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EINSTEIN-LIKE WARPED PRODUCT MANIFOLDS WHICH ARE NOT EINSTEIN

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Abstract. In this paper, we firstly present a necessary and sufficient condition for a warped product manifold to be an Einstein-like manifold. By using this condition, we prove that if a warping function is not constant, then the fiber space of an Einstein-like warped product manifold is an Einstein manifold. Moreover we construct new examples of Einstein-like manifold which are not Einstein.

1. Introduction

Let $(B, {}^B g)$ and $(F, {}^F g)$ be Riemannian manifolds and f a positive smooth function on B. Then a warped product manifold $B \times_f F$ is the product manifold $B \times F$ equipped with the Riemannian metric $g = {}^B g + f^2\, {}^F g$. We say that $(B, {}^B g)$ a base space, $(F, {}^F g)$ a fiber space and f is a warping function.

The notion of warped product manifolds plays important roles in differential geometry. For example, Ejiri [3], Tanno [9] and Ehrlich-Jung-Kim [2] constructed the warped product manifolds with constant scalar curvature, and, in particular, Ejiri showed the existence of counterexamples to a conjecture for a characterization of a Euclidean sphere. On the other hand, Besse [1] derived a necessary and sufficient condition for a warped product manifold to be an Einstein manifold, and he classified complete warped product manifolds whose base space has dimension 1 or 2. Also, Kim and Kim [5] proved that, if an Einstein warped product manifold with a compact base space satisfies some curvature condition, then it is a product manifold (see also [7]).

On the other hand, Gray [4] introduced the class of Einstein-like manifolds which is a notion between the class of Einstein manifolds and the class of Riemmanian manifolds with constant scalar curvarure. He defined the following two classes of manifolds. A Riemannian manifold (M, g) is of class $\mathcal{A}$ if the Ricci tensor Ric is a Killing tensor, that is, for any vector fields ξ, η, ζ on M

$$(\nabla_\xi \mathrm{Ric})(\eta, \zeta) + (\nabla_\eta \mathrm{Ric})(\zeta, \xi) + (\nabla_\zeta \mathrm{Ric})(\xi, \eta) = 0$$

holds, where ∇ stands for the Levi-Civita connection of (M, g). A Riemannian manifold (M, g) is of class $\mathcal{B}$ if the Ricci tensor is a Codazzi tensor, that is, for any vector fields ξ, η, ζ on M

$$(\nabla_\xi \mathrm{Ric})(\eta, \zeta) = (\nabla_\eta \mathrm{Ric})(\xi, \zeta)$$

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holds. Clearly, if a Riemannian manifold has the parallel Ricci tensor, then it belongs to each of these classes.

Recently, for a warped product manifold $B \times_f F$ of a class $\mathcal{A}$ (resp. class $\mathcal{B}$), Mantica and Shenawy [6] derived a necessary and sufficient condition for both $(B, {}^B g)$ and $(F, {}^F g)$ to be manifolds of class $\mathcal{A}$ (resp. class $\mathcal{B}$).

In this paper, we give a necessary and sufficient condition that $(B, {}^B g)$ and $(F, {}^F g)$ satisfy for the warped product manifold $B \times_f F$ to be an Einstein-like manifold. Moreover, if $(F, {}^F g)$ is Ricci flat, then, by using the condition, we construct new examples of manifold of class $\mathcal{B}$ which is not an Einstein manifold. Here, a Riemannian manifold (M, g) is Ricci flat, if its Ricci tensor is identically zero.

2. Preliminaries on a warped product manifold

Let $(B, {}^B g)$ and $(F, {}^F g)$ be Riemannian manifolds of dimension m and n, respectively. We denote by ${}^B\nabla$, ${}^B\mathrm{Ric}$ the Levi-Civita connection and the Ricci tensor of $(B, {}^B g)$, and by ${}^F\nabla$, ${}^F\mathrm{Ric}$ the same objects of $(F, {}^F g)$, respectively. Also, for a local orthonormal frame field $\{\boldsymbol{e}_i\}_{i=1}^m$ on B, the Laplacian $\Delta_B f$ and the gradient $\mathrm{grad}_B f$ of a function f on B are given by

$$\Delta_B f = -\sum_{i=1}^m \{(\boldsymbol{e}_i \cdot \boldsymbol{e}_i) f - ({}^B\nabla_{\boldsymbol{e}_i} \boldsymbol{e}_i) f\}, \quad \mathrm{grad}_B f = \sum_{i=1}^m (\boldsymbol{e}_i \cdot f)\boldsymbol{e}_i,$$

respectively.

Then, the Levi-Civita connection ∇ and the Ricci tensor Ric of the warped product manifold $B \times_f F$ are related to those of $(B, {}^B g)$ and $(F, {}^F g)$ as follows.

Lemma 2.1. ([8]) *Let X, Y be vector fields on B and U, V vector fields on F. Then*

(1) $\nabla_X Y = {}^B\nabla_X Y$.

(2) $\nabla_X U = \nabla_U X = f^{-1}(X \cdot f)U$.

(3) $\nabla_U V = {}^F\nabla_U V - f\,{}^F g(U, V)\mathrm{grad}_B f$.

Lemma 2.2. ([8]) *If X, Y are vector fields on B and U, V are vector fields on F, then*

(1) $\mathrm{Ric}(X, Y) = {}^B\mathrm{Ric}(X, Y) - nf^{-1}H^f(X, Y)$, *where* $H^f(X, Y) = X \cdot (Y \cdot f) - ({}^B\nabla_X Y) \cdot f$.

(2) $\mathrm{Ric}(X, U) = \mathrm{Ric}(U, X) = 0$.

(3) $\mathrm{Ric}(U, V) = {}^F\mathrm{Ric}(U, V) + \varphi\,{}^F g(U, V)$, *where* $\varphi = f\Delta_B f - (n-1)\|\mathrm{grad}_B f\|^2$ *and* $\|X\|^2 = {}^B g(X, X)$ *for a vector field X on B.*

Using these formulas, Besse [1] obtained a necessary and sufficient condition for a warped product manifold to be an Einstein manifold. We present the result in a form that will be used later.

Corollary 2.3. *When* $\dim B = 1$, *the warped product manifold $B \times_f F$ is an Einstein manifold with* $\mathrm{Ric} = \lambda g$ *if and only if $(F, {}^F g)$ and f satisfy the following conditions.*

(1) *$(F, {}^F g)$ is an Einstein manifold with* ${}^F\mathrm{Ric} = \mu {}^F g$ *for a constant μ.*

(2) $f''(t) = -\dfrac{\lambda}{n} f(t)$, *where* $t \in B$.

(3) $\mu = f(t)f''(t) + (n-1)f'(t)^2 + \lambda f(t)^2$.

We mainly use the following formula for the covarinat derivative of the Ricci tensor.

Proposition 2.4. *For vector fields* X, Y, Z *on* B *and* U, V, W *on* F, *we have the following.*

(1) $(\nabla_X \mathrm{Ric})(Y,Z) = ({}^B\nabla_X {}^B\mathrm{Ric})(Y,Z) - n\{{}^B\nabla_X(f^{-1}H^f)\}(Y,Z)$.

(2) $(\nabla_U \mathrm{Ric})(Y,Z) = 0$.

(3) $(\nabla_X \mathrm{Ric})(Y,V) = 0$.

(4) $(\nabla_U \mathrm{Ric})(Y,V) = -f^{-1}(Y\cdot f)^F\mathrm{Ric}(U,V) - \Big\{f^{-1}(Y\cdot f)\varphi$

$$-f\,{}^B\mathrm{Ric}(Y,\mathrm{grad}_B f) + \frac{1}{2}nY\cdot\|\mathrm{grad}_B f\|^2\Big\}{}^F\!g(U,V).$$

(5) $(\nabla_X \mathrm{Ric})(V,W) = -2f^{-1}(X\cdot f)^F\mathrm{Ric}(V,W) + f^2\{X\cdot(f^{-2}\varphi)\}{}^F\!g(V,W)$.

(6) $(\nabla_U \mathrm{Ric})(V,W) = ({}^F\nabla_U {}^F\mathrm{Ric})(V,W)$.

Proof. (1) By using Lemma 2.1(1) and Lemma 2.2(1), we have

$$\begin{aligned}(\nabla_X \mathrm{Ric})(Y,Z) &= X\cdot \mathrm{Ric}(Y,Z) - \mathrm{Ric}(\nabla_X Y, Z) - \mathrm{Ric}(Y,\nabla_X Z)\\ &= X\cdot\Big\{{}^B\mathrm{Ric}(Y,Z) - n(f^{-1}H^f)(Y,Z)\Big\}\\ &\qquad - \mathrm{Ric}({}^B\nabla_X Y, Z) - \mathrm{Ric}(Y, {}^B\nabla_X Z)\\ &= X\cdot {}^B\mathrm{Ric}(Y,Z) - nX\cdot\{(f^{-1}H^f)(Y,Z)f\}\\ &\quad - \{{}^B\mathrm{Ric}({}^B\nabla_X Y, Z) - n(f^{-1}H^f)({}^B\nabla_X Y, Z)f\}\\ &\quad - \{{}^B\mathrm{Ric}(Y, {}^B\nabla_X Z) - n(f^{-1}H^f)(Y, {}^B\nabla_X Z)f\}\\ &= ({}^B\nabla_X {}^B\mathrm{Ric})(Y,Z) - n\{{}^B\nabla_X(f^{-1}H^f)\}(Y,Z).\end{aligned}$$

(2) We have $U\cdot \mathrm{Ric}(Y,Z) = 0$, because $\mathrm{Ric}(Y,Z)$ is a function on the base space B. Thus, from Lemma 2.1(2) and Lemma 2.2(2), it holds that

$$\begin{aligned}(\nabla_U \mathrm{Ric})(Y,Z) &= U\cdot \mathrm{Ric}(Y,Z) - \mathrm{Ric}(\nabla_U Y, Z) - \mathrm{Ric}(Y,\nabla_U Z)\\ &= -f^{-1}(Y\cdot f)\mathrm{Ric}(U,Z) - f^{-1}(Z\cdot f)\mathrm{Ric}(Y,U)\\ &= 0.\end{aligned}$$

(3) From Lemma 2.1(1), (2) and Lemma 2.2 (2), we obtain

$$\begin{aligned}(\nabla_X \mathrm{Ric})(Y,V) &= X\cdot \mathrm{Ric}(Y,V) - \mathrm{Ric}(\nabla_X Y, V) - \mathrm{Ric}(Y,\nabla_X V)\\ &= -\mathrm{Ric}({}^B\nabla_X Y, V) - f^{-1}(X\cdot f)\mathrm{Ric}(Y,V)\\ &= 0.\end{aligned}$$

(4) By using Lemma 2.1(2), (3) and Lemma 2.2 (2), (3), we have

$$\begin{aligned}(\nabla_U \mathrm{Ric})(Y,V) &= U\cdot \mathrm{Ric}(Y,V) - \mathrm{Ric}(\nabla_U Y, V) - \mathrm{Ric}(Y,\nabla_U V)\\ &= -\mathrm{Ric}(f^{-1}(Y\cdot f)U, V) - \mathrm{Ric}(Y, {}^F\nabla_U V - f\,{}^F\!g(U,V)\mathrm{grad}_B f)\\ &= -f^{-1}(Y\cdot f)\mathrm{Ric}(U,V) - \mathrm{Ric}(Y, {}^F\nabla_U V)\\ &\qquad + f\mathrm{Ric}(Y,\mathrm{grad}_B f){}^F\!g(U,V)\end{aligned}$$

$$
\begin{aligned}
&= -f^{-1}(Y\cdot f)\{{}^F\mathrm{Ric}(U,V)+\varphi\,{}^F\!g(U,V)\}\\
&\quad + f\Big\{{}^B\mathrm{Ric}(Y,\mathrm{grad}_B f)-n(f^{-1}H^f)(Y,\mathrm{grad}_B f)\Big\}{}^F\!g(U,V)\\
&= -f^{-1}(Y\cdot f){}^F\mathrm{Ric}(U,V)-\Big\{f^{-1}(Y\cdot f)\varphi\\
&\qquad - f\,{}^B\mathrm{Ric}(Y,\mathrm{grad}_B f)+nH^f(Y,\mathrm{grad}_B f)\Big\}{}^F\!g(U,V).
\end{aligned}
$$

For the last term, from the definition of the Hessian H^f, it holds that

$$
\begin{aligned}
H^f(Y,\mathrm{grad}_B f) &= Y\cdot(\mathrm{grad}_B f)\cdot f-({}^B\nabla_Y\mathrm{grad}_B f)f\\
&= Y\cdot\|\mathrm{grad}_B f\|^2-{}^B g({}^B\nabla_Y\mathrm{grad}_B f,\mathrm{grad}_B f)\\
&= \frac{1}{2}Y\cdot\|\mathrm{grad}_B f\|^2.
\end{aligned}
$$

Thus we have the conclusion.

(5) Since ${}^F\mathrm{Ric}(V,W)$ and ${}^F\!g(V,W)$ are functions on the fiber space F,

$$
X\cdot{}^F\mathrm{Ric}(V,W)=0,\quad X\cdot{}^F\!g(V,W)=0.
$$

Thus, from Lemma 2.1(2) and Lemma 2.2(2),(3), we have

$$
\begin{aligned}
(\nabla_X\mathrm{Ric})(V,W) &= X\cdot\mathrm{Ric}(V,W)-\mathrm{Ric}(\nabla_X V,W)-\mathrm{Ric}(V,\nabla_X W)\\
&= X\cdot\Big\{{}^F\mathrm{Ric}(V,W)+\varphi\,{}^F\!g(V,W)\Big\}\\
&\qquad - f^{-1}(X\cdot f)\mathrm{Ric}(V,W)-f^{-1}(X\cdot f)\mathrm{Ric}(V,W)\\
&= (X\cdot\varphi){}^F\!g(V,W)-2f^{-1}(X\cdot f)\mathrm{Ric}(V,W)\\
&= (X\cdot\varphi){}^F\!g(V,W)\\
&\qquad\quad -2f^{-1}(X\cdot f)\Big\{{}^F\mathrm{Ric}(V,W)+\varphi\,{}^F\!g(V,W)\Big\}\\
&= -2f^{-1}(X\cdot f){}^F\mathrm{Ric}(V,W)+f^2\{X\cdot(f^{-2}\varphi)\}{}^F\!g(V,W).
\end{aligned}
$$

(6) By using Lemma 2.1(3) and Lemma 2.2(2),(3), we obtain

$$
\begin{aligned}
(\nabla_U\mathrm{Ric})(V,W) &= U\cdot\mathrm{Ric}(V,W)-\mathrm{Ric}(\nabla_U V,W)-\mathrm{Ric}(V,\nabla_U W)\\
&= U\cdot\{{}^F\mathrm{Ric}(V,W)+\varphi\,{}^F\!g(V,W)\}\\
&\qquad -\mathrm{Ric}({}^F\nabla_U V-f\,{}^F\!g(U,W)\mathrm{grad}_B f,W)\\
&\qquad\qquad -\mathrm{Ric}(V,{}^F\nabla_U W-f\,{}^F\!g(U,W)\mathrm{grad}_B f)\\
&= U\cdot{}^F\mathrm{Ric}(V,W)+\varphi U\cdot{}^F\!g(V,W)\\
&\qquad -{}^F\mathrm{Ric}({}^F\nabla_U V,W)-\varphi\,{}^F\!g({}^F\nabla_U V,W)\\
&\qquad\qquad -{}^F\mathrm{Ric}(V,{}^F\nabla_U W)-\varphi\,{}^F\!g(V,{}^F\nabla_U W)\\
&= \{U\cdot{}^F\mathrm{Ric}(V,W)-{}^F\mathrm{Ric}({}^F\nabla_U V,W)-{}^F\mathrm{Ric}(V,{}^F\nabla_U W)\}\\
&\quad +\varphi\{U\cdot{}^F\!g(V,W)-{}^F\!g({}^F\nabla_U V,W)-{}^F\!g(V,{}^F\nabla_U W)\}\\
&= ({}^F\nabla_U{}^F\mathrm{Ric})(V,W)+\varphi({}^F\nabla_U{}^F\!g)(V,W)\\
&= ({}^F\nabla_U{}^F\mathrm{Ric})(V,W).
\end{aligned}
$$

This completes the proof. □

3. Einstein-like warped product manifolds

3.1. Warped product manifolds of class $\mathcal{A}$. We recall that a Riemannian manifold (M, g) is of class $\mathcal{A}$ if and only if

$$(\nabla_\xi \mathrm{Ric})(\eta, \zeta) + (\nabla_\eta \mathrm{Ric})(\zeta, \xi) + (\nabla_\zeta \mathrm{Ric})(\xi, \eta) = 0 \tag{3.1}$$

holds for any vector fields ξ, η, ζ on M.

Lemma 3.1. *A warped product manifold $B \times_f F$ is of class $\mathcal{A}$ if and only if the following conditions hold.*

(1) *For any vector fields X, Y, Z on B,*

$$\begin{aligned}({}^B\nabla_X {}^B\mathrm{Ric})(Y, Z) + ({}^B\nabla_Y {}^B\mathrm{Ric})(Z, X) + ({}^B\nabla_Z {}^B\mathrm{Ric})(X, Y)\\ = n\Big[\big\{{}^B\nabla_X(f^{-1}H^f)\big\}(Y, Z) + \big\{{}^B\nabla_Y(f^{-1}H^f)\big\}(Z, X)\\ + \big\{{}^B\nabla_Z(f^{-1}H^f)\big\}(X, Y)\Big].\end{aligned} \tag{3.2}$$

(2) *For any vector fields X on B and U, V on F,*

$$\begin{aligned}4f^{-1}(X \cdot f){}^F\mathrm{Ric}(U, V) = \Big[f^4\big\{X \cdot (f^{-4}\varphi)\big\} + 2f\,{}^B\mathrm{Ric}(X, \mathrm{grad}_B f)\\ - nX \cdot \|\mathrm{grad}_B f\|^2\Big]{}^F g(U, V).\end{aligned} \tag{3.3}$$

(3) *For any vector fields U, V, W on F,*

$$({}^F\nabla_U {}^F\mathrm{Ric})(V, W) + ({}^F\nabla_V {}^F\mathrm{Ric})(W, U) + ({}^F\nabla_W {}^F\mathrm{Ric})(U, V) = 0. \tag{3.4}$$

Hence the fiber space $(F, {}^F g)$ is a manifold of class $\mathcal{A}$.

Proof. Let X, Y, Z be vector fields on B and U, V, W vector fields on F. Firstly, we assume that a warped product manifold $B \times_f F$ is of class $\mathcal{A}$.

(1) If we set $\xi = X, \eta = Y, \zeta = Z$ in (3.1), then (3.2) is a straightforward conclusion of Proposition 2.4(1).

(2) If we set $\xi = X, \eta = U, \zeta = V$ in (3.1), then, by using Proposition 2.4 (4) and (5), it holds that

$$\begin{aligned}0 &= (\nabla_X \mathrm{Ric})(U, V) + (\nabla_U \mathrm{Ric})(V, X) + (\nabla_V \mathrm{Ric})(X, U)\\ &= -2f^{-1}(X \cdot f){}^F\mathrm{Ric}(U, V) + f^2\big\{X \cdot (f^{-2}\varphi)\big\}{}^F g(U, V)\\ &\quad - 2f^{-1}(X \cdot f){}^F\mathrm{Ric}(U, V)\\ &\quad - 2\Big\{f^{-1}(X \cdot f)\varphi - f\,{}^B\mathrm{Ric}(X, \mathrm{grad}_B f) + \frac{1}{2}nX \cdot \|\mathrm{grad}_B f\|^2\Big\}{}^F g(U, V)\\ &= -4f^{-1}(X \cdot f){}^F\mathrm{Ric}(U, V)\\ &\quad + \Big\{-4f^{-1}(X \cdot f)\varphi + X \cdot \varphi + 2f\,{}^B\mathrm{Ric}(X, \mathrm{grad}_B f) - nX \cdot \|\mathrm{grad}_B f\|^2\Big\}{}^F g(U, V)\\ &= -4f^{-1}(X \cdot f){}^F\mathrm{Ric}(U, V)\\ &\quad + \Big[f^4\big\{X \cdot (f^{-4}\varphi)\big\} + 2f\,{}^B\mathrm{Ric}(X, \mathrm{grad}_B f) - nX \cdot \|\mathrm{grad}_B f\|^2\Big]{}^F g(U, V).\end{aligned}$$

Hence we obtain (3.3).

(3) From Proposition 2.4(6), we have

$$(\nabla_U \mathrm{Ric})(V,W) = ({}^F\nabla_U {}^F\mathrm{Ric})(V,W).$$

Thus, for $\xi = U, \eta = V, \zeta = W$, (3.4) is equivalent to (3.1).

Conversly, we assume that the conditions (3.2), (3.3) and (3.4) hold. Since $(\nabla_\xi \mathrm{Ric})(\eta,\zeta)$ is a symmetric in η and ζ, we only need to consider the following four cases for (3.1) .

(i) $\xi = X, \eta = Y, \zeta = Z$. (ii) $\xi = X, \eta = Y, \zeta = U$.

(iii) $\xi = X, \eta = U, \zeta = V$. (iv) $\xi = U, \eta = V, \zeta = W$.

However, from the above discussions, (3.2), (3.3) and (3.4) are equivalent to (3.1) in each case of (i), (iii) and (iv), respectively. For the case (ii), from Proposition 2.4(3), we have

$$(\nabla_X \mathrm{Ric})(Y,U) = (\nabla_Y \mathrm{Ric})(U,X) = 0,$$

and, from Proposition 2.4(2)

$$(\nabla_U \mathrm{Ric})(X,Y) = 0.$$

Thus

$$(\nabla_X \mathrm{Ric})(Y,U) + (\nabla_Y \mathrm{Ric})(U,X) + (\nabla_U \mathrm{Ric})(X,Y) = 0$$

holds and the proof is completed. □

In particular, if f is a constant function, then (3.2) implies that $(B, {}^Bg)$ is a manifold of class $\mathcal{A}$, and (3.3) is obviously satisfied. Thus a Riemannian product manifold $B \times F$ is of class $\mathcal{A}$ if and only if $(B, {}^Bg)$ and $(F, {}^Fg)$ are manifolds of class $\mathcal{A}$.

In contrast, we have the following.

Theorem 3.2. *If a warped product manifold $B \times_f F$ is of class $\mathcal{A}$ and f is not a constant function, then the fiber space $(F, {}^Fg)$ is an Einstein manifold.*

Proof. For simplicity, we set functions h_1, h_2 on B by

$$h_1 = 4f^{-1}(X \cdot f),$$
$$h_2 = f^4\{X \cdot (f^{-4}\varphi)\} + 2f\, {}^B\mathrm{Ric}(X, \mathrm{grad}_B f) - nX \cdot \|\mathrm{grad}_B f\|^2.$$

Since f is not a constant function, there exists a vector field X on B and a point $b_0 \in B$ such that $(X \cdot f)(b_0) \neq 0$. Hence, for an open neighborhood $D \subset B$ of b_0, $h_1 \neq 0$ on D. Thus, from (3.3), we have

$$\frac{{}^F\mathrm{Ric}(U,U)}{{}^Fg(U,U)} = \frac{h_2}{h_1}$$

holds on $D \times F$ for any non zero vector field U on F. Since h_1, h_2 are functions on B, and ${}^F\mathrm{Ric}(U,U), {}^Fg(U,U)$ are functions on F, there exists a constant μ such that

$${}^F\mathrm{Ric}(U,U) = \mu\, {}^Fg(U,U)$$

holds on F. Hence, using the polarization identity, we can show that $(F, {}^Fg)$ is an Einstein manifold. □

Corollary 3.3. *Let B be an open interval in the real line and $t \in B$. If a warped product manifold $B \times_f F$ is of class $\mathcal{A}$, and f is not a constant function, then $B \times_f F$ is an Einstein manifold.*

Proof. Since $\dim B = 1$, taking $X = Y = Z = \partial/\partial t$, where $t \in B$, in (3.2), then we have

$$\frac{d}{dt}\{f^{-1}(t)f''(t)\} = 0.$$

Hence

$$f''(t) = cf(t) \tag{3.5}$$

for some constant c, thus f is given by

$$f(t) = \begin{cases} c_1 e^{\sqrt{c}t} + c_2 e^{-\sqrt{c}t} & (c > 0), \\ c_1 t + c_2 & (c = 0), \\ c_1 \sin(\sqrt{-c}t) + c_2 \cos(\sqrt{-c}t) & (c < 0), \end{cases}$$

where c_1 and c_2 are constants. On the other hand, by substituting $X = \partial/\partial t$ and $\varphi(t) = -f(t)f''(t) - (n-1)f'(t)^2$ into (3.3) and using (3.5), we have

$$4f(t)^{-1}f'(t){}^F\mathrm{Ric}(U,V) = -4(n-1)f(t)^{-1}f'(t)\{cf(t)^2 - f'(t)^2\}{}^F g(U,V).$$

Thus it holds that

$$f'(t){}^F\mathrm{Ric}(U,V) = -(n-1)f'(t)\{cf(t)^2 - f'(t)^2\}{}^F g(U,V).$$

If we set μ by

$$\mu = -(n-1)\{cf(t)^2 - f'(t)^2\},$$

then, from (3.5), μ is a constant and

$$f'(t){}^F\mathrm{Ric}(U,V) = \mu f'(t){}^F g(U,V).$$

Since f is not a constant function, we can see that

$${}^F\mathrm{Ric}(U,V) = \mu {}^F g(U,V)$$

holds for any vector fields U, V on F. Therefore $(F, {}^F g)$ is an Einstein manifolds. On the other hand, if we define λ by $\lambda = -cn$, then, from (3.5) and the definition of μ, we obtain the following.

$$\begin{aligned} f''(t) &= -\frac{\lambda}{n}f(t), \\ \mu &= f(t)f''(t) + (n-1)f'(t)^2 + \lambda f(t)^2. \end{aligned}$$

Therefore $(F, {}^F g)$ and f satisfy the conditions of Corollary 2.3, so that, for above f, the warped product manifold $B \times_f F$ is an Einstein manifold. □

3.2. Warped product manifolds of class $\mathcal{B}$. We recall that a Riemannian manifold (M, g) is of class $\mathcal{B}$ if and only if

$$(\nabla_\xi \mathrm{Ric})(\eta, \zeta) = (\nabla_\eta \mathrm{Ric})(\xi, \zeta) \tag{3.6}$$

holds for any vector fields ξ, η, ζ on M.

Lemma 3.4. *A warped product manifold $B \times_f F$ is of class $\mathcal{B}$ if and only if the following conditions hold.*

(1) *For any vector fields X, Y, Z on B,*

$$\begin{aligned} &({}^B\nabla_X {}^B\mathrm{Ric})(Y,Z) - n\{{}^B\nabla_X(f^{-1}H^f)\}(Y,Z) \\ &= ({}^B\nabla_Y {}^B\mathrm{Ric})(X,Z) - n\{{}^B\nabla_Y(f^{-1}H^f)\}(X,Z). \end{aligned} \tag{3.7}$$

(2) *For any vector fields X on B and V, W on F,*

$$f^{-1}(Xf)\,{}^{F}\mathrm{Ric}(V,W) = \Big[f\big\{X\cdot(f^{-1}\varphi)\big\} - f\,{}^{B}\mathrm{Ric}(X,\mathrm{grad}_B f) + \frac{1}{2}nX\cdot\|\mathrm{grad}_B f\|^2\Big]\,{}^{F}\!g(V,W). \tag{3.8}$$

(3) *For any vector fields U, V, W on F,*

$$({}^{F}\nabla_U\,{}^{F}\mathrm{Ric})(V,W) = ({}^{F}\nabla_V\,{}^{F}\mathrm{Ric})(U,W). \tag{3.9}$$

Hence the fiber space $(F, {}^{F}\!g)$ is a manifold of class $\mathcal{B}$.

Proof. Let X, Y, Z be vector fields on B and U, V, W vector fields on F. We assume that a warped product manifold $B\times_f F$ is of class $\mathcal{B}$.
(1) If we set $\xi = X, \eta = Y, \zeta = Z$ in (3.6), then from Proposition 2.4 (1) we can easily show that (3.7) holds.
(2) If we set $\xi = X, \eta = V, \zeta = W$ in (3.6), then from Proposition 2.4 (4) and (5) it holds that

$$\begin{aligned}
0 &= (\nabla_X\mathrm{Ric})(V,W) - (\nabla_V\mathrm{Ric})(X,W)\\
&= -2f^{-1}(X\cdot f)\,{}^{F}\mathrm{Ric}(V,W) + f^2\big\{X\cdot(f^{-2}\varphi)\big\}\,{}^{F}\!g(V,W)\\
&\quad + f^{-1}(X\cdot f)\,{}^{F}\mathrm{Ric}(V,W)\\
&\qquad + \Big[f^{-1}(X\cdot f)\varphi - f\,{}^{B}\mathrm{Ric}(X,\mathrm{grad}_B f) + \frac{1}{2}nX\cdot\|\mathrm{grad}_B f\|^2\Big]\,{}^{F}\!g(V,W)\\
&= -f^{-1}(X\cdot f)\,{}^{F}\mathrm{Ric}(V,W)\\
&\quad + \Big[-f^{-1}(X\cdot f)\varphi + X\cdot\varphi - f\,{}^{B}\mathrm{Ric}(X,\mathrm{grad}_B f) + \frac{1}{2}nX\cdot\|\mathrm{grad}_B f\|^2\Big]\,{}^{F}\!g(V,W)\\
&= -f^{-1}(Xf)\,{}^{F}\mathrm{Ric}(V,W)\\
&\quad + \Big[f\big\{X\cdot(f^{-1}\varphi)\big\} - f\,{}^{B}\mathrm{Ric}(X,\mathrm{grad}_B f) + \frac{1}{2}nX\cdot\|\mathrm{grad}_B f\|^2\Big]\,{}^{F}\!g(V,W).
\end{aligned}$$

Therefore we obtain (3.8).
(3) From Proposition 2.4(6), we can see that (3.9) is equivalent to (3.6) for $\xi = U, \eta = V$ and $\zeta = W$.

In the other direction, we need to consider the folowing six cases for (3.6).

(i) $\xi = X, \eta = Y, \zeta = Z$. (ii) $\xi = X, \eta = U, \zeta = Z$.
(iii) $\xi = U, \eta = V, \zeta = Z$. (iv) $\xi = X, \eta = Y, \zeta = W$.
(v) $\xi = X, \eta = V, \zeta = W$. (vi) $\xi = U, \eta = V, \zeta = W$.

However, from the above discussions, (3.7), (3.8) and (3.9) are equivalent to (3.6) in the cases of (i), (v) and (vi), respectively. For the cases (ii) and (iv), we can verify that

$$(\nabla_X\mathrm{Ric})(U,Z) = 0 = (\nabla_U\mathrm{Ric})(X,Z)$$

and

$$(\nabla_X\mathrm{Ric})(Y,W) = 0 = (\nabla_Y\mathrm{Ric})(X,W)$$

hold from Proposition 2.4 (2) and (3). Moreover, from Proposition 2.4(4), $(\nabla_U\mathrm{Ric})(Z,V)$ is symmetric in U and V. Therefore, we have

$$(\nabla_U\mathrm{Ric})(V,Z) = (\nabla_U\mathrm{Ric})(Z,V) = (\nabla_V\mathrm{Ric})(Z,U) = (\nabla_V\mathrm{Ric})(U,Z),$$

which implies that (3.6) also holds for the case (iii). □

In particular, if f is a constant function, then (3.7) implies that $(B, {}^{B}g)$ is a manifold of class $\mathcal{B}$, and (3.8) is obviously satisfied. Thus a Riemannian product manifold $B \times F$ is of class $\mathcal{B}$ if and only if $(B, {}^{B}g)$ and $(F, {}^{F}g)$ are manifolds of class $\mathcal{B}$.

In contrast, we have the following.

Theorem 3.5. *If a warped product manifold $B \times_f F$ is of class $\mathcal{B}$ and f is not a constant function, then the fiber space $(F, {}^{F}g)$ is an Einstein manifold.*

Proof. Since a warped product manifold $B \times_f F$ is of class $\mathcal{B}$, the identity (3.8) holds. Then by using the similar argument in the proof of Theorem 3.2, we can show that $(F, {}^{F}g)$ is an Einstein manifold. □

Corollary 3.6. *Let B be an open interval and $(F, {}^{F}g)$ an Einstein manifold with ${}^{F}\mathrm{Ric} = \mu {}^{F}g$ for a constant μ.*

(1) *If $n = \dim F = 3$, then there exists a warped product manifold $B \times_f F$ of class $\mathcal{B}$ which is not an Einstein manifold.*

(2) *If $(F, {}^{F}g)$ is a Ricci flat manifold, then there exists a warped product manifold $B \times_f F$ of class $\mathcal{B}$ which is not an Einstein manifold.*

Proof. Since $(F, {}^{F}g)$ ia an Einstein manifold, it is of class $\mathcal{B}$. Moreover, since $\dim B = 1$, (3.7) is clearly satisfied for any function f. Thus we show the existence of a positive function f satisfying (3.8).

By taking $X = \partial/\partial t$, where $t \in B$, and substituting $\varphi(t) = -f(t)f''(t) - (n-1)f'(t)^2$ in (3.8), we obtain

$$\begin{aligned} f^{-1}(t)f'(t){}^{F}\mathrm{Ric}(U,V) &= \left[f\frac{d}{dt}\{f^{-1}(t)\varphi(t)\} + \frac{1}{2}n\frac{d}{dt}f'(t)^2 \right] {}^{F}g(U,V) \\ &= -f(t)^2\frac{d}{dt}\left[f(t)^{-2}\left\{ f(t)f''(t) + \frac{1}{2}(n-1)f'(t)^2 \right\} \right] \\ &\quad \times {}^{F}g(U,V). \end{aligned}$$

Thus

$$\frac{d}{dt}\{f(t)^{-2}\}{}^{F}\mathrm{Ric}(U,V) = \frac{d}{dt}\Big[f(t)^{-2}\{2f(t)f''(t) + (n-1)f'(t)^2\} \Big] {}^{F}g(U,V).$$

Since ${}^{F}\mathrm{Ric} = \mu {}^{F}g$, we have

$$\frac{d}{dt}\{\mu f(t)^{-2}\} = \frac{d}{dt}\Big[f(t)^{-2}\{2f(t)f''(t) + (n-1)f'(t)^2\} \Big].$$

Integrating the both sides, we obtain

$$\mu = 2f(t)f''(t) + (n-1)f'(t)^2 + cf(t)^2, \tag{3.10}$$

where c is a constant.

(1) According to the method due to [2] and [9], we can reduce the equation (3.10) to the following ordinary deifferential equation for $f(t)^2$.

$$\mu = \frac{d^2}{dt^2}\{f(t)^2\} + cf(t)^2.$$

Thus we have

$$f(t)=\begin{cases}\left[c_1\sin(\sqrt{c}t+c_2)+\dfrac{\mu}{c}\right]^{\frac{1}{2}} & (c>0),\\ \left[\dfrac{\mu}{2}t^2+c_1t+c_2\right]^{\frac{1}{2}} & (c=0),\\ \left[c_1e^{\sqrt{-c}t}+c_2e^{-\sqrt{-c}t}+\dfrac{\mu}{c}\right]^{\frac{1}{2}} & (c<0),\end{cases}\tag{3.11}$$

where c_1 and c_2 are constants. We choose an interval B and constants μ, c_1, c_2 appropriately so that f is positive on B (see Remark 3.7(1)). For each function f, the warped product manifold $B\times_f F$ is of class $\mathcal{B}$. On the other hand, f does not satisfy the condition (2) of Corollary 2.3, thus the warped product manifold $B\times_f F$ is not an Einstein manifold.

(2) Since $\mu=0$, (3.10) is changed into the following form.

$$2\frac{f''(t)}{f(t)}+(n-1)\left\{\frac{f'(t)}{f(t)}\right\}^2+c=0.\tag{3.12}$$

If we define a function w by

$$w(t)=\frac{f'(t)}{f(t)},$$

then w satisfies the following ordinary differential equation.

$$2w'(t)+(n+1)w(t)^2+c=0.$$

By using the quadrature method, we have a solution w, and obtain a solution f to (3.12) as follows.

$$f(t)=\begin{cases}c_1\left[\cos\left(\dfrac{\sqrt{(n+1)c}}{2}t+c_2\right)\right]^{\frac{2}{n+1}} & (c>0),\\ (c_1t+c_2)^{\frac{2}{n+1}} & (c=0),\\ \left[c_1e^{\frac{\sqrt{-(n+1)c}}{2}t}+c_2e^{-\frac{\sqrt{-(n+1)c}}{2}t}\right]^{\frac{2}{n+1}} & (c<0),\end{cases}\tag{3.13}$$

where c_1 and c_2 are constants. We choose an interval B and constants c_1, c_2 appropriately so that f is positive on B (see Remark 3.7(2)). For each function f, the warped product manifold $B\times_f F$ is of class $\mathcal{B}$. On the other hand, f does not satisfy the condition (2) of Corollary 2.3, thus the warped product manifold $B\times_f F$ is not an Einstein manifold. □

Remark 3.7. (1) *For the function f given by* (3.11), *we can choose constants μ, c_1 and c_2 so that f is a nonconstant, positive and smooth function on $B=\mathbf{R}$ in the following way*([9]).

For $c>0$,

$$\mu>0,\ 0<c_1<\frac{\mu}{c}.$$

For $c=0$,

$$\mu>0,\ c_2-\frac{c_1^2}{2\mu}>0.$$

For $c < 0$,

(i) $\mu \leq 0,\ c_1 > 0,\ c_2 = 0,$

(ii) $\mu \leq 0,\ c_1 = 0,\ c_2 > 0,$

(iii) $c_1 > 0,\ c_2 > 0,\ 2\sqrt{c_1 c_1} > -\dfrac{\mu}{c}.$

Thus, for each case, we have a warped product manifold $\mathbf{R} \times_f F$ *of class* $\mathcal{B}$, *which is not an Einstein manifold.*

(2) *For the function* f *given by* (3.13), *if we choose constants* c_1, c_2 *and an interval* B *in the following way, then* f *is a nonconstant, positive and smooth function on* B.

For $c > 0$,

$$c_1 > 0, c_2 = 0, \text{and } B = \left(0, \frac{\pi}{\sqrt{(n+1)c}}\right).$$

For $c = 0$,

$$c_1 > 0, c_2 = 0, \text{and } B = (0, \infty).$$

For $c < 0$,

$$c_1 > 0, c_2 \geq 0, \text{or } c_1 \geq 0, c_2 > 0, \text{and } B = \mathbf{R}.$$

Thus, for each case, we have a warped product manifold $B \times_f F$ *of class* $\mathcal{B}$, *which is not an Einstein manifold.*

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