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ANALYTICAL STUDY ON THE EXISTENCE, UNIQUENESS AND STABILITY OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS INVOLVING THE MODIFIED ATANGANA-BALEANU OPERATOR

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Abstract. The recently constructed Modified Atangana-Baleanu (MAB) fractional derivative, whose kernel is given by the generalised Mittag-Leffler function, is the subject of a class of integro-fractional differential equations that are thoroughly examined in this paper. Classical or even conventional fractional derivatives are unable to fully capture nonlocal dynamics and memory effects, but this derivative operator offers a potent tool for doing so. Complex physical, biological, and engineering systems exhibiting anomalous diffusion or hereditary behaviour can be best described by the MAB operator's versatility. We rigorously prove the existence and uniqueness of solutions to the suggested integro-fractional problem inside a suitable Banach space framework by utilising fixed-point techniques like Schauder's fixed-point theorem and the Banach contraction principle. A tangible example is provided to further bolster the analytical findings, proving the applicability and accuracy of the theoretical framework. Furthermore, the concepts of Hyers-Ulam stability and generalised Hyers-Ulam stability are examined in the stability analysis of the obtained solutions. The results of this work add to the body of knowledge on fractional differential equations and aid in the continuous creation of new fractional operators that better represent nonlocal occurrences. The proven results also pave the way for new research directions, such as numerical simulation, control, and optimisation of systems controlled by modified kernel-based fractional integro-differential equations.

1. Introduction

An important and quickly developing field of mathematics is the study of derivatives and integrals of arbitrary, non-integer order. By extending the traditional ideas of differentiation and integration, this field known as fractional calculus offers a potent foundation for simulating processes that display memory, nonlocality, and hereditary behaviour in a variety of biological, engineering, and physical systems. Because of the impressive results that were obtained when techniques from this branch were applied to real-life models and events, it has been gradually gaining respect among researchers

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from a variety of fields [12]. The range of fractional operators in this theory is one of its attractive features. Because of this large selection, investigators can choose the best operator to produce more precise or useful data. In fractional calculus, the Riemann-Liouville is one of the oldest and most studied operator. It uses convolution type integral operators to define fractional derivatives and integrals [23]. However, the Caputo derivative, which is closely related to the Riemann-Liouville concept, is frequently used to describe specific physical processes. Systems with memory and hereditary characteristics are well described by these operators. However, a typical drawback of these operators is that their kernels are single, which may limit how useful they are in some situations.

Even though they have been utilised extensively, the Riemann-Liouville and Caputo operators are occasionally insufficient for modelling intricate theoretical and physical problems involving powerful memory effects. There are also more fractional differentiation types in literature that resemble the Riemann-Liouville and Caputo types. The well-known Hadamard and Riemann-Liouville fractional derivatives are reduced to a single form by the authors [15], who present a novel fractional derivative. Two representations of the generalised derivative in question are also obtained by the author. To get around this problem, Caputo and Fabrizio [8] created a fractional derivative with a nonsingular kernel. More recently, Atangana and Baleanu developed a novel derivative defined using the generalised Mittag-Leffler function, sometimes known as the (ML-function) [4]. The Riemann–Liouville and Caputo variants of this operator, called the Atangana–Baleanu derivative (AB derivative) or ABC derivative, respectively, are called ABR and ABC derivatives. There are still numerous initialisation issues with Atangana-Baleanu integral operators. The Mittag-Leffler function was used as a crucial non-singular kernel in the Modified Atangana-Baleanu operators, which were offered as a solution to the problems with classical fractional operators. This ensured improved mathematical stability and physical consistency [13]. With the help of these operators, many problems are fixed. Numerous generalisations of the M-L function are possible by increasing the number of parameters.

Numerous works have developed the theoretical underpinnings and applications of the AB derivative, which has garnered significant attention [1]. In addition to natural models governed by exponential and Mittag-Leffler laws [5], it has been applied to epidemiological models like rubella [16] and H1N1 influenza [3], electrical circuits [11], chemical pollution and groundwater contamination [2], ordinary differential equations [10], heat conduction [17], oxygen diffusion [19], chaotic and hyperchaotic systems [21], smoking dynamics [32], and interspecies interactions [22]. Furthermore, models based on this operators perform better than integer order derivatives; readers are encouraged to examine, for example [7]. Atangana and Gómez Aguilar in [6] introduced a numerical approximation for fractional differentiation derived from the definition of Riemann-Liouville, broadening its application beyond the power law kernel to the exponential decay and generalized Mittag-Leffler kernels, resulting in the Atangana-Baleanu model. The authors of [18] examined a fractional nonlinear SIR epidemic model that included the Atangana-Baleanu derivative in the Caputo sense. Using fixed point theory, they proved the system's existence, uniqueness, and Hyers-Ulam stability. The SEI fractional model of COVID-19 in India was developed and evaluated by the author using the Adomian decomposition approach [14]. In [30], the authors used actual data from March 2020 to propose a fractional-order Bats-Hosts-Reservoir-People model for COVID-19. Alongside the analysis of equilibrium state stability, the existence and stability of solutions were validated. Numerical simulations and sensitivity analysis showed how biological characteristics and control measures affect the spread of disease. Memory effects and nonlocal interactions in the system were successfully captured in [28] when a mathematical model explaining the transmission kinetics of HIV/AIDS within an aware society was developed and analysed using a fractional differential operator with a Mittag-Leffler kernel. The fixed point technique is applied to determine the model's uniqueness and existence criteria. Author employed a new method called the iterative Laplace transform approach to get the approximate solution of the Atagana-Baleanu operator-based mathematical model of HIV/AIDS. In a recent paper [29], the author thoroughly examined the nonlocal Cauchy problem using the Atangana-Baleanu fractional derivative, offering a framework for examining the existence, uniqueness, and stability of its solutions. A thorough illustrative case was provided in [31], where the authors established the existence and uniqueness of solutions using Schauder's fixed-point and Arzelà Ascoli theorems. The Atangana-Baleanu operator was also used in the study to capture nonlocal effects and changed wave dynamics in the time-fractional Burgers-Fisher equation. In addition, fractional Friedmann-Robertson-Walker (FRW) cosmological models with accelerating expansion under phantom, Chaplygin gas, and tachyon fields were studied, and the controllability of nonlinear fractional dynamical systems with a Mittag-Leffler kernel was examined using the Laplace transform and fixed-point techniques. The authors in [33] demonstrated the effective use of the finite difference strategy for fractional equations that are nonlinear and of different orders by combining it with the Atangana-Baleanu operator.

When analysing fractional differential equations, qualitative characteristics like stability are just as important as the existence of solutions and uniqueness also. The framework of AB fractional integrals has specifically established Gronwall-type inequalities [27], which offer crucial resources for researching Ulam-Hyers stability and associated characteristics. Using the AB framework, Ravichandran et al. in [24] proposed a novel framework for coupled classes of fractional integro-differential equations with non densely defined linear operators in Banach spaces. By the use of theory of measures of non-compactness, the author investigated and prove that the solution exists to the given problem. The authors examine a class of Caputo fractional derivative-based Volterra-Fredholm integro-differential equations in [20]. They use Schaefer's fixed point theorem to demonstrate the existence of at least one solution and Banach's contraction principle to establish the existence and uniqueness of the solution. They also address the system's Ulam stability and offer an example to support their analytical findings. The authors of [25] established controllability results for fractional neutral integro-differential equations with nonlocal conditions in Banach spaces by using the measure of non-compactness, Mönch's fixed-point theorem, and other fractional calculus approaches. In systems with memory effects and nonlocal interactions, this method offers a rigorous framework for examining the existence, uniqueness, and control aspects of solutions, demonstrating the flexibility of fractional operators in handling intricate dynamical behaviours. In their work, they examined such equations in Banach spaces and used fixed point theorems under appropriate Lipschitz-type conditions to establish existence and uniqueness results. They also examined the Ulam-Hyers stability of solutions. They provided a tangible example to demonstrate the theoretical conclusions and show how the fractional order affects solution behaviour.

Motivated by these developments, this work focuses on fractional integro-differential equations by applying Modified Atangana-Baleanu derivatives. Since, recently constructed Modified Atangana-Baleanu (MAB) fractional derivative, whose kernel is given by the generalised Mittag-Leffler function, is the subject of a class of integro-fractional differential equations. Classical or even conventional fractional derivatives are unable to fully capture non-local dynamics and memory effects, but this derivative operator offers a potent tool for doing so. We demonstrate that the solution exists, is unique, and is stable. Also we provide an appropriate examples to explain the theoretical results for the following FDE,

$$\begin{aligned} {}^{MABD}D^{\sigma}y(\tau) &= g\Big(\tau, y(\tau), \int_0^{\tau} f_1(\tau,\zeta,y(\zeta))d\zeta, \int_0^{w} f_2(\tau,\zeta,y(\zeta))d\zeta\Big), \\ y(0) &= y_0. \end{aligned} \tag{1}$$

with $\tau \in T = [0,1],\ 0 < \sigma < 1,\ 0 \leq w \leq 1$ where ${}^{MABD}D^{\sigma}$ is Modified Atangana-Baleanu derivative operator, $y(\tau)$,
${}^{MABD}D^{\sigma}, g \in C(T),\ g(0, y(0), 0, \int_0^w f_2(0,\zeta,y(\zeta))d\zeta) = 0,\ g : [0,1]\times\mathcal{R}\times\mathcal{R}\times\mathcal{R} \to \mathcal{R}$
and
$f_1,\ f_2 : [0,1]\times[0,1]\times\mathcal{R} \to \mathcal{R}$.

Consider $I_1 y(\tau) = \int_0^{\tau} f_1\big(\tau,\zeta,y(\zeta)\big)\,d\zeta$ and $I_2 y(\tau) = \int_0^{w} f_2\big(\tau,\zeta,y(\zeta)\big)\,d\zeta$. Then (1) becomes

$$\begin{aligned} {}^{MABD}D^{\sigma}y(\tau) &= g\big(\tau,\, y(\tau),\, I_1y(\tau), I_2y(\tau)\big), \\ y(0) &= y_0. \end{aligned} \tag{2}$$

2. Background Information

A few fundamental definitions of fractional calculus are included in this section.

Definition 2.1. [9] Let $0 < \sigma < 1$ be a fractional order, the Modified differential operator given by Atangana-Baleanu in caputo sense is defined as

$${}^{MABD}D^{\sigma}\mathcal{F}(\tau) = \frac{\rho(\sigma)}{1-\sigma}\Big[\mathcal{F}(\tau) - E_{\sigma}(-\mu_{\sigma}\tau^{\sigma})\mathcal{F}(0) - \mu_{\sigma}\int_0^{\tau}(\tau-x)^{(\sigma-1)}E_{\sigma,\sigma}(-\mu_{\sigma}(\tau-x)^{\sigma})\mathcal{F}(x)\Big]$$

where $\mu_{\sigma} = \dfrac{\sigma}{1-\sigma}$, the normalizing factor $\rho(\sigma)$ such that $\rho(0) = \rho(1) = 1$.

Definition 2.2. [26] For $\mathcal{F} \in L^1(0,\infty),\ \sigma \in (0,1)$, the Modified Atangana-Baleanu integral operator is given by

$${}^{MABD}I^{\sigma}\mathcal{F}(\tau) = \frac{1-\sigma}{\rho(\sigma)}\Big(\mathcal{F}(\tau) - \mathcal{F}(0)\Big) + \frac{\mu_{\sigma}}{\Gamma(\sigma)}\int_0^{\tau}\Big(\frac{\mathcal{F}(x)-\mathcal{F}(0)}{(\tau-x)^{1-\sigma}}\Big)dx$$

where $\mu_{\sigma} = \dfrac{\sigma}{1-\sigma}$.

Definition 2.3. [29](Equicontinuous)$\mathcal{F}_1 \subset C(Y)$ is equicontinuous if for every $\epsilon > 0$, $\exists\ \delta > 0$ (which depends only on ϵ) such that for $a, b \in Y, d(a-b) < \delta$ implies $|f(a) - f(b)| < \epsilon, \forall f \in \mathcal{F}_1$, where d is metric on Y.

Theorem 2.4. [29](Arzela-Ascoli Theorem) Consider the compact metric space (Y, d). Let $\mathcal{F}$ be a subset of $C(Y)$. $\mathcal{F}$ is relatively compact if and only if $\mathcal{F}$ is uniformely bounded and equicontinuous.

Theorem 2.5. *(Schauder's Fixed Point Theorem) Let (E, d) be a complete metric space, let U be a closed convex subset of E, and Let $A : U \to U$ be a mapping such that the set $\{Au : u \in U\}$ is relativelty compact in E. Then A has at least one fixed point.*

Theorem 2.6. *(Banach's Fixed Point Theorem)* Let $(V \neq \phi, d)$ be a complete metric space with $0 \le \sigma \le 1$, and the function $T : V \to V$ holds the inequality

$$d(Tv_1, Tv_2) \le \sigma\ d(v_1, v_2), \forall\ v_1, v_2 \in V$$

Then, v^* is uniquely determined fixed point of T. Further, for any $v_0 \in V$, the sequence $(T^j v_0)_{j=1}^{\infty}$ converges to this fixed point v^*.

3. Important Results:

First, we examine the existence and uniqueness of problem (2) using a fixed point technique, which is a Banach contraction principle with a recently defined Modified AB-derivative. To obtained the required results, we have to consider the following assumptions:

A_1 There exist positive constants $\theta_1, \theta_2, \theta_3, \theta_{I_1}$, θ_{I_2} such that for all $\mathrm{x}_1, \mathrm{y}_1, \mathrm{z}_1, \mathrm{x}_2, \mathrm{y}_2, \mathrm{z}_2 \in C[[0,1], \mathcal{R}]$,

$$\|g(\tau, \mathrm{x}_1, \mathrm{y}_1, \mathrm{z}_1) - g(\tau, \mathrm{x}_2, \mathrm{y}_2, \mathrm{z}_2)\| \le \theta_1\|\mathrm{x}_1 - \mathrm{x}_2\| + \theta_2\|\mathrm{y}_1 - \mathrm{y}_2\| + \theta_3\|\mathrm{z}_1 - \mathrm{z}_2\|.$$

A_2 $\forall \mathrm{y}_1, \mathrm{y}_2 \in C[[0,1], \mathcal{R}]$, $\exists \theta_{I_1} > 0$,

$$\|I_1(\tau, s, \mathrm{y}_1) - I_1(\tau, s, \mathrm{y}_2)\| \le \theta_{I_1}\|\mathrm{y}_1 - \mathrm{y}_2\|.$$

A_3 $\forall \mathrm{y}_1, \mathrm{y}_2 \in C[[0,1], \mathcal{R}]$, $\exists \theta_{I_2} > 0$,

$$\|I_2(\tau, s, \mathrm{y}_1) - I_2(\tau, s, \mathrm{y}_2)\| \le \theta_{I_2}\|\mathrm{y}_1 - \mathrm{y}_2\|.$$

A_4 The functions f_1, f_2 are continuous functions and there exists a positive constant $\eta_0,\ \eta_1,\ \eta_2,\ \eta_3$ such that
$|f_1(\tau, \zeta, y)| \le \eta_0 + \eta_1\|y\|$.
$|f_2(\tau, \zeta, y)| \le \eta_2 + \eta_3\|y\|$.

A_5 Let $0 < \lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3 < 1$ such that

$$\|g(\tau, y(\tau), I_1y(\tau), I_2y(\tau))\| \le \lambda_0 + \lambda_1\|y\| + \lambda_2\|I_1y(\tau)\| + \lambda_3\|I_2y(\tau)\|.$$

The following theorem proves that the solution of given FIDE (1) is equivalent to integral equation defined below.

Theorem 3.1. Let $y(\tau) \in C[0,1]$, $0 < \sigma < 1$, ${}^{MABD}D^{\sigma}$ is Modified Atangana-Baleanu derivative operator, let a continuous function $g : [0,1] \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ along with
$g(0, y(0), 0, \int_0^w f_2(0, \zeta, y(\zeta))d\zeta) = 0$, then the solution of

$$ {}^{MABD}D^{\sigma}y(\tau) = g\bigg(\tau, y(\tau), I_1y(\tau), I_2y(\tau)\bigg)$$

$$y(0) = y_0$$

is equivalent to

$$y(\tau) = y_0 + \frac{1-\sigma}{\rho(\sigma)} g(\tau, y(\tau), I_1 y(\tau), I_2 y(\tau)) + \frac{\mu_\sigma}{\Gamma(\sigma)} \int_0^\tau (\tau - s)^{\sigma-1} g(s, y(s), I_1 y(s), I_2 y(s)) ds$$

The following theorem gives the existence of solution of given FIDE (2).

THEOREM 3.2. If $\left[\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right](\lambda_1 + \lambda_2\eta_1 + \lambda_3\eta_3) < 1$, then the given fractional differential equation (2) has atleast one solution, with

$$\frac{\|y_0\| + \left[\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right](\lambda_0 + \lambda_2\eta_0 + \lambda_3\eta_2)}{1 - \left[\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right](\lambda_1 + \lambda_2\eta_1 + \lambda_3\eta_3)} \leq r.$$

Proof. To solve the fractional differential equation (2), we define the fixed point operator $P : C[[0,1], \mathcal{R}]$ as

(3)

$$Py(\tau) = y_0 + \frac{1-\sigma}{\rho(\sigma)} g\Big(\tau, y(\tau), I_1 y(\tau), I_2 y(\tau)\Big) + \frac{\mu_\sigma}{\Gamma(\sigma)} \int_0^\tau (\tau - s)^{\sigma-1} g\Big(s, y(s), I_1 y(s), I_2 y(s)\Big)$$

Now consider $\mathcal{N}_r(0) = \{y \in C([0,1], \mathcal{R}), \|y\| \leq r\}$ be an open ball with radius r and centered at origin.
Clearly $\mathcal{N}_r(0)$ is convex, closed and bounded.
We prove the above theorem in the several steps.
Step1: Let us first prove that P is continuous operator.
Let $y_n,\ y \in [0,1]$

$$\begin{aligned}
\|Py_n(\tau) - Py(\tau)\| = \Big\| & y_0 + \frac{1-\sigma}{\rho(\sigma)} g\Big(\tau, y_n(\tau), I_1 y_n(\tau), I_2 y_n(\tau)\Big) \\
& + \frac{\mu_\sigma}{\Gamma(\sigma)} \int_0^\tau (\tau - s)^{\sigma-1} g\Big(s, y_n(s), I_1 y_n(s), I_2 y_n(s)\Big) \\
& - y_0 - \frac{1-\sigma}{\rho(\sigma)} g\Big(\tau, y(\tau), I_1 y(\tau), I_2 y(\tau)\Big) \\
& - \frac{\mu_\sigma}{\Gamma(\sigma)} \int_0^\tau (\tau - s)^{\sigma-1} g\Big(s, y(s), I_1 y(s), I_2 y(s)\Big)\Big\|
\end{aligned}$$

$$\begin{aligned}
\|Py_n(\tau) - Py(\tau)\| \leq & \frac{1-\sigma}{\rho(\sigma)} \Big\| g\Big(\tau, y_n(\tau), I_1 y_n(\tau), I_2 y_n(\tau)\Big) - g\Big(\tau, y(\tau), I_1 y(\tau), I_2 y(\tau)\Big)\Big\| \\
& + \frac{\mu_\sigma}{\Gamma(\sigma)} \int_0^\tau (\tau - s)^{\sigma-1} \Big\| g\Big(s, y_n(s), I_1 y_n(s), I_2 y_n(s)\Big) - g\Big(s, y(s), I_1 y(s), I_2 y(s)\Big)\Big\| ds
\end{aligned}$$

$$\|Py_n(\tau) - Py(\tau)\| \leq \frac{1-\sigma}{\rho(\sigma)} \Big\| g\Big(\tau, y_n(\tau), I_1 y_n(\tau), I_2 y_n(\tau)\Big) - g\Big(\tau, y(\tau), I_1 y(\tau), I_2 y(\tau)\Big)\Big\|$$

$$+\frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\left\|g\Big(s,y_n(s),I_1y_n(s),I_2y_n(s)\Big)-g\Big(s,y(s),I_1y(s),I_2y(s)\Big)\right\|$$

as $y_n \to y$, $\|Py_n(\tau)-Py(\tau)\| \to 0$. $\therefore$ The operator P is continuous.
Step 2: $P(\mathcal{N}_r(0))$ is equicontinuous.
Let $\tau_1,\ \tau_2 \in [0,1]$ such that $0 \le \tau_1 \le \tau_2 \le 1$.
Consider

$$\begin{aligned}\|Py(\tau_2)-Py(\tau_1)\| = \Bigg\| & y_0+\frac{1-\sigma}{\rho(\sigma)}g\Big(\tau_2,y(\tau_2),I_1y(\tau_2),I_2y(\tau_2)\Big)\\ &+\frac{\mu_\sigma}{\Gamma(\sigma)}\int_0^{\tau_2}(\tau_2-s)^{\sigma-1}g\Big(s,y(s),I_1y(s),I_2y(s)\Big)-y_0\\ &-\frac{1-\sigma}{\rho(\sigma)}g\Big(\tau_1,y(\tau_1),I_1y(\tau_1),I_2y(\tau_1)\Big)\\ &-\frac{\mu_\sigma}{\Gamma(\sigma)}\int_0^{\tau_1}(\tau_1-s)^{\sigma-1}g\Big(s,y(s),I_1y(s),I_2y(s)\Big)\Bigg\|\end{aligned}$$

$$\begin{aligned}\|Py(\tau_2)-Py(\tau_1)\| \le & \frac{1-\sigma}{\rho(\sigma)}\left\|g\Big(\tau_2,y(\tau_2),I_1y(\tau_2),I_2y(\tau_2)\Big)-g\Big(\tau_1,y(\tau_1),I_1y(\tau_1),I_2y(\tau_1)\Big)\right\|\\ &-\frac{\mu_\sigma}{\sigma\Gamma(\sigma)}(\tau_2^\sigma-\tau_1^\sigma)\left\|g\Big(s,y(s),I_1y(s),I_2y(s)\Big)\right\|\end{aligned}$$

as $\tau_1 \to \tau_2$, $\|Py(\tau_2)-Py(\tau_1)\| \to 0$. Hence the result.
Step 3: The operator P is uniformely bounded.
Claim: $P(\mathcal{N}_r(0)) \subset \mathcal{N}_r(0)$.
Let us consider

$$\begin{aligned}\|Py(\tau)\| &= \left\|y_0+\frac{1-\sigma}{\rho(\sigma)}g\Big(\tau,y(\tau),I_1y(\tau),I_2y(\tau)\Big)+\frac{\mu_\sigma}{\Gamma(\sigma)}\int_0^{\tau}(\tau-s)^{\sigma-1}g\Big(s,y(s),I_1y(s),I_2y(s)\Big)\right\|\\ &\le \|y_0\|+\frac{1-\sigma}{\rho(\sigma)}\left\|g\Big(\tau,y(\tau),I_1y(\tau),I_2y(\tau)\Big)\right\|+\frac{\mu_\sigma}{\Gamma(\sigma)}\int_0^{\tau}(\tau-s)^{\sigma-1}ds\left\|g\Big(s,y(s),I_1y(s),I_2y(s)\Big)\right\|\\ &\le \|y_0\|+\frac{1-\sigma}{\rho(\sigma)}\left\|g\Big(\tau,y(\tau),I_1y(\tau),I_2y(\tau)\Big)\right\|+\frac{\mu_\sigma}{\Gamma(\sigma)}\frac{\tau^\sigma}{\sigma}\left\|g\Big(s,y(s),I_1y(s),I_2y(s)\Big)\right\|\\ &\le \|y_0\|+\left[\frac{1-\sigma}{\rho(\sigma)}+\frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right]\left\|g\Big(\tau,y(\tau),I_1y(\tau),I_2y(\tau)\Big)\right\|\\ &\le \|y_0\|+\left[\frac{1-\sigma}{\rho(\sigma)}+\frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right]\Big(\lambda_0+\lambda_1\|y_1\|+\lambda_2\|I_1y(\tau)\|+\lambda_3\|I_2y(\tau)\|\Big)\\ &\le \|y_0\|+\left[\frac{1-\sigma}{\rho(\sigma)}+\frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right]\Big((\lambda_0+\lambda_2\eta_0+\lambda_3\eta_0)+(\lambda_1+\lambda_2\eta_1+\lambda_3\eta_1)\|y\|\Big)\end{aligned}$$

$\because$, $\|y\| \leq r$

$$\|Py(\tau)\| \leq \|y_0\| + \left[\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right]\Big((\lambda_0 + \lambda_2\eta_0 + \lambda_3\eta_2) + (\lambda_1 + \lambda_2\eta_1 + \lambda_3\eta_3)r\Big) \leq r$$

Hence $P(\mathcal{N}_r(0)) \subset \mathcal{N}_r(0)$.
$P(\mathcal{N}_r(0))$ is relatively compact in $\mathcal{R}$, by Arzela-Ascoli theorem . Hence it is proved that there exist a solution to the given fractional differential equation (2), by applying Schauder fixed point theorem. □

Now, we prove the uniqueness of solution of the given FIDE (1). Also, we prove our solution is converges to a fixed point of operator P.

THEOREM 3.3. Assume that the assumptions $A_1 - A_3$ holds. If $\left(\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right)\Big(\theta_1 + \theta_2\theta_{I_1} + \theta_3\theta_{I_2}\Big) < 1$, then the given fractional differential equation (2) has a unique solution in $C[0,1]$. Furthermore, the iterative sequence $\{y_n(\tau)\}_{n=0}^{\infty}$ generated by $y_{n+1}(\tau) = P(y_n(\tau))$, with the initial approximation $y_0(\tau) = y_0$ is convergent to the fixed point of the operator P.

Proof. For $y_1,\ y_2 \in C[0,1]$,

$$\begin{aligned}\|Py_1(\tau) - Py_2(\tau)\| \leq \Bigg\| & y_0 + \frac{1-\sigma}{\rho(\sigma)} g(\tau, y_1(\tau), I_1y_1(\tau), I_2y_1(\tau)) \\ & + \frac{\mu_\sigma}{\Gamma(\sigma)} \int_0^\tau (\tau - s)^{\sigma-1} g(s, y_1(s), I_1y_1(s), I_2y_1(s)) ds \\ & - y_0 - \frac{1-\sigma}{\rho(\sigma)} g(\tau, y_2(\tau), I_1y_2(\tau), I_2y_2(\tau)) \\ & - \frac{\mu_\sigma}{\Gamma(\sigma)} \int_0^\tau (\tau - s)^{\sigma-1} g(s, y_2(s), I_1y_2(s), I_2y_2(s)) ds \Bigg\|\end{aligned}$$

$$\begin{aligned}\|Py_1(\tau) - Py_2(\tau)\| \leq & \frac{1-\sigma}{\rho(\sigma)} \Big\| g(\tau, y_1(\tau), I_1y_1(\tau), I_2y_1(\tau)) - g(\tau, y_2(\tau), I_1y_2(\tau), I_2y_2(\tau)) \Big\| \\ & + \frac{\mu_\sigma}{\Gamma(\sigma)} \int_0^\tau (\tau - s)^{\sigma-1} \Big\| g(s, y_1(s), I_1y_1(s), I_2y_1(s)) \\ & \qquad - g(s, y_2(s), I_1y_2(s), I_2y_2(s)) \Big\| ds\end{aligned}$$

$$\begin{aligned}\|Py_1(\tau) - Py_2(\tau)\| \leq & \frac{1-\sigma}{\rho(\sigma)} \Big(\theta_1\|y_1 - y_2\| + \theta_2\|I_1y_1(\tau) - I_1y_2(\tau)\| + \theta_3\|I_2y_1(\tau) - I_2y_2(\tau)\|\Big) \\ & + \frac{\mu_\sigma t^\sigma}{\Gamma(\sigma)\sigma} \Big(\theta_1\|y_1 - y_2\| + \theta_2\|I_1y_1(\tau) - I_1y_2(\tau)\| + \theta_3\|I_2y_1(\tau) - I_2y_2(\tau)\|\Big)\end{aligned}$$

$$\|Py_1(\tau) - Py_2(\tau)\| \leq \left(\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right) \Big(\theta_1\|y_1 - y_2\| + \theta_2\theta_{I_1}\|y_1 - y_2\| + \theta_3\theta_{I_2}\|y_1 - y_2\|\Big)$$

$$\|Py_1(\tau) - Py_2(\tau)\| \leq \left(\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right) (\theta_1 + \theta_2\theta_{I_1} + \theta_3\theta_{I_2})\|y_1 - y_2\|$$

$$\because \quad \left(\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right)(\theta_1 + \theta_2\theta_{I_1} + \theta_3\theta_{I_2}) < 1$$

$$\|Py_1(\tau) - Py_2(\tau)\| \le \|y_1 - y_2\|$$

$\therefore$ P is contraction mapping on $[0,1]$. Hence (2) has unique solution, by Banach contraction principle.
Next, for convergence we consider the sequence y_m in $C([0,1])$:

$$\begin{aligned}
\|y_n(\tau) - y_m(\tau)\| &= \|P(y_{n-1}(\tau)) - P(y_{m-1}(\tau))\| \\
&= \bigg\| y_0 + \frac{1-\sigma}{\rho(\sigma)} g\big(\tau, y_{n-1}(\tau), I_1 y_{n-1}(\tau), I_2 y_{n-1}(\tau)\big) \\
&\quad + \frac{\mu_\sigma}{\Gamma(\sigma)} \int_0^\tau (\tau - s)^{\sigma-1} g\big(s, y_{n-1}(s), I_1 y_{n-1}(s), I_2 y_{n-1}(s)\big) ds \\
&\quad - y_0 - \frac{1-\sigma}{\rho(\sigma)} g\big(\tau, y_{m-1}(\tau), I_1 y_{m-1}(\tau), I_2 y_{m-1}(\tau)\big) \\
&\quad - \frac{\mu_\sigma}{\Gamma(\sigma)} \int_0^\tau (\tau - s)^{\sigma-1} g\big(s, y_{m-1}(s), I_1 y_{m-1}(s), I_2 y_{m-1}(s)\big) ds \bigg\| \\
&\le \frac{1-\sigma}{\rho(\sigma)} \Big\| g\big(\tau, y_{n-1}(\tau), I_1 y_{n-1}(\tau), I_2 y_{n-1}(\tau)\big) \\
&\qquad\qquad - g\big(\tau, y_{m-1}(\tau), I_1 y_{m-1}(\tau), I_2 y_{m-1}(\tau)\big) \Big\| \\
&\quad + \frac{\mu_\sigma}{\Gamma(\sigma)} \int_0^\tau (\tau - s)^{\sigma-1} \Big\| g\big(s, y_{n-1}(s), I_1 y_{n-1}(s), I_2 y_{n-1}(s)\big) \\
&\qquad\qquad - g\big(s, y_{m-1}(s), I_1 y_{m-1}(s), I_2 y_{m-1}(s)\big) \Big\| ds \\
&\le \left(\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma \tau^\sigma}{\sigma\Gamma(\sigma)}\right) \Big(\theta_1 \|y_{n-1}(\tau) - y_{m-1}(\tau)\| + \theta_2 \|I_1 y_{n-1}(\tau) - I_1 y_{m-1}(\tau)\| \\
&\qquad\qquad + \theta_3 \|I_2 y_{n-1}(\tau) - I_2 y_{m-1}(\tau)\| \Big) \\
&\le \left(\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma \tau^\sigma}{\sigma\Gamma(\sigma)}\right) (\theta_1 + \theta_2\theta_{I_1} + \theta_3\theta_{I_2}) \|y_{n-1}(\tau) - y_{m-1}(\tau)\|
\end{aligned}$$

Let

$$\lambda = \left(\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma \tau^\sigma}{\sigma\Gamma(\sigma)}\right)(\theta_1 + \theta_2\theta_{I_1} + \theta_3\theta_{I_2}) \quad \text{and} \quad \|y_n(\tau) - y_m(\tau)\| \le \lambda \|y_{n-1}(\tau) - y_{m-1}(\tau)\|.$$

Let $n = m+1$. Then

$$\|y_{m+1}(\tau) - y_m(\tau)\| \le \lambda \|y_m(\tau) - y_{m-1}(\tau)\| \le \lambda^2 \|y_{m-1}(\tau) - y_{m-2}(\tau)\| \le \cdots \le \lambda^m \|y_1(\tau) - y_0(\tau)\|$$

For $p > q$, by triangle inequality and sum of geometric series, we obtain

$$\begin{aligned}
\|y_n(\tau) - y_m(\tau)\| &\le \|y_{m+1}(\tau) - y_m(\tau)\| + \|y_{m+2}(\tau) - y_{m+1}(\tau)\| + \cdots + \|y_n(\tau) - y_{n-1}(\tau)\| \\
&\le [\lambda^m + \lambda^{m+1} + \cdots + \lambda^{n-1}] \|y_1(\tau) - y_0(\tau)\| \\
&\le \lambda^m [1 + \lambda + \lambda^2 + \cdots + \lambda^{n-m-1}] \|y_1 - y_0\|
\end{aligned}$$

$$\leq \lambda^m \left[\frac{1-\lambda^{n-m}}{1-\lambda}\right] \|y_1 - y_0\|$$

since $0 < \lambda < 1$, so $1 - \lambda^{n-m} < 1$

$$\|y_n(\tau) - y_m(\tau)\| \leq \left[\frac{\lambda^m}{1-\lambda}\right] \|y_1 - y_0\|$$

But $\|y_1 - y_0\| < \infty$, so as $m \to \infty$, then $\|y_n(\tau) - y_m(\tau)\| \to 0$. Hence we can conclude that y_n is cauchy sequence in $C([0,1]))$. Hence convergent. □

4. Stability Results

In [29], The Gronwall inequality was stated as follows within the context of the Riemann-Liouville fractional integral.

THEOREM 4.1. Let $\sigma > 0$, $a(\tau) \geq 0$ is non-decreasing and locally-integrable on $a_1 < \tau < b_1 < \infty$, $b(\tau) \leq M$ and $z(\tau) \geq 0$ is locally-integrable on $[a_1, b_1)$ with

$$z(\tau) \leq a(\tau) + b(\tau)(I^\sigma z)(\tau), \text{ on } [a_1, b_1). \tag{4}$$

Then,

$$z(\tau) \leq a(\tau) E_\sigma(b(\tau)(\tau - a_1)^\sigma), \ \forall \tau \in [a_1, b_1). \tag{5}$$

REMARK 4.2. A particular case of the Gronwall inequality mentioned in [31] is the Gronwall inequality in Theorem (4.1). The most generic form of the inequality (5) is obtained if $a(\tau)$ in Theorem (4.1) is not necessarily nondecreasing.

$$z(\tau) \leq a(\tau) + \int_0^\tau \sum_{m=1}^{\infty} \left[\frac{(b(\tau))^m}{\Gamma(m\sigma)} (\tau - s)^{m\sigma - 1}\right] ds, \quad a_1 \leq \tau \leq b_1. \tag{6}$$

Using the Modified Atangana Baleanu integral operator, we will now discuss about the Gronwall inequality.

THEOREM 4.3. Let $\sigma > 0$. Suppose that $\dfrac{i(\tau)\rho(\sigma)}{\rho(\sigma) - (1-\sigma)h(\tau)} \geq 0$ is nondecreasing and locally integrable on $[a_1, b_1)$, and that $\dfrac{\mu_\sigma \rho(\sigma) h(\tau)}{\Gamma(\sigma)(\rho(\sigma) - (1-\sigma)h(\tau))} \geq 0$ is bounded on $[a_1, b_1)$. Furthermore, let $z(\tau)$ is nonnegative and locally integrable on $[a_1, b_1)$ with

$$z(\tau) \leq i(\tau) + h(\tau)(^{MABD}I^\sigma z)(\tau), \text{ on } [a_1, b_1). \tag{7}$$

Then,

$$z(\tau) \leq \frac{i(\tau)\rho(\sigma)}{\rho(\sigma) - (1-\sigma)h(\tau)} E_\sigma\left(\frac{\mu_\sigma \rho(\sigma) h(\tau)(\tau - a_1)^\sigma}{\Gamma(\sigma)(\rho(\sigma) - (1-\sigma)h(\tau))}\right), \ \forall \tau \in [a_1, b_1). \tag{8}$$

Proof. By using Modified Atangana-Baleanu integral definition, we get

$$z(\tau) \leq i(\tau) + h(\tau)\left[\frac{(1-\sigma)z(\tau)}{\rho(\sigma)} + \frac{\mu_\sigma}{\Gamma(\sigma)} \int_0^s (s-\tau)^{\sigma-1} z(\tau) d\tau\right],$$

$$z(\tau) \leq i(\tau) + h(\tau)\frac{(1-\sigma)z(t)}{\rho(\sigma)} + h(\tau)\frac{\mu_\sigma}{\Gamma(\sigma)} I^\sigma z(\tau).$$

After, simplifying we have

$$z(\tau) \leq \frac{i(\tau)\rho(\sigma)}{\rho(\sigma)-(1-\sigma)h(\tau)} + \frac{\mu_\sigma}{\Gamma(\sigma)}\left(\frac{\rho(\sigma)h(\tau)}{\rho(\sigma)-(1-\sigma)h(\tau)}\right) I^\sigma z(\tau).$$

Now, take $a(\tau) = \dfrac{i(\tau)\rho(\sigma)}{\rho(\sigma)-(1-\sigma)h(\tau)}$, $b(\tau) = \dfrac{\mu_\sigma}{\Gamma(\sigma)}\left(\dfrac{\rho(\sigma)h(\tau)}{\rho(\sigma)-(1-\sigma)h(\tau)}\right)$

By using Theorem (4.1), we get

$$z(\tau) \leq \frac{i(\tau)\rho(\sigma)}{\rho(\sigma)-(1-\sigma)h(\tau)} E_\sigma\left(\frac{\mu_\sigma}{\Gamma(\sigma)} \frac{\rho(\sigma)h(\tau)}{(\rho(\sigma)-(1-\sigma)h(\tau))}\right).$$

□

Remark 4.4. According to Remark (4.2), the inequality (5) will have $h(\tau)$ in the generic form (8) with $a(\tau) = \dfrac{i(\tau)\rho(\sigma)}{\rho(\sigma)-(1-\sigma)h(\tau)}$, $b(\tau) = \dfrac{\mu_\sigma}{\Gamma(\sigma)}\left(\dfrac{\rho(\sigma)h(\tau)}{\rho(\sigma)-(1-\sigma)h(\tau)}\right)$, if $\sigma > 0$, $\dfrac{i(\tau)\rho(\sigma)}{\rho(\sigma)-(1-\sigma)h(\tau)}$ is not necessarily non-decreasing. It's also worth noting that the functions $a(\tau)$ and $b(\tau)$ in Theorem (4.3) can be non-negative for $i(\tau) \geq 0$ and $0 \leq h(\tau) \leq \dfrac{\rho(\sigma)}{1-\sigma}$.

The Ulam-Stability and Hyers Ulam-Stability of equation (2) are examined in this section. To attain the intended outcomes, we establish a few definitions and remarks. For $\epsilon > 0$, the following inequality is taken into account.

$$|{}^{MABD}D^\sigma y(\tau) - g(\tau, y(\tau), I_1 y(\tau), I_2 y(\tau))| \leq \epsilon, \ \forall\ \tau \in [0,1]. \tag{9}$$

Definition 4.5. The given FDE (2) are Ulam-Hyers stable, if $\exists\ \phi \in \mathcal{R}$, $\phi > 0$ such that for each $\epsilon > 0$, for each $\overline{y}(\tau) \in C([0,1], X)$ which is solution of inequality (9), $\exists$ a solution $y \in C([0,1], X)$ of (2) which is unique with

$$|\overline{y}(\tau) - y(\tau)| \leq \phi\epsilon, \ \tau \in [0,1].$$

The given FDE (2) are generalized Ulam-Hyers stable if we can find $\psi : \mathcal{R}^+ \to \mathcal{R}^+$ with $\psi(0) = 0$ such that

$$|\overline{y}(\tau) - y(\tau)| \leq \psi(\epsilon), \ \tau \in [0,1].$$

Remark 4.6. Let $\overline{y}(\tau) \in C([0,1], X)$ be the solution of inequality (9) iff we have a function $G \in C([0,1], X)$ dependant on $\overline{y}(\tau)$ such that

(1) $|G(\tau)| \leq \epsilon, \ \forall \tau \in [0,1]$,
(2) ${}^{MABD}D^\sigma y(\tau) = g(\tau, y(\tau), I_1 y(\tau), I_2 y(\tau)) + G(\tau), \ \tau \in [0,1]$.

Lemma 4.7. $\overline{y}(\tau)$ is the solution of the following inequality, if $\overline{y}(\tau) \in C([0,1], \mathcal{R})$ is the solution of (2).

$$|\overline{y}(\tau) - \overline{y}(0) -{}^{MABD} I^\sigma g(\tau, y(\tau), I_1 y(\tau), I_2 y(\tau))| \leq \delta\epsilon, \ \delta > 0 \tag{10}$$

where $\delta = \left(\dfrac{1-\sigma}{\rho(\sigma)} + \dfrac{1}{(1-\sigma)\Gamma(\sigma)}\right)$.

Proof. By applying Remark (4.6), we have

$$\begin{aligned} {}^{MABD}D^\sigma y(\tau) &= g(\tau, y(\tau), I_1 y(\tau), I_2 y(\tau)) + G(\tau), \ \tau \in [0,1], \\ \overline{y}(0) &= \overline{y}_0 \end{aligned}$$

by Theorem (3.1)

$$\begin{aligned}\overline{y}(\tau) &= \overline{y}_0 + \frac{1-\sigma}{\rho(\sigma)}\bigg(g(\tau,\overline{y}(\tau),I_1\overline{y}(\tau),I_2\overline{y}(\tau)) + G(\tau)\bigg)\\ &+ \frac{\mu_\sigma}{\Gamma(\sigma)}\int_0^\tau (\tau-s)^{\sigma-1}\bigg(g(s,\overline{y}(s),I_1\overline{y}(s),I_2\overline{y}(s)) + G(\tau)\bigg)ds\end{aligned}$$

$$\overline{y}(\tau) - \overline{y}_0 = \frac{1-\sigma}{\rho(\sigma)}g(\tau,\overline{y}(\tau),I_1\overline{y}(\tau),I_2\overline{y}(\tau)) + \frac{\mu_\sigma}{\Gamma(\sigma)}\int_0^\tau (\tau-s)^{\sigma-1}g(s,\overline{y}(s),I_1\overline{y}(s),I_2\overline{y}(s))ds$$

$$+ \frac{1-\sigma}{\rho(\sigma)}G(\tau) + \frac{\mu_\sigma}{\Gamma(\sigma)}\int_0^\tau G(\tau)(\tau-s)^{\sigma-1}$$

$$\left|\overline{y}(\tau) - \overline{y}_0 -^{MABD} I^\sigma g(\tau,\overline{y}(\tau),I_1\overline{y}(\tau),I_2\overline{y}(\tau))\right| \leq \frac{1-\sigma}{\rho(\sigma)}|G(\tau)| + \frac{\mu_\sigma}{\Gamma(\sigma)}\frac{\tau^\sigma}{\sigma}|G(\tau)|$$

$$\left|\overline{y}(\tau) - \overline{y}_0 -^{MABD} I^\sigma g(\tau,\overline{y}(\tau),I_1\overline{y}(\tau),I_2\overline{y}(\tau))\right| \leq \frac{1-\sigma}{\rho(\sigma)}\epsilon + \frac{\mu_\sigma}{\Gamma(\sigma)}\frac{\tau^\sigma}{\sigma}\epsilon$$

$$\left|\overline{y}(\tau) - \overline{y}_0 -^{MABD} I^\sigma g(\tau,\overline{y}(\tau),I_1\overline{y}(\tau),I_2\overline{y}(\tau))\right| \leq \left(\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\Gamma(\sigma)}\frac{\tau^\sigma}{\sigma}\right)\epsilon$$

$$\left|\overline{y}(\tau) - \overline{y}_0 -^{MABD} I^\sigma g(\tau,\overline{y}(\tau),I_1\overline{y}(\tau),I_2\overline{y}(\tau))\right| \leq \left(\frac{1-\sigma}{\rho(\sigma)} + \frac{1}{(1-\sigma)\Gamma(\sigma)}\right)\epsilon$$

□

THEOREM 4.8. The given FDE (2) is Ulam-Hyers Stable, if all the assumptions of the existence of solution holds with $(\theta_1 + \theta_2\theta_{I_1} + \theta_3\theta_{I_2}) \leq \frac{\rho(\sigma)}{1-\sigma}$.

Proof. Let $\epsilon > 0$, $\overline{y} \in C[0,1]$ be a function which satisfies (9) and let the unique solution of the given FDE (2) is $y \in C[0,1]$ with $y(0) = \overline{y}(0)$.
Consider

$$\begin{aligned}\|\overline{y}(\tau) - y(\tau)\| &= \bigg\|\overline{y}(\tau) - y_0 - \frac{1-\sigma}{\rho(\sigma)}g(\tau,y(\tau),I_1y(\tau),I_2y(\tau))\\ &- \frac{\mu_\sigma}{\Gamma(\sigma)}\int_0^\tau g(s,y(s),I_1y(s),I_2y(s))(\tau-s)^{\sigma-1}ds\bigg\|\end{aligned}$$

$$\|\overline{y}(\tau) - y(\tau)\| = \left\|\overline{y}(\tau) - \overline{y}(0) -^{MABD} I^\sigma g(\tau,y(\tau),I_1y(\tau),I_2y(\tau))\right\|$$

$$\|\overline{y}(\tau) - y(\tau)\| = \left\|\overline{y}(\tau) - \overline{y}(0) -^{MABD} I^\sigma g(\tau,\overline{y}(\tau),I_1\overline{y}(\tau),I_2\overline{y}(\tau))\right\|$$

$$+ \left\|^{MABD} I^\sigma g(\tau,\overline{y}(\tau),I_1\overline{y}(\tau),I_2\overline{y}(\tau)) -^{MABD} I^\sigma g(\tau,y(\tau),I_1y(\tau),I_2y(\tau))\right\|$$

$$\|\overline{y}(\tau)-y(\tau)\| \leq \delta\epsilon + {}^{MABD}I^{\sigma}\left\|g(\tau,\overline{y}(\tau),I_1\overline{y}(\tau),I_2\overline{y}(\tau)) - g(\tau,y(\tau),I_1y(\tau),I_2y(\tau))\right\|ds$$

$\|\overline{y}(\tau)-y(\tau)\| \leq \delta\epsilon + (\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})^{MABD}I^{\sigma}\|y-\overline{y}\|$

By using Theorem (4.3), take $i(\tau)=\delta\epsilon,\ h(\tau)=(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})$, we have

$$\|\overline{y}(\tau)-y(\tau)\| \leq \frac{\delta\epsilon\rho(\sigma)}{\rho(\sigma)-(1-\sigma)(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})}E_{\sigma}\left(\frac{\mu_{\sigma}\rho(\sigma)(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})}{\Gamma(\sigma)[\rho(\sigma)-(1-\sigma)(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})]}\right)$$

Therefore $\|\overline{y}(\tau)-y(\tau)\| \leq \phi\epsilon$, where

$$\phi = \frac{\delta\rho(\sigma)}{\rho(\sigma)-(1-\sigma)(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})}E_{\sigma}\left(\frac{\mu_{\sigma}\rho(\sigma)(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})}{\Gamma(\sigma)[\rho(\sigma)-(1-\sigma)(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})]}\right),$$

with $\delta = \left(\frac{1-\sigma}{\rho(\sigma)}+\frac{1}{(1-\sigma)\Gamma(\sigma)}\right)$. □

Corollary 4.9. *Under the hypothesis of Theorem (4.8), if there exists $\psi:\mathcal{R}^+\to\mathcal{R}^+$ such that $\psi(0)=0$, then the given fractional integro-differential DE (2) has generalized Ulam-Hyers stability.*

Proof. Choose

$$\psi(\epsilon) = \frac{\delta\epsilon\rho(\sigma)}{\rho(\sigma)-(1-\sigma)(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})}E_{\sigma}\left(\frac{\mu_{\sigma}\rho(\sigma)(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})}{\Gamma(\sigma)[\rho(\sigma)-(1-\sigma)(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})]}\right),$$

$\psi(0)=0$

, from Theorem (4.8), we obtain

$$|\overline{y}(\tau)-y(\tau)| \leq \psi(\epsilon).$$

□

To validate the obtained results, we provide the following illustrations.

Example 4.10.

$$(11)\qquad {}^{MABD}D^{\frac{1}{2}}y(\tau) = \tau + \frac{1}{5}y(\tau) + \frac{1}{5}\int_0^{\tau} y(\zeta)d\zeta + \frac{1}{5}\int_0^{T} \tau y(\zeta)d\zeta$$
$$y(0)=0,\ \tau\in[0,1].$$

Here, one can have $\sigma=\frac{1}{2}$,

$$g\left(\tau,\, y(\tau),\, \int_0^{\tau} f_1(\tau,\zeta,y(\zeta))\,d\zeta,\, \int_0^{T} f_2(\tau,\zeta,y(\zeta))\,d\zeta\right) = \tau+\frac{1}{5}y(\tau)+\frac{1}{5}\int_0^{\tau}y(\zeta)d\zeta+\frac{1}{5}\int_0^{T}\tau y(\zeta)d\zeta,$$

$f_1(\tau,\zeta,y(\zeta)) = \frac{1}{5}y(\zeta), f_2(\tau,\zeta,y(\zeta)) = \frac{1}{5}\tau y(\zeta)$.

At $\tau=0,\ g(0,y(0),I_1y(0),I_2y(0))=0$,

$$\begin{aligned}\|g\left(\tau,y(\tau),I_1y(\tau),I_2y(\tau)\right)\| &= \left\|\tau+\frac{1}{5}y(\tau)+\frac{1}{5}\int_0^{\tau}y(\zeta)\,d\zeta+\frac{1}{5}\int_0^{T}\tau y(\zeta)\,d\zeta\right\| \\ &\leq \|\tau\|+\frac{1}{5}\|y(\tau)\|+\frac{1}{5}\|I_1y(\tau)\|+\frac{1}{5}\|\tau\|\|I_2y(\tau)\|\end{aligned}$$

$$\leq 1 + \frac{1}{5}\|y\| + \frac{1}{5}\|I_1 y\| + \frac{1}{5}\|I_2 y\|.$$

So, $\lambda_0 = 1,\ \lambda_1 = \frac{1}{5}.\ \lambda_2 = \frac{1}{5},\ \lambda_3 = \frac{1}{5}$. Also, we get $\eta_0 = 0,\ \eta_1 = 1,\ \eta_2 = 0,\ \eta_3 = 1$. Now,

$$\|f_1(\tau,\zeta,y_1(\zeta)) - f_1(\tau,\zeta,y_2(\zeta))\| \leq \frac{1}{5}\|y_1(\zeta) - y_2(\zeta)\|,$$
$$\|f_2(\tau,\zeta,y_1(\zeta)) - f_2(\tau,\zeta,y_2(\zeta))\| \leq \frac{1}{5}\|y_1(\zeta) - y_2(\zeta)\|,\ \because \tau \leq 1.$$

Hence, $\theta_{I_1} = \frac{1}{5},\ \theta_{I_2} = \frac{1}{5}$. After that,

$$\left\|g\left(\tau, y_1(\tau), I_1y_1(\tau), I_2y_1(\tau)\right) - g\left(\tau, y_2(\tau), I_1y_2(\tau), I_2y_2(\tau)\right)\right\|$$
$$\leq \frac{1}{5}\|y_1 - y_2\| + \frac{1}{5}\|I_1y_1 - I_1y_2\| + \frac{1}{5}\|I_2y_1 - I_2y_2\|.$$

$\therefore\ \theta_1 = \frac{1}{5},\ \theta_2 = \frac{1}{5},\ \theta_3 = \frac{1}{5}$
By theorem (3.2),

$$\left[\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right](\lambda_1 + \lambda_2\eta_1 + \lambda_3\eta_3) = \left[1 - \frac{1}{2} + \frac{1}{(1-\frac{1}{2})\Gamma\frac{1}{2}}\right]\left(\frac{3}{5}\right) = 0.977027 < 1$$

$$\frac{\|y_0\| + \left[\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right](\lambda_0 + \lambda_2\eta_0 + \lambda_3\eta_2)}{1 - \left[\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right](\lambda_1 + \lambda_2\eta_1 + \lambda_3\eta_3)} = \frac{0 + \left[1 - \frac{1}{2} + \frac{1}{(1-\frac{1}{2})\Gamma\frac{1}{2}}\right](1+0+0)}{1 - \left[1 - \frac{1}{2} + \frac{1}{(1-\frac{1}{2})\Gamma\frac{1}{2}}\right]\left(\frac{3}{5}\right)} = 70.8838 \leq r$$

Hence, by theorem (3.2), there exists a solution for given FDE with r=71.
For uniqueness, Consider

$$\left(\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right)\left(\theta_1 + \theta_2\theta_{I_1} + \theta_3\theta_{I_2}\right) = \left[1 - \frac{1}{2} + \frac{1}{(1-\frac{1}{2})\Gamma\frac{1}{2}}\right]\left(\frac{1}{5} + \frac{1}{25} + \frac{1}{25}\right) = 0.4559 < 1$$

According to the theorem (3.3), there is only one solution to the given problem. For stability, our FDE (11) is Hyer-Ulam and Generalized Hyer-Ulam stable with

$$\phi = \frac{\delta\rho(\sigma)}{\rho(\sigma) - (1-\sigma)(\theta_1 + \theta_2\theta_{I_1} + \theta_3\theta_{I_2})} E_\sigma\left(\frac{\mu_\sigma\rho(\sigma)(\theta_1 + \theta_2\theta_{I_1} + \theta_3\theta_{I_2})}{\Gamma(\sigma)[\rho(\sigma) - (1-\sigma)(\theta_1 + \theta_2\theta_{I_1} + \theta_3\theta_{I_2})]}\right)$$

with $\delta = \left(\frac{1-\sigma}{\rho(\sigma)} + \frac{1}{(1-\sigma)\Gamma(\sigma)}\right)$.

$$\phi = \left(\frac{25\sqrt{\pi} + 100}{43\sqrt{\pi}}\right) E_{\frac{1}{2}}\left(\frac{14}{43\sqrt{\pi}}\right) = 2.3599 > 0.$$

EXAMPLE 4.11.

$$ {}^{MABD}D^{\frac{1}{3}} y(\tau) = \tau + \frac{1}{4}y(\tau) + \frac{1}{3}\int_0^\tau y(\zeta)d\zeta + \frac{1}{2}\int_0^T \tau y(\zeta)d\zeta \tag{12}$$
$$y(0) = 0,\ \tau \in [0,1].$$

Here, one can have $\sigma = \dfrac{1}{3}$,

$$\begin{aligned} & g\left(\tau,\, y(\tau),\, \int_0^\tau f_1(\tau,\zeta,y(\zeta))\,d\zeta,\, \int_0^T f_2(\tau,\zeta,y(\zeta))\,d\zeta\right) \\ & = \tau + \frac{1}{4}y(\tau) + \frac{1}{3}\int_0^\tau y(\zeta)d\zeta + \frac{1}{2}\int_0^T \tau y(\zeta)d\zeta, f_1(\tau,\zeta,y(\zeta)) \\ & = \frac{1}{3}y(\zeta), f_2(\tau,\zeta,y(\zeta)) = \frac{1}{2}\tau y(\zeta) \end{aligned}$$

At $\tau = 0,\ g(0, y(0), I_1y(0)), I_2y(0)) = 0.$

$$\begin{aligned} \|g\,(\tau,\, y(\tau),\, I_1y(\tau), I_2y(\tau))\,\| &= \left\|\tau + \frac{1}{4}y(\tau) + \frac{1}{3}\int_0^\tau y(\zeta)d\zeta + \frac{1}{2}\int_0^T \tau y(\zeta)d\zeta\right\|, \\ &\le \|\tau\| + \frac{1}{4}\|y(\tau)\| + \frac{1}{3}\|I_1y(\tau)\| + \frac{1}{2}\|\tau\|\|I_2y(\tau)\| \\ &\le 1 + \frac{1}{4}\|y\| + \frac{1}{3}\|I_1y\| + \frac{1}{2}\|I_2y\|. \end{aligned}$$

So, $\lambda_0 = 1,\ \lambda_1 = \dfrac{1}{4}.\ \lambda_2 = \dfrac{1}{3},\ \lambda_3 = \dfrac{1}{2},$
Also, we get $\eta_0 = 0,\ \eta_1 = \dfrac{1}{3},\ \eta_2 = 0,\ \eta_3 = \dfrac{1}{2}.$
Now,

$$\begin{aligned} \|f_1(\tau,\zeta,y_1(\zeta)) - f_1(\tau,\zeta,y_2(\zeta))\| &\le \frac{1}{3}\|y_1(\zeta) - y_2(\zeta)\|, \\ \|f_2(\tau,\zeta,y_1(\zeta)) - f_2(\tau,\zeta,y_2(\zeta))\| &\le \frac{1}{2}\|y_1(\zeta) - y_2(\zeta)\|,\ \because \tau \le 1. \end{aligned}$$

Hence, $\theta_{I_1} = \dfrac{1}{3},\ \theta_{I_2} = \dfrac{1}{2}.$
After that,

$$\begin{aligned} & \left\|g\Big(\tau, y_1(\tau), I_1y_1(\tau), I_2y_1(\tau)\Big) - g\Big(\tau, y_2(\tau), I_1y_2(\tau), I_2y_2(\tau)\Big)\right\| \\ & \qquad \le \frac{1}{4}\|y_1 - y_2\| + \frac{1}{3}\|I_1y_1 - I_1y_2\| + \frac{1}{2}\|I_2y_1 - I_2y_2\| \end{aligned}$$

$\therefore\ \theta_1 = \dfrac{1}{4},\ \theta_2 = \dfrac{1}{3},\ \theta_3 = \dfrac{1}{2}.$
By theorem (3.2),

$$\left[\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right](\lambda_1 + \lambda_2\eta_1 + \lambda_3\eta_3) = \left[1 - \frac{1}{3} + \frac{1}{(1-\frac{1}{3})\Gamma\frac{1}{3}}\right]\left(\frac{11}{18}\right) = 0.74958 < 1$$

$$\frac{\|y_0\| + \left[\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right](\lambda_0 + \lambda_2\eta_0 + \lambda_3\eta_2)}{1 - \left[\frac{1-\sigma}{\rho(\sigma)} + \frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right](\lambda_1 + \lambda_2\eta_1 + \lambda_3\eta_3)} = \frac{0 + \left[1 - \frac{1}{3} + \frac{1}{(1-\frac{1}{3})\Gamma\frac{1}{3}}\right](1+0+0)}{1 - \left[1 - \frac{1}{3} + \frac{1}{(1-\frac{1}{3})\Gamma\frac{1}{3}}\right]\left(\frac{22}{36}\right)} = 4.8981 \le r.$$

Hence, by theorem (3.2), there exists a solution for given FDE with r=5.
For uniqueness, Consider

$$\left(\frac{1-\sigma}{\rho(\sigma)}+\frac{\mu_\sigma}{\sigma\Gamma(\sigma)}\right)\left(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2}\right)=\left[1-\frac{1}{3}+\frac{1}{(1-\frac{1}{3})\Gamma\frac{1}{3}}\right]\left(\frac{1}{4}+\frac{1}{9}+\frac{1}{4}\right)=0.74958<1.$$

According to the theorem (3.3), there is only one solution to the given problem. For stability, our FDE (12) is Hyer-Ulam and Generalized Hyer-Ulam stable with

$$\phi=\frac{\delta\rho(\sigma)}{\rho(\sigma)-(1-\sigma)(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})}E_\sigma\left(\frac{\mu_\sigma\rho(\sigma)(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})}{\Gamma(\sigma)[\rho(\sigma)-(1-\sigma)(\theta_1+\theta_2\theta_{I_1}+\theta_3\theta_{I_2})]}\right)$$

with $\delta=\left(\frac{1-\sigma}{\rho(\sigma)}+\frac{1}{(1-\sigma)\Gamma(\sigma)}\right)$.

$$\phi=\frac{9(4\Gamma\frac{1}{3}+9)}{32\Gamma\frac{1}{3}}*E_{\frac{1}{3}}\left(\frac{44}{243\Gamma\frac{1}{3}}\right)=2.2377>0.$$

5. Conclusion

In recent years, new fractional operators with nonsingular kernels have drawn the interest of researchers. Among them, the Modified Atangana–Baleanu fractional derivative has garnered more attention due to its adaptability and improved kernel structure, which makes analysis easier and increases the range of applications for fractional models. However, this operator's theoretical component is still being developed and needs more research. A class of integro-fractional differential equations containing the Modified AB derivative has been examined in this study, along with the existence and uniqueness of solutions. We provided adequate conditions that ensured the problem was well-posed under appropriate assumptions. In addition, the stability of solutions in the sense of Ulam–Hyers and generalized Ulam-Hyers within the Modified Atangana–Baleanu fractional derivative framework was discussed. A concrete examples employing the Modified AB derivative was provided to demonstrate the theoretical conclusions' relevance. In these examples, we proved that the requirements derived in the analysis are satisfied by confirming the assumptions of our key theorems. This attests to the solution's uniqueness and the practical validity of the theoretical findings.

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