derivation. Moreover, at the end of Theorem 2.2, we provide an example (2.16) of a nontrivial map that satisfies the condition 1.

Remark 1.2. The preceding claim reveals an important difference between the present result and our earlier result in [19]. In [19], which concerns nonlinear λ-Jordan triple $*$-derivations on prime $*$-algebras, the self-adjointness of $\Phi(I)$ was imposed as an additional hypothesis to conclude that Φ is a $*$-derivation. In the present n-tuple setting, however, the n-tuple λ-Jordan $*$-derivation identity, together with the additivity of Ω, implies $\Omega(iI)^* = -\Omega(iI)$ and $\Omega(iI) = i\Omega(I)$. Consequently, $\Omega(I)^* = \Omega(I)$, so the self-adjointness of $\Omega(I)$ follows automatically and need not be assumed separately. Thus, in this respect, the present result improves the corresponding triple-product result obtained in [19].

2. Main Results

By making use of an interesting result of Martindale [13], we have the following partition of the von Neumann algebra $\mathfrak{A}$. Let P_1 be a nontrivial projection in $\mathfrak{A}$ and $P_2 = I_{\mathfrak{A}} - P_1$. Denote $\mathfrak{A}_{ij} = P_i\mathfrak{A}P_j,\ i, j = 1, 2$, then $\mathfrak{A} = \sum_{i,j=1}^{2} \mathfrak{A}_{ij}$. For every $A \in \mathfrak{A}$ we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follow, when we write A_{ij}, it indicates that $A_{ij} \in \mathfrak{A}_{ij}$.

Our first theorem is as follows:

Theorem 2.1. *Let $\mathfrak{A}$ be a factor von Neumann algebra. Then the map $\Omega : \mathfrak{A} \to \mathfrak{A}$ satisfies in the following condition*

$$\begin{aligned}\Omega(A_1 \triangleright A_2 \triangleright \ldots \triangleright A_n) &= \Omega(A_1) \triangleright A_2 \triangleright \ldots \triangleright A_n + A_1 \triangleright \Omega(A_2) \triangleright \ldots \triangleright A_n \\ &\quad + \ldots + A_1 \triangleright A_2 \triangleright \ldots \triangleright \Omega(A_n)\end{aligned} \tag{1}$$

for all $A_1, A_2, \ldots, A_n \in \mathfrak{A}$ where $A_i \triangleright A_j = A_iA_j + \lambda A_jA_i^*$ for $\lambda \in \mathbb{R}$, $\lambda \notin \{0, \pm 1\}$, is additive.

Proof. We give some claims that prove Ω is additive on each $\mathfrak{A}_{ij},\ i, j = 1, 2$.

Claim 2.2. We show that $\Omega(0) = 0$.

We write

$$\begin{aligned}\Omega(0) &= \Omega(0 \triangleright 0 \triangleright \ldots \triangleright 0) \\ &= \Omega(0) \triangleright 0 \triangleright \ldots \triangleright 0 + 0 \triangleright \Omega(0) \triangleright 0 \triangleright \ldots \triangleright 0 + 0 \triangleright 0 \triangleright \ldots \triangleright \Omega(0) = 0.\end{aligned}$$

Claim 2.3. For each $A_{11} \in \mathfrak{A}_{11}$ and $A_{21} \in \mathfrak{A}_{21}$ we have

$$\Omega(A_{11} + A_{21}) = \Omega(A_{11}) + \Omega(A_{21}).$$

Let $T = \Omega(A_{11} + A_{21}) - \Omega(A_{11}) - \Omega(A_{21})$. We show that $T = 0$. It is easy to check that

$$\begin{aligned}
&\Omega(I) \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (A_{11} + A_{21}) \triangleright P_2 + I \triangleright \Omega(I) \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (A_{11} + A_{21}) \triangleright P_2 \\
&\quad + \ \ldots \\
&\quad + \ I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright \Omega(A_{11} + A_{21}) \triangleright P_2 + I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (A_{11} + A_{21}) \triangleright \Omega(P_2) \\
&= \Omega(I \triangleright I \ldots \triangleright I \triangleright P_2 \triangleright (A_{11} + A_{21}) \triangleright P_2) + \Omega(I) \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright A_{11} \triangleright P_2) \\
&\quad + \ \Omega(I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright A_{21} \triangleright P_2) + \Omega(I) \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (A_{11} + A_{21}) \triangleright P_2 \\
&\quad + \ I \triangleright \Omega(I) \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (A_{11} + A_{21}) \triangleright P_2 \\
&\quad + \ \ldots \\
&\quad + \ I \triangleright I \triangleright \ldots \triangleright \Omega(P_2) \triangleright (A_{11} + A_{21}) \triangleright P_2 + I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (\Omega(A_{11}) + \Omega(A_{21})) \triangleright P_2 \\
&\quad + \ I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (A_{11} + A_{21}) \triangleright \Omega(P_2).
\end{aligned}$$

It follows that $I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright T \triangleright P_2 = 0$. Since $T = T_{11} + T_{12} + T_{21} + T_{22}$, so $T_{12} = T_{21} = 0$.
On the other hand, we have

$$\begin{aligned}
&\Omega(I) \triangleright I \triangleright \ldots \triangleright I \triangleright X_{21} \triangleright (A_{11} + A_{21}) \triangleright P_1 + I \triangleright \Omega(I) \triangleright \ldots \triangleright I \triangleright X_{21} \triangleright (A_{11} + A_{21}) \triangleright P_1 \\
&\quad + \ \ldots \\
&\quad + \ I \triangleright I \triangleright \ldots \triangleright I \triangleright X_{21} \triangleright \Omega(A_{11} + A_{21}) \triangleright P_1 + I \triangleright I \triangleright \ldots \triangleright I \triangleright X_{21} \triangleright (A_{11} + A_{21}) \triangleright \Omega(P_1) \\
&= \Omega(I \triangleright I \ldots \triangleright I \triangleright X_{21} \triangleright (A_{11} + A_{21}) \triangleright P_1) + \Omega(I) \triangleright I \triangleright \ldots \triangleright I \triangleright X_{21} \triangleright (A_{11} + A_{21}) \triangleright P_1) \\
&\quad + \ I \triangleright \Omega(I) \triangleright \ldots \triangleright I \triangleright X_{21} \triangleright (A_{11} + A_{21}) \triangleright P_1) + I \triangleright I \triangleright \ldots \triangleright I \triangleright X_{21} \triangleright (\Omega(A_{11}) + \Omega(A_{21})) \triangleright P_1 \\
&\quad + \ I \triangleright I \triangleright \ldots \triangleright I \triangleright X_{21} \triangleright (A_{11} + A_{21}) \triangleright \Omega(P_1).
\end{aligned}$$

Then, we obtain $I \triangleright I \triangleright \ldots \triangleright X_{21} \triangleright T \triangleright P_1 = 0$. So, $T_{11} = 0$. Similarly, by applying X_{12} instead of X_{21} in the above, we have $T_{21} = 0$.

Claim 2.4. For each $A_{12} \in \mathfrak{A}_{12}$ and $A_{21} \in \mathfrak{A}_{21}$, we have $\Omega(A_{12} + A_{21}) = \Omega(A_{12}) + \Omega(A_{21})$.

Let $T = \Omega(A_{12} + A_{21}) - \Omega(A_{12}) - \Omega(A_{21})$. As in the preceding claim, applying the n-tuple λ-Jordan $*$-derivation identity gives $I \triangleright \cdots \triangleright I \triangleright P_2 \triangleright T \triangleright P_2 = 0$. Since $I \triangleright X = (1 + \lambda)X$, we obtain $(P_2 \triangleright T) \triangleright P_2 = 0$. Writing

$$T = \sum_{i,j=1}^{2} T_{ij},$$

a direct computation yields $\lambda T_{12} + \lambda^2 T_{12}^* + (1 + \lambda)\big(T_{22} + \lambda T_{22}^*\big) = 0$. Comparing the Peirce components, we get $T_{12} = 0, T_{22} + \lambda T_{22}^* = 0$. Taking adjoints gives $T_{22}^* + \lambda T_{22} = 0$, and hence $(1 - \lambda^2)T_{22} = 0$. Since $\lambda \neq \pm 1$, it follows that $T_{22} = 0$.

Replacing P_2 by P_1 in the above argument gives $T_{11} = T_{21} = 0$. Therefore, $T = 0$, and consequently $\Omega(A_{12} + A_{21}) = \Omega(A_{12}) + \Omega(A_{21})$.

Claim 2.5. For each $A_{11} \in \mathfrak{A}_{11}$, $A_{12} \in \mathfrak{A}_{12}$, $A_{21} \in \mathfrak{A}_{21}$, we have

$$\Omega(A_{11} + A_{12} + A_{21}) = \Omega(A_{11}) + \Omega(A_{12}) + \Omega(A_{21}).$$

We show that for T in $\mathfrak{A}$ the following holds

$$T = \Omega(A_{11} + A_{12} + A_{21}) - \Omega(A_{11}) - \Omega(A_{12}) - \Omega(A_{21}) = 0. \tag{2}$$

We have

$$\begin{aligned}
&\Omega\,(I) \triangleright I \triangleright \ldots \triangleright (P_2) \triangleright (A_{11} + A_{12} + A_{21}) \triangleright P_2\\
&\quad + \quad I \triangleright \Omega(I) \triangleright I \triangleright \ldots \triangleright P_2 \triangleright (A_{11} + A_{12} + A_{21}) \triangleright P_2\\
&\quad + \quad \ldots\\
&\quad + \quad I \triangleright I \triangleright \ldots \triangleright \Omega(P_2) \triangleright (A_{11} + A_{12} + A_{21}) \triangleright P_2\\
&\quad + \quad I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright \Omega(A_{11} + A_{12} + A_{21}) \triangleright P_2\\
&\quad + \quad I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (A_{11} + A_{12} + A_{21}) \triangleright \Omega(P_2)\\
&= \Omega\,(I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (A_{11} + A_{12} + A_{21}) \triangleright P_2)\\
&= \Omega\,(I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright A_{11} \triangleright P_2)\\
&\quad + \quad \Omega\,(I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright A_{12} \triangleright P_2)\\
&\quad + \quad \Omega\,(I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright A_{21} \triangleright P_2)\\
&= \Omega\,(I) \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (A_{11} + A_{21} + A_{12}) \triangleright P_2\\
&\quad + \quad I \triangleright \Omega\,(I) \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (A_{11} + A_{12} + A_{21}) \triangleright P_2\\
&\quad + \quad \ldots\\
&\quad + \quad I \triangleright I \triangleright \ldots \triangleright I \triangleright \Omega(P_2) \triangleright (A_{11} + A_{12} + A_{21}) \triangleright P_2\\
&\quad + \quad I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (\Omega(A_{11}) + \Omega(A_{12}) + \Omega(A_{21})) \triangleright P_2\\
&\quad + \quad I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright (A_{11} + A_{12} + A_{21}) \triangleright \Omega(P_2).
\end{aligned}$$

Then, we have

$$I \triangleright I \triangleright \ldots \triangleright I \triangleright P_2 \triangleright T \triangleright P_2 = 0.$$

So $T_{12} = T_{22} = 0$. By applying P_1 instead of P_2 in the above relation and Claims 2.3 and 2.4, we have $T_{11} = T_{21} = 0$.

Claim 2.6. For each $A_{11} \in \mathfrak{A}_{11}$, $A_{12} \in \mathfrak{A}_{12}$, $A_{21} \in \mathfrak{A}_{21}$, $A_{22} \in \mathfrak{A}_{22}$ we have

$$\Omega(A_{11} + A_{12} + A_{21} + A_{22}) = \Omega(A_{11}) + \Omega(A_{12}) + \Omega(A_{21}) + \Omega(A_{22}).$$

We show that for T in $\mathfrak{A}$ the following holds

$$T = \Omega(A_{11} + A_{12} + A_{21} + A_{22}) - \Omega(A_{11}) - \Omega(A_{12}) - \Omega(A_{21}) - \Omega(A_{22}) = 0. \tag{3}$$

From Claim 2.5, we write

$$\begin{aligned}
&\Omega\,(I) \triangleright I \triangleright I \triangleright \ldots \triangleright P_1 \triangleright (A_{11} + A_{12} + A_{21} + A_{22})\\
&\quad + \quad I \triangleright \Omega\,(I) \triangleright I \triangleright \ldots \triangleright P_1 \triangleright (A_{11} + A_{12} + A_{21} + A_{22})\\
&\quad + \quad \ldots\\
&\quad + \quad I \triangleright \ldots \triangleright \Omega(P_1) \triangleright (A_{11} + A_{12} + A_{21} + A_{22})\\
&\quad + \quad I \triangleright I \triangleright \ldots \triangleright P_1 \triangleright \Omega(A_{11} + A_{12} + A_{21} + A_{22})\\
&= \Omega\,(I \triangleright I \triangleright \ldots \triangleright P_1 \triangleright (A_{11} + A_{12} + A_{21} + A_{22}))\\
&= \Omega\,(I \triangleright I \triangleright \ldots \triangleright P_1 \triangleright (A_{12} + A_{21} + A_{11})) + \Omega\,(I \triangleright I \triangleright \ldots \triangleright P_1 \triangleright A_{22})\\
&= \Omega\,(I \triangleright I \triangleright \ldots \triangleright P_1 \triangleright A_{12}) + \Omega\,(I \triangleright I \triangleright \ldots \triangleright P_1 \triangleright A_{21})\\
&\quad + \quad \Omega\,(I \triangleright I \triangleright \ldots \triangleright P_1 \triangleright A_{11}) + \Omega\,(I \triangleright I \triangleright \ldots \triangleright P_1 \triangleright A_{22})\\
&= \Omega\,(I) \triangleright I \ldots \triangleright P_1 \triangleright (A_{11} + A_{12} + A_{21} + A_{22})\\
&\quad + \quad I \triangleright \Omega\,(I) \triangleright I \triangleright \ldots \triangleright P_1 \triangleright (A_{11} + A_{12} + A_{21} + A_{22})\\
&\quad + \quad \ldots
\end{aligned}$$

$$+ \quad I \triangleright I \triangleright \ldots \triangleright \Omega(P_1) \triangleright (A_{11} + A_{12} + A_{21} + A_{22}).$$
$$+ \quad I \triangleright I \triangleright \ldots \triangleright P_1 \triangleright (\Omega(A_{11}) + \Omega(A_{12}) + \Omega(A_{21}) + \Omega(A_{22}))$$

So,

$$I \triangleright I \triangleright I \triangleright \ldots \triangleright P_1 \triangleright T = 0.$$

Then $(1+\lambda)^{n-2} P_1 \triangleright T = 0$ we have $(1+\lambda)T_{11} + T_{12} + \lambda T_{21} = 0$. Since $\lambda \notin \{0, \pm 1\}$ then $T_{11} = T_{21} = T_{12} = 0$. Similarly, we can show that $T_{22} = 0$.

CLAIM 2.7. For each $A_{ij}, B_{ij} \in \mathfrak{A}_{ij}$ such that $i \neq j$, we have

$$\Omega(A_{ij} + B_{ij}) = \Omega(A_{ij}) + \Omega(B_{ij}).$$

It is easy to check that

$$\begin{aligned}
&(1+\lambda)^{n-2}((P_i + R_{ij})(P_j + S_{ij}) + \lambda(P_j + S_{ij})(P_i + R^*_{ij})) \\
&= (1+\lambda)^{n-2}(R_{ij} + S_{ij} + \lambda S^*_{ij} + \lambda S_{ij} R^*_{ij})
\end{aligned}$$

From the above equation, we have

$$\begin{aligned}
&\Omega((1+\lambda)^{n-2}(R_{ij} + S_{ij})) + \Omega(\lambda(1+\lambda)^{n-2} S^*_{ij}) + \Omega(\lambda(1+\lambda)^{n-2} R^*_{ij}) \\
&= \Omega(I \triangleright \ldots \triangleright I \triangleright (P_i + R_{ij}) \triangleright (P_j + S_{ij})) \\
&= \Omega(I) \triangleright I \triangleright \ldots \triangleright (P_i + R_{ij}) \triangleright (P_j + S_{ij}) \\
&\quad + \quad I \triangleright \Omega(I) \triangleright I \triangleright \ldots \triangleright (P_i + R_{ij}) \triangleright (P_j + S_{ij}) \\
&\quad + \quad \ldots + I \triangleright I \triangleright \ldots \triangleright \Omega(P_i + R_{ij}) \triangleright (P_j + S_{ij}) \\
&\quad + \quad I \triangleright I \triangleright \ldots \triangleright (P_i + R_{ij}) \triangleright \Omega(P_j + S_{ij}) \\
&= \Omega(I) \triangleright I \triangleright \ldots \triangleright (P_i + R_{ij}) \triangleright (P_j + S_{ij}) \\
&\quad + \quad I \triangleright \Omega(I) \triangleright I \triangleright \ldots \triangleright (P_i + R_{ij}) \triangleright (P_j + S_{ij}) \\
&\quad + \quad \ldots + I \triangleright I \triangleright \ldots \triangleright (\Omega(P_i) + \Omega(R_{ij})) \triangleright P_j + S_{ij} \\
&\quad + \quad I \triangleright I \triangleright \ldots \triangleright (P_i + R_{ij}) \triangleright (\Omega(P_j) \\
&\quad + \quad \Omega(S_{ij})) = \Omega(I \triangleright I \triangleright \ldots \triangleright P_i \triangleright P_j) \\
&\quad + \quad \Omega(I \triangleright I \triangleright \ldots \triangleright R_{ij} \triangleright S_{ij}) + \Omega(I \triangleright I \triangleright \ldots \triangleright P_i \triangleright S_{ij}) \\
&\quad + \quad \Omega(I \triangleright I \triangleright \ldots \triangleright R_{ij} \triangleright P_j) = \Omega((1+\lambda)^{n-2} R_{ij}) + \Omega((1+\lambda)^{n-2} S_{ij}) \\
&\quad + \quad \Omega(\lambda(1+\lambda)^{n-2} S^*_{ij}) + \Omega(\lambda(1+\lambda)^{n-2} S_{ij} R^*_{ij})
\end{aligned}$$

So,

$$\Omega((1+\lambda)^{n-2}(R_{ij} + S_{ij})) = \Omega((1+\lambda)^{n-2} R_{ij}) + \Omega((1+\lambda)^{n-2} S_{ij}).$$

Let

$$A_{ij} = (1+\lambda)^{n-2} R_{ij}\ ,\ B_{ij} = (1+\lambda)^{n-2} S_{ij}.$$

Hence

$$\Omega(A_{ij} + B_{ij}) = \Omega(A_{ij}) + \Omega(B_{ij}).$$

CLAIM 2.8. For each $A_{ii}, B_{ii} \in \mathfrak{A}_{ii}$ such that $1 \leq i \leq 2$, we have

$$\Omega(A_{ii} + B_{ii}) = \Omega(A_{ii}) + \Omega(B_{ii}).$$

We show that

$$T = \Omega(A_{ii} + B_{ii}) - \Omega(A_{ii}) - \Omega(B_{ii}) = 0.$$

We write

$$\Omega\,(I) \triangleright I \triangleright \ldots \triangleright P_j \triangleright (A_{ii} + B_{ii}) + I \triangleright \Omega\,(I) \triangleright I \triangleright \ldots \triangleright P_j \triangleright (A_{ii} + B_{ii})$$

$$
\begin{aligned}
&+ \ \ldots \\
&+ \ I \triangleright I \triangleright \ldots \triangleright \Omega(P_j) \triangleright (A_{ii} + B_{ii}) + I \triangleright I \triangleright I \triangleright \ldots \triangleright P_j \triangleright \Omega(A_{ii} + B_{ii}) \\
&= \Omega\left(I \triangleright I \triangleright I \triangleright \ldots \triangleright P_j \triangleright (A_{ii} + B_{ii})\right) \\
&= \Omega\left(I \triangleright I \triangleright I \triangleright \ldots \triangleright P_j \triangleright A_{ii}\right) + \Omega\left(I \triangleright I \triangleright I \triangleright \ldots \triangleright P_j \triangleright B_{ii}\right) \\
&= \Omega\left(I\right) \triangleright I \triangleright I \triangleright \ldots \triangleright P_j \triangleright (A_{ii} + B_{ii}) + I \triangleright \Omega\left(I\right) \triangleright I \triangleright \ldots \triangleright P_j \triangleright (A_{ii} + B_{ii}) \\
&+ \ \ldots \\
&+ \ I \triangleright I \triangleright \ldots \triangleright \Omega(P_j) \triangleright (A_{ii} + B_{ii}) + I \triangleright I \triangleright I \triangleright \ldots \triangleright P_j \triangleright (\Omega(A_{ii}) + \Omega(B_{ii})).
\end{aligned}
$$

Therefore, $I \triangleright I \triangleright I \triangleright P_j \triangleright T = 0$. Since $(1+\lambda)^{n-2} P_j \triangleright T = 0$ we have $T_{jj} = T_{ji} = T_{ij} = 0$. From Claim 2.7, for every $C_{ij} \in \mathfrak{A}_{ij}$ we have

$$
\begin{aligned}
&\Omega(I) \triangleright I \triangleright I \triangleright \ldots \triangleright P_i \triangleright (A_{ii} + B_{ii}) \triangleright C_{ij} + I \triangleright \Omega(I) \triangleright I \triangleright \ldots \triangleright P_i \triangleright (A_{ii} + B_{ii}) \triangleright C_{ij} \\
&+ \ \ldots \\
&+ \ I \triangleright I \triangleright \ldots \triangleright \Omega(P_i) \triangleright (A_{ii} + B_{ii}) \triangleright C_{ij} + I \triangleright I \triangleright \ldots \triangleright P_i \triangleright \Omega(A_{ii} + B_{ii}) \triangleright C_{ij} \\
&+ \ I \triangleright I \triangleright \ldots \triangleright P_i \triangleright (A_{ii} + B_{ii}) \triangleright \Omega(C_{ij}) \\
&= \Omega(I \triangleright I \triangleright \ldots \triangleright P_i \triangleright A_{ii} \triangleright C_{ij}) + \Omega(I \triangleright I \triangleright \ldots \triangleright P_i \triangleright B_{ii} \triangleright C_{ij}) \\
&= \Omega(I) \triangleright I \triangleright I \triangleright \ldots \triangleright P_i \triangleright (A_{ii} + B_{ii}) \triangleright C_{ij} + I \triangleright \Omega(I) \triangleright I \triangleright \ldots \triangleright P_i \triangleright (A_{ii} + B_{ii}) \triangleright C_{ij} \\
&+ \ \ldots \\
&+ \ I \triangleright I \triangleright \ldots \triangleright \Omega(P_i) \triangleright A_{ii} + B_{ii} \triangleright C_{ij} + I \triangleright I \triangleright \ldots \triangleright P_i \triangleright (\Omega(A_{ii}) + \Omega(B_{ii})) \triangleright C_{ij} \\
&+ \ I \triangleright I \triangleright \ldots \triangleright P_i \triangleright A_{ii} + B_{ii} \triangleright \Omega(C_{ij})
\end{aligned}
$$

So, $I \triangleright I \triangleright \ldots \triangleright P_i \triangleright T \triangleright C_{ij} = 0$. We have $(1+\lambda)^{n-3} P_i \triangleright T \triangleright C_{ij} = 0$, then $(1+\lambda)^{n-2} T_{ii} C_{ij} = 0$. So $(1+\lambda)^{n-2} T_{ii} C P_j = 0$ for every $C \in \mathfrak{A}$. Since the algebra is factor, $P_j \neq 0$ and $\lambda \neq -1$ we obtain $T_{ii} = 0$.

Hence, the additivity of Ω comes from the above claims.

In the rest of this paper, we prove that Ω is $*$-derivation.

Theorem 2.9. *Let $\mathfrak{A}$ be a factor von Neumann algebra and the map $\Omega : \mathfrak{A} \to \mathfrak{A}$ satisfy in the following condition*

$$
\begin{aligned}
\Omega(A_1 \triangleright A_2 \triangleright \ldots \triangleright A_n) \ = \ & \Omega(A_1) \triangleright A_2 \triangleright \ldots \triangleright A_n + A_1 \triangleright \Omega(A_2) \triangleright \ldots \triangleright A_n \\
& + \ldots + A_1 \triangleright A_2 \triangleright \ldots \triangleright \Omega(A_n)
\end{aligned} \tag{4}
$$

for all $A_1, A_2, \ldots, A_n \in \mathfrak{A}$ where $A_i \triangleright A_j = A_i A_j + \lambda A_j A_i^*$ for $i, j \in \mathbb{N}, \lambda \in \mathbb{R}$, $\lambda \notin \{0, \pm 1\}$, then Ω is $*$-derivation.

Proof. We present the proof by several claims.

Claim 2.10. The element $\Omega(I)$ is self-adjoint.

Proof. By Theorem 2.1, the map Ω is additive. In particular,

$$\Omega(-A) = -\Omega(A)$$

for every $A \in \mathfrak{A}$. All iterated products below are associated from the left.

First, observe that

$$\underbrace{I \triangleright \cdots \triangleright I}_{n-2} \triangleright iI \triangleright iI = \underbrace{I \triangleright \cdots \triangleright I}_{n-3} \triangleright iI \triangleright iI \triangleright I,$$

because both sides are equal to $(1+\lambda)^{n-2}(\lambda-1)I$. Applying the n-tuple λ-Jordan $*$-derivation identity to both sides and subtracting the resulting equalities, we obtain

$$2i\lambda(1+\lambda)^{n-3}\big(\Omega(iI)+\Omega(iI)^*\big)=0.$$

Since $\lambda\neq 0,-1$, it follows that

$$\Omega(iI)^*=-\Omega(iI). \tag{5}$$

Next, we have

$$\underbrace{I\triangleright\cdots\triangleright I}_{n-3}\triangleright iI\triangleright iI\triangleright iI=-\underbrace{I\triangleright\cdots\triangleright I}_{n-2}\triangleright iI\triangleright I.$$

Indeed,

$$\underbrace{I\triangleright\cdots\triangleright I}_{n-3}\triangleright iI\triangleright iI\triangleright iI=i(\lambda-1)(1+\lambda)^{n-2}I,$$

whereas

$$\underbrace{I\triangleright\cdots\triangleright I}_{n-2}\triangleright iI\triangleright I=-i(\lambda-1)(1+\lambda)^{n-2}I.$$

Applying the defining identity to the preceding equality and using the additivity of Ω, we obtain

$$2(1+\lambda)^{n-3}\Big[(\lambda^2+\lambda-1)\Omega(iI)+\lambda\Omega(iI)^*-i(\lambda^2-1)\Omega(I)\Big]=0.$$

By (5), this becomes

$$2(1+\lambda)^{n-3}(\lambda^2-1)\big(\Omega(iI)-i\Omega(I)\big)=0.$$

Equivalently,

$$2(\lambda-1)(1+\lambda)^{n-2}\big(\Omega(iI)-i\Omega(I)\big)=0.$$

Since $\lambda\neq\pm1$, we conclude that

$$\Omega(iI)=i\Omega(I). \tag{6}$$

Taking adjoints in (6), we obtain

$$\Omega(iI)^*=-i\Omega(I)^*.$$

On the other hand, by (5) and (6),

$$\Omega(iI)^*=-\Omega(iI)=-i\Omega(I).$$

Consequently, $-i\Omega(I)^*=-i\Omega(I)$, and hence $\Omega(I)^*=\Omega(I)$. Therefore, $\Omega(I)$ is self-adjoint. □

Claim 2.11. If $\Omega(I)$ is self-adjoint then $\Omega\,(I)=0$.

Clearly, we have $\Omega(I\triangleright I\triangleright\ldots\triangleright I\triangleright iI\triangleright iI)=\Omega(I\triangleright I\triangleright\ldots\triangleright I\triangleright iI\triangleright iI\triangleright I)$. Then

$$\begin{aligned}&\Omega(I)\triangleright I\triangleright\ldots\triangleright I\triangleright iI\triangleright iI+I\triangleright\Omega(I)\triangleright\ldots\triangleright iI\triangleright iI\\&+\ \ldots\\&+\ I\triangleright I\triangleright\ldots\triangleright iI\triangleright\Omega(iI)=\Omega(I)\triangleright I\triangleright\ldots\triangleright I\triangleright iI\triangleright iI\triangleright I\\&+\ I\triangleright\Omega(I)\triangleright I\triangleright\ldots\triangleright iI\triangleright iI\triangleright I+\ldots\\&+\ I\triangleright I\triangleright\ldots\triangleright I\triangleright iI\triangleright iI\triangleright\Omega(I).\end{aligned}$$

Since $\Omega(I)$ is self-adjoint from above equation we have

$$(1+\lambda)^{n-3}(i\Omega(iI)+\lambda i\Omega(iI)^*+\lambda i\Omega(iI)+\lambda^2 i\Omega(iI)^*+i\Omega(iI)$$

$$
\begin{aligned}
&- \lambda i\Omega(iI) + \lambda i\Omega(iI) - \lambda^2 i\Omega(iI))\\
&= (1+\lambda)^{n-3}(i\Omega(iI) + \lambda i\Omega(iI)^* - \lambda i\Omega(iI)^* - \lambda^2 i\Omega(iI) + i\Omega(iI)\\
&\quad - i\lambda\Omega(iI)^* - i\lambda\Omega(iI) + i\lambda^2\Omega(iI)^*).
\end{aligned}
$$

Since $\lambda \neq -1$, from the above equation we have $\Omega(iI) + \Omega(iI)^* = 0$, it follows that

$$\Omega(iI)^* = -\Omega(iI). \tag{7}$$

We have

$$\Omega(I \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright iI \triangleright iI) = -\Omega(I \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright I).$$

Equivalently,

$$
\begin{aligned}
&\Omega(I) \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright iI \triangleright iI + I \triangleright \Omega(I) \triangleright \ldots \triangleright I \triangleright iI \triangleright iI \triangleright iI\\
&\quad + \ldots + I \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright iI \triangleright \Omega(iI)\\
&= -(\Omega(I) \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright I + I \triangleright \Omega(I) \triangleright \ldots \triangleright I \triangleright iI \triangleright I\\
&\quad + \ldots + I \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright \Omega(I)).
\end{aligned}
$$

So,

$$
\begin{aligned}
&I \triangleright I \triangleright \ldots \triangleright I \triangleright \Omega(iI) \triangleright iI \triangleright iI + I \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright \Omega(iI) \triangleright iI\\
&\quad + I \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright iI \triangleright \Omega(iI)\\
&= -(I \triangleright I \triangleright \ldots \triangleright I \triangleright \Omega(I) \triangleright iI \triangleright I + I \triangleright I \triangleright \ldots \triangleright I \triangleright \Omega(iI) \triangleright I\\
&\quad + I \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright \Omega(I)).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&(1+\lambda)^{n-2}(\lambda\Omega(iI) - \Omega(iI)) + (1+\lambda)^{n-3}(-\Omega(iI) + \lambda\Omega(iI) + \lambda\Omega(iI)^* - \lambda^2\Omega(iI)^*)\\
&+(1+\lambda)^{n-2}(\Omega(iI) - \lambda\Omega(iI))\\
&= -((1+\lambda)^{n-2}(i\Omega(I) - \lambda i\Omega(I)) + (1+\lambda)^{n-2}(\Omega(iI) + \lambda\Omega(iI)^*)\\
&+(1+\lambda)^{n-2}(i\Omega(I) - \lambda i\Omega(I))).
\end{aligned}
$$

Since $\lambda \neq -1$ from (7) we have

$$
\begin{aligned}
&\lambda\Omega(iI) - \Omega(iI) + \lambda^2\Omega(iI) - \lambda\Omega(iI) - \Omega(iI) + \lambda\Omega(iI) + \lambda\Omega(iI)^*\\
&-\lambda^2\Omega(iI)^* + \Omega(iI) - \lambda\Omega(iI) + \lambda\Omega(iI) - \lambda^2\Omega(iI)\\
&= -i\Omega(I) + \lambda i\Omega(I) - \lambda i\Omega(I) + \lambda^2 i\Omega(I) - \Omega(iI) - \lambda\Omega(iI)^* - \lambda\Omega(iI)\\
&-\lambda^2\Omega(iI)^* - i\Omega(I) + \lambda i\Omega(I) - \lambda i\Omega(I) + \lambda^2 i\Omega(I).
\end{aligned}
$$

Then we have

$$\lambda\Omega(iI) + \lambda\Omega(iI)^* + i\Omega(I) - \lambda^2 i\Omega(I) = 0. \tag{8}$$

From (7) and (8) we have $(1-\lambda^2)\Omega(I) = 0$. We obtain $\Omega(I) = 0$.

Claim 2.12. If $\Omega(I)$ is self-adjoint then $\Omega(iI) = 0$.

For showing that $\Omega\,(iI) = 0$, we write

$$-\Omega(I \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright iI \triangleright iI) = \Omega(I \triangleright I \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright I).$$

So

$$
\begin{aligned}
&I \triangleright I \triangleright \ldots \triangleright I \triangleright \Omega(iI) \triangleright iI \triangleright iI + I \triangleright I \triangleright \ldots \triangleright iI \triangleright \Omega(iI) \triangleright iI\\
&+I \triangleright I \triangleright \ldots \triangleright iI \triangleright iI \triangleright \Omega(iI) = -(I \triangleright I \triangleright \ldots \triangleright I \triangleright \Omega(iI) \triangleright I).
\end{aligned}
$$

It follows that

$$3(1+\lambda)^{n-2}(\lambda-1)\Omega(iI) \quad = \quad (1+\lambda)^{n-2}(\lambda-1)\Omega(iI).$$

Finally we have

$$3\Omega(iI)=\Omega(iI). \tag{9}$$

From (9) we deduce that $\Omega\,(iI)$=0.

Claim 2.13. Ω preserves star.

Since $\Omega(I)=\Omega(iI)=0$, then

$$\Omega(I \triangleright I \triangleright \ldots \triangleright A \triangleright iI \triangleright iI) = I \triangleright I \triangleright \ldots \triangleright \Omega(A) \triangleright iI \triangleright iI.$$

Therefore

$$\Omega((1+\lambda)^{n-2}(\lambda-1)A)=(1+\lambda)^{n-2}(\lambda-1)\Omega(A). \tag{10}$$

On the other hand

$$\Omega(I \triangleright I \triangleright \ldots \triangleright I \triangleright I \triangleright A) = I \triangleright I \triangleright \ldots \triangleright I \triangleright \Omega(A).$$

Therefore

$$\Omega((1+\lambda)^{n-2}(1+\lambda)A)=(1+\lambda)^{n-2}(1+\lambda)\Omega(A). \tag{11}$$

By summing and subtracting (10) and (11) respectively, we have

$$\Omega((1+\lambda)^{n-2}A) \quad = \quad (1+\lambda)^{n-2}\Omega(A) \tag{12}$$

$$\Omega(\lambda(1+\lambda)^{n-2}A) \quad = \quad \lambda(1+\lambda)^{n-2}\Omega(A). \tag{13}$$

Therefore

$$\Omega(I \triangleright I \triangleright \ldots \triangleright I \triangleright A \triangleright I) = I \triangleright I \triangleright \ldots \triangleright I \triangleright \Omega(A) \triangleright I.$$

It follows that

$$\Omega((1+\lambda)^{n-2}(A+\lambda A^*)) = (1+\lambda)^{n-2}\Omega(A)+\lambda(1+\lambda)^{n-2}\Omega(A)^*.$$

From (12), (13) and the additivity of Ω we have $\Omega(A^*)=\Omega(A)^*$.

Claim 2.14. We show that $\Omega(iA)=i\Omega(A)$ for every $A\in\mathfrak{A}$.

From (12) and (13) we have

$$\begin{aligned}(1+\lambda)^{n-2}(1-\lambda)\Omega(iA) &= \Omega(I \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright A)\\ &= I \triangleright I \triangleright \ldots \triangleright I \triangleright iI \triangleright \Omega(A)\\ &= (1+\lambda)^{n-2}(1-\lambda)i\Omega(A).\end{aligned}$$

Then

$$(1+\lambda)^{n-2}(1-\lambda)\Omega(iA)=(1+\lambda)^{n-2}(1-\lambda)i\Omega(A).$$

Since $\lambda\notin\{0,\pm1\}$ we have $\Omega(iA)=i\Omega(A)$

Claim 2.15. Ω is a derivation.

For every $A, B \in \mathfrak{A}$ we have

$$\begin{aligned}
\Omega((1+\lambda)^{n-2}(AB+\lambda BA^*)) &= \Omega\,(I \triangleright I \triangleright \ldots \triangleright I \triangleright A \triangleright B) \\
&= I \triangleright I \triangleright \ldots \triangleright I \triangleright \Omega(A) \triangleright B + I \triangleright I \triangleright \ldots \triangleright I \triangleright A \triangleright \Omega(B) \\
&= (1+\lambda)^{n-2}(\Omega(A)B + \lambda B\Omega(A)^*) \\
&\quad +(1+\lambda)^{n-2}(A\Omega(B) + \lambda\Omega(B)A^*).
\end{aligned}$$

It follows that

$$\begin{aligned}
\Omega((1+\lambda)^{n-2}(AB+\lambda BA^*)) &= (1+\lambda)^{n-2}(\Omega(A)B + \lambda B\Omega(A)^*) \\
&\quad +(1+\lambda)^{n-2}(A\Omega(B) + \lambda\Omega(B)A^*).
\end{aligned}$$

(14)

On the other hand

$$\begin{aligned}
\Omega((1+\lambda)^{n-2}(AB-\lambda BA^*)) &= \Omega(I \triangleright I \triangleright I \triangleright \ldots \triangleright (iA) \triangleright (-iB)) \\
&= I \triangleright \ldots \triangleright I \triangleright \Omega(iA) \triangleright (-iB) \\
&\quad +I \triangleright \ldots \triangleright I \triangleright iA \triangleright \Omega(-iB) \\
&= (1+\lambda)^{n-2}(\Omega(A)B - \lambda B\Omega(A)^*) \\
&\quad +(1+\lambda)^{n-2}(A\Omega(B) - \lambda\Omega(B)A^*).
\end{aligned}$$

So

$$\begin{aligned}
&\Omega((1+\lambda)^{n-2}(AB-\lambda BA^*)) \\
&= (1+\lambda)^{n-2}(\Omega(A)B - \lambda B\Omega(A)^* + (1+\lambda)^{n-2}(A\Omega(B) - \lambda\Omega(B)A^*).
\end{aligned}$$

(15)

From (14) and (15) we obtain $\Omega((1+\lambda)^{n-2}AB) = (1+\lambda)^{n-2}(\Omega(A)B + A\Omega(B))$. From (12) we have $(1+\lambda)^{n-2}\Omega(AB) = (1+\lambda)^{n-2}(\Omega(A)B + A\Omega(B))$. Since $\lambda \notin \{0, \pm 1\}$, we have $\Omega(AB) = \Omega(A)B + A\Omega(B)$. $\square$

Lastly, we provide an example of a nontrivial map Ψ on $\mathscr{F}$ that satisfies

$$\Psi(A_1 \triangleright \cdots \triangleright A_n) = \Psi(A_1) \triangleright \cdots \triangleright A_n + \cdots + A_1 \triangleright \cdots \triangleright \Psi(A_n)$$

for all $A_i \in \mathscr{F}, i = 1, 2, ...n$.

Example 2.16. Consider $\mathfrak{A} = \mathcal{M}_2(\mathbb{C})$, the factor von Neumann algebra of all 2×2 complex matrices, equipped with the usual involution given by the conjugate transpose. Let

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Define the map $\Omega : \mathfrak{A} \to \mathfrak{A}$ by

$$\Omega(A) = XA - AX, \qquad A \in \mathfrak{A}.$$

Equivalently, for

$$A = \begin{bmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{bmatrix},$$

we have

$$\Omega(A) = \begin{bmatrix} -\eta_2 - \eta_3 & \eta_1 - \eta_4 \\ \eta_1 - \eta_4 & \eta_2 + \eta_3 \end{bmatrix}.$$

Since $X^* = -X$, the map Ω is an inner $*$-derivation. Indeed, for all $A, B \in \mathfrak{A}$, $\Omega(AB) = \Omega(A)B + A\Omega(B)$ and $\Omega(A^*) = \Omega(A)^*$. Hence, for the product $A \triangleright B = AB + \lambda BA^*$, we obtain

$$\begin{aligned}\Omega(A \triangleright B) &= \Omega(AB + \lambda BA^*)\\ &= \Omega(A)B + A\Omega(B) + \lambda\Omega(B)A^* + \lambda B\Omega(A^*)\\ &= \Omega(A)B + \lambda B\Omega(A)^* + A\Omega(B) + \lambda\Omega(B)A^*\\ &= \Omega(A) \triangleright B + A \triangleright \Omega(B).\end{aligned}$$

Therefore, by induction, Ω satisfies

$$\Omega(A_1 \triangleright A_2 \triangleright \cdots \triangleright A_n) = \Omega(A_1) \triangleright A_2 \triangleright \cdots \triangleright A_n + \cdots + A_1 \triangleright A_2 \triangleright \cdots \triangleright \Omega(A_n)$$

for all $A_1, A_2, \ldots, A_n \in \mathfrak{A}$.

Moreover,

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is a nontrivial projection in $\mathfrak{A}$, and Ω is a nonzero map.

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