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SCATTEREDNESS OF HYPERSPACES

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Abstract. In this paper, we study separation properties on the space of closed subsets of a totally disconnected compact space. Thus we show that a topological space X is scattered if and only if the corresponding hyperspace 2^X is also scattered under the condition of local finiteness in X. We also provide characterizations concerning separation properties of the hyperspace via continuous real-valued functions. Furthermore, we give some examples related to our results.

1. Introduction and preliminaries

Coornaert [2] studied the relationship between the class of zero-dimensional spaces and other classes of highly-disconnected topological spaces such as scattered spaces, the class of totally disconnected spaces, and the class of totally separated spaces. Thus Coornaert established some equivalent conditions characterizing when a topological(or accessible Lindelöf) space is zero-dimensional(see [2, Theorem 2.3.1 and Corollary 2.4.22]). Koo and Lee provided some characterizations of zero-dimensional spaces by using the class of continuous real-valued functions and a partition of unity(see [5, Theorem 2.3]). These results motivate us to extend such characterizations to the context of hyperspaces.

For mention of our results, we recall some basic notions and facts related to topological dimension and hyperspaces of a topological space X. First, we recall the topological dimension of a topological space X(see [2]). When E is a set, $\sharp E$ denotes the cardinality of E if E is finite or the symbol ∞ otherwise. Let $\alpha = \{A_i\}_{i\in I}$ be a family of subsets of X indexed by a set I. For each $x \in X$, let $\operatorname{ord}_x(\alpha) := -1 + \sharp\{i \in I \mid x \in A_i\}$ be the order of α at the point x. We define the order $\operatorname{ord}(\alpha)$ of the family α by $\operatorname{ord}(\alpha) = \sup_{x\in X} \operatorname{ord}_x(\alpha)$. Let X be a topological space and $\alpha = \{U_i\}_{i\in I}$ be a finite open cover of X. We recall that the *quantity* $D(\alpha)$ of α is defined by $D(\alpha) = \min_\beta \operatorname{ord}(\beta)$, where β runs over all finite open covers of X that are finer than α. Also we define the *topological dimension* $\dim(X)$ of X by $\dim(X) = \sup_\alpha D(\alpha)$, where α runs over all finite open covers of X and $\dim(X) \in \{-1, 0\} \cup \mathbb{N} \cup \{\infty\}$.

Next, we recall the elementary notions about separation properties of a topological space. We say that a topological space X is *accessible* if every subset of X that is reduced to a single point is closed in X. We say that a space X is *scattered* if it admits a base

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consisting of clopen subsets of X. Every scattered accessible space is Hausdorff(see [2, Proposition 2.2.5]).

Next, we recall the notion and basic facts concerning hyperspaces of a topological space X with topology $\mathcal{T}$(see [3, 7]).

Let $I_n = \{1, \ldots, n\}$ be a finite indexed set for each $n \in \mathbb{N}$. Let 2^X be the collection of nonempty closed subsets of X. If $\{S_i\}_{i \in I_n}$ is a collection of subsets of X, then

$$\langle S_1, \cdots, S_n \rangle = \{A \in 2^X \mid A \subset \bigcup_{i \in I_n} S_i \text{ and } A \cap S_i \neq \emptyset \text{ for each } i \in I_n\}.$$

Then the largest hyperspace of X is 2^X with the Vietoris topology $\mathcal{T}_V$ generated by a base $\mathcal{B}_V$ given by

$$\mathcal{B}_V = \bigcup_{n \in \mathbb{N}} \{\langle U_1, \cdots, U_n \rangle \mid U_i \in \mathcal{T} \text{ for each } i \in I_n\}.$$

We collect some well-known facts about hyperspaces as follows:

- If U is an open subset of a topological space X, then $\{A \in 2^X \mid A \subset U\} = \langle U \rangle := 2^U$ is open in 2^X.
- If B is a closed subset of a topological space X, then $\{A \in 2^X \mid A \subset B\}$ is closed in 2^X(see [7, Lemma 2.2.1]).
- For any finitely many open subsets $U_1, \cdots, U_n$ of X, we have
$$\overline{< U_1, \cdots, U_n >} = < \overline{U_1}, \cdots, \overline{U_n}) >,$$
where $\overline{U}$ denotes the closure of a set U(see [7, Lemma 2.3.2]).
- We denote by $\mathcal{F}_1(X) = \{\{x\} \in 2^X \mid x \in X\}$ the family of all singletons of a topological space X.

In this paper, we study separation properties of hyperspaces of a totally disconnected compact space. Thus we provide characterizations of the scattered hyperspace via the notion of finite open partitions and some classes of continuous real-valued functions. We also give some examples related to our results.

2. Main results

In this section we study the equivalent conditions about separation axioms of topological spaces. Thus we provide characterizations of scatteredness and topological dimension for a topological space and its hyperspace, and illustrate these results with examples. We say that a topological space X satisfies the following property:

(UP) For every pair of disjoint closed subsets A and B of X, there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$;

We say that A and B can be separated by a continuous function. The Urysohn lemma means that a normal space X satisfies (UP)(see [9]). It is well known that the following conditions are equivalent(see [8]):

(1) A space X is normal;
(2) A space X satisfies (UP).

The following result provides some equivalent conditions for zero-dimensional spaces.

Proposition 2.1. *[5, Theorem 2.3] Let X be a topological space. Then the following conditions are equivalent.*

(1) X is zero-dimensional.
(2) For every pair of disjoint closed subsets A and B of X, there exists a continuous function $f : X \to [0, 1]$ with $f(X) \neq [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

(3) For every pair of disjoint closed subsets A and B of X, then there exists a continuous function $f : X \to [0,1]$ and a clopen subset $\mathrm{supp}(f)$ *of X such that $f(A) = \{1\}$ and $A \subset \mathrm{supp}(f) \subset B^c$, where* $\mathrm{supp}(f) = \overline{\{x \in X | f(x) \neq 0\}}$.

(4) Let $\mathcal{A} = \{U_i\}_{i=1}^{l}$ be a finite open cover of X. Then there exists a family $\{\phi_i : X \to [0,1]\}_{i=1}^{l}$ of continuous functions such that

(i) $\mathrm{supp}(\phi_i)$ *is clopen(or $\phi_i(X) \neq [0,1]$) for all $1 \leq i \leq l$;*

(ii) $\mathrm{supp}(\phi_i) \subset U_i$ *for all $1 \leq i \leq l$;*

(iii) $\sum_{i=1}^{l} \phi_i(x) = 1$ for all $x \in X$.

The following result is an extension of [2, Theorem 2.3.1] to the case of the hyperspace of a topological space.

Proposition 2.2. [4, Theorem 3.9] Let 2^X be the hyperspace of a topological space X. Then the following conditions are equivalent.

(1) 2^X is zero-dimensional.

(2) For every pair of disjoint closed subsets $< A >, < B >$ of 2^X, there exist open subsets (clopen) $< U >, < V >$ of 2^X such that $< A > \subset < U >, < B > \subset < V >$ and $\{< U >, < V >, < U, V >\}$ is a finite open partition of 2^X.

(3) For every closed subset $< A >$ of 2^X and every open subset $< U >$ of 2^X containing $< A >$, there exists a clopen subset $< V >$ of 2^X such that $< A > \subset < V > \subset < U >$.

Corollary 2.3. [5, Corollary 2.4] If a topological space X satisfies (UP) with surjection f, then X is not zero-dimensional.

We recall that a family $\mathcal{A}$ of subsets of a topological space is *locally finite* if each point of the space has a neighborhood which intersects only finitely many members of $\mathcal{A}$. It follows from the definition that the closure of the union of all members of $\mathcal{A}$ is the union of the closures of all members of $\mathcal{A}$; that is, $\overline{\cup_{A \in \mathcal{A}} A} = \cup_{A \in \mathcal{A}} \overline{A}$.

Remark 2.4. We note that every paracompact Hausdorff space is normal. Every regular Lindelöf space is paracompact.

Lemma 2.5. [4, Lemma 3.2] Let 2^X be the hyperspace of a topological space X; let A be a subset of X. Then A is closed(or open, resp.) in X if and only if $< A >$ is closed(or open, resp.) in 2^X.

Lemma 2.5 implies that a subset A of a topological space X is clopen in X if and only if $< A >$ is also clopen in 2^X. From Lemma 2.5, we obtain the following result concerning the scatteredness of the hyperspace.

Theorem 2.6. *Let X be a topological space with a locally finite basis. A topological space X is scattered if and only if 2^X is scattered.*

Proof. Suppose that a topological space X is scattered and $\mathcal{B}_X$ is a locally finite base for a topology on X consisting of clopen subsets of X. It is well known in general topology that if $\mathcal{A} = \{A_\alpha\}_{\alpha \in \Lambda}$ is a locally finite collection of subsets of X, then

$$\overline{\cup_{\alpha \in \Lambda} A_\alpha} = \cup_{\alpha \in \Lambda} \overline{A_\alpha}.$$

Given an open subset U of X, we have that U is closed since

$$\overline{U} = \overline{\bigcup_{\substack{x \in U \\ B_x \subset U \\ B_x \in \mathcal{B}}} B_x} = \bigcup_{\substack{x \in U \\ B_x \subset U \\ B_x \in \mathcal{B}}} \overline{B_x} = \bigcup_{\substack{x \in U \\ B_x \subset U \\ B_x \in \mathcal{B}}} B_x = U.$$

Put

$$\mathcal{B}_{2^X} = \bigcup_{n \in \mathbb{N}} \{< U_1, \cdots, U_n > \mid U_i \text{ is open in } X \text{ for each } i \in I_n\}.$$

Then $< U_1, \cdots, U_n >$ is clopen in 2^X because

$$\begin{aligned}\overline{< U_1, \cdots, U_n >} &= < \overline{U_1}, \cdots, \overline{U_n} > = < U_1, \cdots, U_n > \\ &= < U_1, \cdots, U_n >^\circ = < U_1^\circ, \cdots, U_n^\circ >\end{aligned}$$

by Lemma 2.5. Here A° denotes the interior of a set A. Thus $\mathcal{B}_{2^X}$ is a base consisting of clopen subsets of 2^X. Hence 2^X is scattered.

Suppose that $\mathcal{B}_{2^X}$ is a base consisting of clopen subsets of subsets of 2^X. Let U be an open subset of X. We claim that U is closed in X. Then $< U >$ is a basis element in $\mathcal{B}_{2^X}$ and so $< U >$ is clopen in 2^X from the scatteredness of 2^X. From Lemma 2.5, we can see that U is also clopen in X and so every basis element is clearly clopen in X. Hence X is scattered. This completes the proof. □

Corollary 2.7. *Let X be an accessible Lindelöf space. Then X is scattered if and only if 2^X is also scattered.*

Proof. Suppose X is scattered. Since X is an accessible Lindelöf space, it follows from [2, Corollary 2.4.22] that X is zero-dimensional. 2^X is also an accessible zero-dimensional space by [3, Proposition 8.6] and [7, Theorem 4.9.2]. Moreover, 2^X is scattered by [2, Corollary 2.3.3].

The converse is proved by similar argument as in the proof of Theorem 2.6. □

Corollary 2.8. *Let 2^X be a Lindelöf space of an accessible space X. Then X is scattered if and only if 2^X is also scattered.*

Proof. Suppose that 2^X is Lindelöf. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. Then $\{< X, U_\alpha >\}_{\alpha\in\Lambda}$ is an open cover of 2^X, whence there exists a countable open subcover $\{< X, U_i >\}_{i=1}^\infty$ of 2^X. Let $x \in X$. Then $\{x\} \in 2^X$ and so $\{x\} \in < X, U_i >$, i.e., $x \in U_i$, for some $i \in \mathbb{N}$, which implies that $X \subset \cup_{i=1}^\infty U_i$. Thus $\{U_i\}_{i=1}^\infty$ is a countable subcover of X. Hence X is Lindelöf. So we immediately obtain the result by Corollary 2.7. □

It is well known that every compact space X is paracompact and so locally finite. Thus we immediately obtain the following result.

Corollary 2.9. *Let X be an accessible space. Then X is a scattered compact space if and only if 2^X is also a scattered compact space.*

Proof. We note that a topological space X is compact if and only if 2^X is also compact(see [6, Theorem 4.2]). Combining Theorem 2.6 with [6, Theorem 4.2], we get the result. □

The following result presents an equivalent condition between normality of a topological space and a continuous real-valued function on its hyperspace.

Theorem 2.10. *Let $[0, 1]$ be the closed interval in the real line $\mathbb{R}$. Let 2^X be the hyperspace of a topological space X. Then the following conditions are equivalent.*

(1) A topological space X is normal.
(2) For every pair of disjoint closed subsets $< A >$ and $< B >$ of 2^X there exists a continuous function $F : 2^X \to [0, 1]$ such that $F(< A >) = \{0\}$ and $F(< B >) = \{1\}$.

Proof. (1) $\Leftarrow$ (2) Let A and B be disjoint closed subsets of X. Then $< A >$ and $< B >$ are disjoint closed subsets of 2^X. Thus there exists a continuous function $F : 2^X \to [0, 1]$ such that $F(< A >) = \{0\}$ and $F(< B >) = \{1\}$. Since X and $\mathcal{F}_1(X)$ are homeomorphic, there exists an imbedding $j : X \to 2^X$ given by $j(x) = \{x\}$ such that $j(X) = \mathcal{F}_1(X)$ is a subspace of 2^X(see [3, Exercise 1.15]). Define a map $f : X \to [0,1]$ by $F \circ j$. Then f is continuous and also satisfies the following:

$$f(a) = (F \circ j)(a) = F(\{a\}) = 0 \text{ for all } a \in A$$

and

$$f(b) = (F \circ j)(b) = F(\{b\}) = 1 \text{ for all } b \in B,$$

since $F(\{a\}) = 0$ for $\{a\} \in < A >$ and $F(\{b\}) = 1$ for $\{b\} \in < B >$, respectively. So X is normal.

(1) $\Rightarrow$ (2) Suppose that for every pair of disjoint closed subsets A and B of X there exists a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Define a map $F : 2^X \to [0,1]$ by

$$F(E) = \sup_{x \in E} f(x).$$

Then the function F is continuous by the continuity of f and clearly satisfies $F(< A >) = \{0\}$ and $F(< B >) = \{1\}$(see [6, Proposition 4.7]). For each $E \in < A >$ we have $E \in 2^X$ with $E \subset A$ and $f(A) = \{0\}$. Thus we also obtain $F(E) = \sup_{x \in E} f(x) = 0$. Similarly, we obtain $F(< B >) = \{1\}$ for each $E \in < B >$. This completes the proof. $\square$

By Theorem2.10 and [7, Theorem 4.9.5], we obtain the following result.

Corollary 2.11. *Let 2^X be the hyperspace of a topological space X. Then the following conditions are equivalent.*

(1) 2^X is completely regular.

(2) For every pair of disjoint closed subsets $< A >$ and $< B >$ of 2^X there exists a continuous function $F : 2^X \to [0,1]$ such that $F(< A >) = \{0\}$ and $F(< B >) = \{1\}$.

Proof. We note that the normality of X and the condition (2) are equivalent by Theorem 2.10. Also, it follows from [7, Theorem 4.9.5] that a topological space X is normal if and only if 2^X is completely regular. Thus we can see that two items (1) and (2) are equivalent. $\square$

The following example illustrates Theorem 2.10.

Example 2.12. Let $\mathbb{R}_l$ be the real line with the lower limit topology. Let $< A >$ and $< B >$ be disjoint closed subsets of $2^{\mathbb{R}_l}$. Then there is a continuous function $F : 2^{\mathbb{R}_l} \to [0,1]$ with $F(2^{\mathbb{R}_l}) \neq [0,1]$ such that $F(< A >) = \{0\}$ and $F(< B >) = \{1\}$.

Proof. We note that $\mathbb{R}_l$ is a zero-dimensional Hausdorff space that is not compact, and so normal(see [2, Corollary 2.3.2]).

By Theorem 2.10 and Proposition 2.1, we see that for every pair of disjoint closed subsets $< A >$ and $< B >$ of $2^{\mathbb{R}_l}$, there exists a continuous map $F : 2^{\mathbb{R}_l} \to [0,1]$ with $F(2^{\mathbb{R}_l}) \neq [0,1]$ such that $F(< A >) = \{0\}$ and $F(< B >) = \{1\}$. $\square$

The following result shows that there exists a totally disconnected compact space 2^X with $\dim(2^X) \neq 0$ that is not Hausdorff.

Proposition 2.13. [5, Proposition 2.5] Let X be an infinite set. Let us fix two distinct points $a, b \in X$ and let $Y = X \setminus \{a, b\}$. Let $\mathcal{T}$ denotes the topology on X consisting of all $U \subset X$ satisfying one of the following two conditions.

(1) $U \subset Y$;

(2) $U = U_1 \cup U_2$, where U_1 is a non-empty subset of $\{a, b\}$ and $U_2 \subset Y$ is such that $Y \setminus U_2$ is a finite set.

Then the hyperspace 2^X of X is a totally disconnected compact space with $\dim(2^X) \neq 0$ that is not Hausdorff.

Proof. It follows from [5, Proposition 2.5] that X is a totally disconnected compact non-Hausdorff space with $\dim(X) = 1$. For the completion of proof, we need the following well-known facts:

- A topological space X is totally disconnected compact if and only if so is the hyperspace 2^X(see [7, Theorem 4.2 and Proposition 4.13.2]).
- Let X be an accessible space. Then X is zero-dimensional if and only if 2^X is also zero-dimensional(see [3, Proposition 8.6]).
- Let X be a compact space. A topological space X is Hausdorff if and only if 2^X is also Hausdorff(see [7, Theorem 4.9.8]).

From the above results, we can see that the hyperspace 2^X is also totally disconnected compact with $\dim(2^X) \neq 0$ that is not Hausdorff. □

For example, we briefly recall the Stone-Čech compactification $\beta(\mathbb{N})$ of $\mathbb{N}$. Then it is well known that the subspace $\mathbb{N}$ of the completely regular space $\mathbb{R}$ is also completely regular and so there exists the compact Hausdorff space $\beta(\mathbb{N})$ such that $\overline{\mathbb{N}} = \beta(\mathbb{N})$(see [1, Theorem 3.1, p. 779]).

The following example illustrates Theorem 2.10 and Corollary 2.11.

Example 2.14. Let $2^{\beta(\mathbb{N})}$ be the hyperspace of the compact Hausdorff space $\beta(\mathbb{N})$; let $< A >$ and $< B >$ be disjoint closed subsets of $2^{\beta(\mathbb{N})}$. Then there exists a continuous map $F : 2^{\beta(\mathbb{N})} \to [0, 1]$ such that $F(< A >) = \{0\}$ and $F(< B >) = \{1\}$.

Proof. We note that the compact Hausdorff space $\beta(\mathbb{N})$ is a totally disconnected space that is not metrizable(see [8, Exercise 38, ♯ 7 and ♯ 9, p. 242]). We can easily see that the totally disconnectedness of the compact Hausdorff space $\beta(\mathbb{N})$ implies zero-dimension and so scatteredness(see [2, Theorem 2.7.1]). Thus the hyperspace $2^{\beta(\mathbb{N})}$ is also a scattered compact Hausdorff space with $\dim(2^X) = 0$(see Corollary 2.7, [2, Proposition 8.6] and [7, Theorem 4.9.6]). Since X is compact Hausdorff, X is normal and so 2^X is completely regular(see [7, Theorem 4.9.5]). So it follows from Theorem 2.10 that for disjoint closed subsets $< A >$ and $< B >$ of $2^{\beta(\mathbb{N})}$, there is a continuous map $F : 2^{\beta(\mathbb{N})} \to [0, 1]$ such that $F(< A >) = \{0\}$ and $F(< B >) = \{1\}$. □

Remark 2.15. We collect some well-known results about topological dimension and separation properties of the hyperspace 2^X of a topological space X as follows.

- If the hyperspace 2^X is normal, then the base space X is also normal, but the converse is false(see [6, Theorem 4.9.5]).
- Every zero-dimensional space X is normal(see [2, Corollary 2.3.2]). So the hyperspace 2^X of an accessible zero-dimensional space is also normal since 2^X is also zero-dimensional(see [3, Proposition 8.6]).
- Let X be an accessible Lindelöf space. Then the topological space X is zero-dimensional if and only if X is scattered(see [2, Corollary 2.4.22]).

Table 1. Suppose that X is compact Hausdorff(see [4, Proposition 3.13]).

$\dim(2^X) = 0$	$\iff$	scattered	$\iff$	totally separated	$\iff$	totally disconnected

TABLE 2. Suppose that X has a locally finite basis or X is a Lindelöf space(Theorem 2.5 and Corollary 2.7).

$$\begin{array}{ccc} \dim(2^X)=0 & \Longleftrightarrow & 2^X \text{ scattered} \\ \Updownarrow & & \Updownarrow \\ \dim(X)=0 & \Longleftrightarrow & X \text{ scattered} \end{array}$$

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