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ON THE BIRKHOFF INTEGRAL OF FUZZY MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce the Birkhoff integral of fuzzy mappings in Banach spaces in terms of the Birkhoff integral of set-valued mappings and investigate some properties of the Birkhoff integrals of set-valued mappings and fuzzy mappings in Banach spaces.

1. Introduction

Birkhoff [2] introduced the Birkhoff integral for Banach space valued functions. Birkhoff integrability lies strictly between Bochner and Pettis integrability when the range space X is nonseparable [2, 8]. Lately, Several authors [4,7,9] have investigated the Birkhoff integral for Banach space valued functions. Several types of integrals of set-valued mappings were introduced by many authors. In particular, Cascales and Rodriguez [3] introduced the Birkhoff integral of $CWK(X)$-valued mappings by means of a certain embedding of $CWK(X)$ into a Banach space. Several authors introduced the integrals of fuzzy mappings in Banach spaces in terms of the integrals of set-valued mappings. In particular, Xue, Ha and Ma [10] and Xue, Wang and Wu [11] introduced integrals of fuzzy mappings in Banach spaces in terms of Aumann-Pettis and Aumann-Bochner integrals of set-valued mappings.

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In this paper, we introduce the Birkhoff integral of fuzzy mappings in Banach spaces in terms of the Birkhoff integral of set-valued mappings and investigate some properties of the Birkhoff integrals of set-valued mappings and fuzzy mappings in Banach spaces and obtain convergence theorems for set-valued mappings and fuzzy mappings in Banach spaces.

2. Preliminaries

Throughout this paper, (Ω, Σ, μ) denotes a complete finite measure space and $(X, \|\cdot\|)$ a Banach space with dual X^*. The closed unit ball of X^* is denoted by B_{X^*}. $CL(X)$ denotes the family of all nonempty closed subsets of X and $CWK(X)$ the family of all nonempty convex weakly compact subsets of X. For $A \subseteq X$ and $x^* \in X^*$, let $s(x^*, A) = \sup\{x^*(x) : x \in A\}$, the support function of A. For $A, B \in CL(X)$, let $H(A, B)$ denote the Hausdorff metric of A and B defined by

$$H(A, B) = \max\left(\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right),$$

where $d(a, B) = \inf_{b \in B} \|a - b\|$ and $d(b, A) = \inf_{a \in A} \|a - b\|$. Especially,

$$H(A, B) = \sup_{x^* \in B_{X^*}} |s(x^*, A) - s(x^*, B)|$$

whenever A, B are convex sets.

Note that $(CWK(X), H)$ is a complete metric space with the following properties:

(1) $H(\lambda A, \lambda B) = |\lambda| H(A, B)$ for all $A, B \in CWK(X)$ and $\lambda \in \mathbb{R}$;

(2) $H(A + C, B + C) = H(A, B)$ for all $A, B, C \in CWK(X)$;

(3) $H(A + C, B + D) \leq H(A, B) + H(C, D)$ for all $A, B, C, D \in CWK(X)$.

The number $\|A\|$ is defined by $\|A\| = H(A, \{0\}) = \sup_{x \in A} \|x\|$.

Let $u : X \to [0, 1]$. We denote $[u]^r = \{x \in X : u(x) \geq r\}$ for $r \in (0, 1]$ and $[u]^0 = cl\{x \in X : u(x) > 0\}$. u is called a *generalized fuzzy number* on X if for each $r \in (0, 1]$, $[u]^r \in CWK(X)$. Let $\mathcal{F}(X)$ denote the set of all generalized fuzzy numbers on X. For $u, v \in \mathcal{F}(X)$ and $\lambda \in \mathbb{R}$, we define $u + v$ and λu as follows:

$$(u + v)(x) = \sup_{x = y + z} \min(u(y), v(z)),$$

$$(\lambda u)(x) = u\left(\frac{1}{\lambda}x\right), \ \lambda \neq 0$$

$$\lambda u = \tilde{0}, \ \lambda = 0, \ \text{where } \tilde{0} = \chi_{\{0\}}.$$

For $u, v \in \mathcal{F}(X)$ and $\lambda \in \mathbb{R}$, $[u+v]^r = [u]^r + [v]^r$ and $[\lambda u]^r = \lambda[u]^r$ for each $r \in (0,1]$. Hence $u+v, \lambda u \in \mathcal{F}(X)$. For $u, v \in \mathcal{F}(X)$, we define $u \leq v$ as follows:

$$u \leq v \ \text{ if } \ u(x) \leq v(x) \ \text{ for all } x \in X.$$

For $u, v \in \mathcal{F}(X)$, $u \leq v$ if and only if $[u]^r \subseteq [v]^r$ for each $r \in (0,1]$.

Define $D : \mathcal{F}(X) \times \mathcal{F}(X) \to [0, +\infty]$ by the equation

$$D(u,v) = \sup_{r \in (0,1]} H([u]^r, [v]^r).$$

Then D is a metric on $\mathcal{F}(X)$. The norm $\|u\|$ of $u \in \mathbf{F}(X)$ is defined by

$$\|u\| = D(u, \tilde{0}) = \sup_{r \in (0,1]} H([u]^r, \{0\}) = \sup_{r \in (0,1]} \|[u]^r\|.$$

The mapping $F : \Omega \to CL(X)$ is called a *set-valued mapping.* F is said to be *scalarly measurable* if for every $x^* \in X^*$, the real-valued function $s(x^*, F(\cdot))$ is measurable. F is said to be *Effros measurable* (or *measurable* for short) if $F^{-1}(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \phi\} \in \Sigma$ for every open subset U of X. Note that measurability is stronger than scalar measurability.

Let $F : \Omega \to CL(X)$. Then the following statements are equivalent:

(1) $F : \Omega \to CL(X)$ is measurable;

(2) $F^{-1}(A) = \{\omega \in \Omega : F(\omega) \cap A \neq \phi\} \in \Sigma$ for every $A \in CL(X)$;

(3) (Castaing representation) there exists a sequence (f_n) of measurable functions $f_n : \Omega \to X$ such that $F(\omega) = cl\{f_n(\omega)\}$ for all $\omega \in \Omega$.

$F : \Omega \to CL(X)$ is said to be *weakly integrably bounded* if the real-valued function $|x^*F| : \Omega \to \mathbb{R}, |x^*F|(\omega) = \sup\{|x^*(x)| : x \in F(\omega)\}$, is integrable for every $x^* \in X^*$. $F : \Omega \to CL(X)$ is said to be *integrably bounded* if there exists an integrable real-valued function h such that for each $\omega \in \Omega$, $\|x\| \leq h(\omega)$ for all $x \in F(\omega)$. $F : \Omega \to CL(X)$ is said to be *scalarly integrable* on Ω if for every $x^* \in X^*$, $s(x^*, F(\cdot))$ is integrable on Ω. $F : \Omega \to CL(X)$ is said to be *scalarly uniformly integrable* if the set $\{s(x^*, F(\cdot)) : x^* \in B_{X^*}\}$ is uniformly integrable. $f : \Omega \to X$ is called a *measurable selector* of $F : \Omega \to CL(X)$ if f is measurable and $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

A measurable set-valued mapping $F : \Omega \to CWK(X)$ is said to be *Pettis integrable* on Ω if $F : \Omega \to CWK(X)$ is scalarly integrable on Ω and for each $A \in \Sigma$ there exists $(P)\int_A F d\mu \in CWK(X)$ such that $s(x^*, (P)\int_A F d\mu) = \int_A s(x^*, F) d\mu$ for all $x^* \in X^*$. In this case, $(P)\int_A F d\mu$ is called the *Pettis integral* of F over A [6].

A function $f : \Omega \to X$ is called *summable* with respect to a given countable partition $\Gamma = (A_n)$ of Ω in Σ if $f|_{A_n}$ is bounded whenever $\mu(A_n) > 0$ and the set

$$J(f, \Gamma) = \left\{ \sum_n \mu(A_n) f(t_n) : t_n \in A_n \right\}$$

is made up of unconditionally convergent series.

Definition 2.1.[2]. A function $f : \Omega \to X$ is said to be *Birkhoff integrable* on Ω if for every $\epsilon > 0$ there exists a countable partition Γ of Ω in Σ for which f is summable and $\|\cdot\|$-diam $(J(f, \Gamma)) < \epsilon$. In this case, the *Birkhoff integral* $(B)\int_\Omega f d\mu$ of f is the only point in the intersection

$$\cap \left\{ \overline{co(J(f, \Gamma))} : f \text{ is summable with respect to } \Gamma \right\}.$$

If $f : \Omega \to X$ is Birkhoff integrable on Ω, then $f : \Omega \to X$ is Birkhoff integrable on every $A \in \Sigma$. Birkhoff integrability lies strictly between Bochner and Pettis integrability. If $f : \Omega \to X$ is Birkhoff integrable, then $(B)\int_\Omega f d\mu = (P)\int_\Omega f d\mu$. When the range space X is separable, Birkhoff and Pettis integrability are the same. In the definition of the Birkhoff integral, if the respective series

$$J(f, \Gamma) = \left\{ \sum_n \mu(A_n) f(t_n) : t_n \in A_n \right\}$$

is made up of absolutely convergent series, then $f : \Omega \to X$ is said to be *absolutely Birkhoff integrable* on Ω [1].

Theorem 2.2.[5]. *Let $\ell_\infty(B_{X^*})$ be the Banach space of bounded real-valued functions defined on B_{X^*} endowed with the supremum norm*

$\|\cdot\|_\infty$. *Then the map* $j : CWK(X) \longrightarrow \ell_\infty(B_{X^*})$ *given by* $j(A) := s(\cdot, A)$ *satisfies the following properties:*

(1) $j(A+B) = j(A) + j(B)$ for every $A, B \in CWK(X)$;
(2) $j(\lambda A) = \lambda j(A)$ for every $\lambda \geq 0$ and $A \in CWK(X)$;
(3) $H(A, B) = \|j(A) - j(B)\|_\infty$ for every $A, B \in CWK(X)$;
(4) $j(CWK(X))$ is closed in $\ell_\infty(B_{X^*})$.

Definition 2.3.[3]. A set-valued mapping $F : \Omega \to CWK(X)$ is said to be *Birkhoff integrable* on Ω if the composition $j \circ F : \Omega \to \ell_\infty(B_{X^*})$ is Birkhoff integrable on Ω. In this case, for each $A \in \Sigma$ there exists a unique element $(B)\int_A F d\mu \in CWK(X)$, that is called the *Birkhoff integral* of F on A, such that $j((B)\int_A F d\mu) = (B)\int_A j \circ F d\mu$.

3. Results

A mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is called a *fuzzy mapping* in a Banach space X. In this case, $\tilde{F}^r : \Omega \to CWK(X)$ defined by $\tilde{F}^r(\omega) = [\tilde{F}(\omega)]^r$ is a set-valued mapping for each $r \in (0, 1]$. A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *measurable* (resp., *scalarly measurable*) if $\tilde{F}^r : \Omega \to CWK(X)$ is measurable (resp., scalarly measurable) for each $r \in (0, 1]$.

Definition 3.1. A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *Birkhoff integrable* on Ω if for each $A \in \Sigma$ there exists $u_A \in \mathcal{F}(X)$ such that $[u_A]^r = (B)\int_A \tilde{F}^r d\mu$ for each $r \in (0, 1]$. In this case, $u_A = (B)\int_A \tilde{F} d\mu$ is called the *Birkhoff integral* of $\tilde{F}$ on A.

Theorem 3.2. *Let* $\tilde{F} : \Omega \to \mathcal{F}(X)$ *and* $\tilde{G} : \Omega \to \mathcal{F}(X)$ *be Birkhoff integrable on* Ω *and* $\lambda \geq 0$. *Then*

(1) $\tilde{F} + \tilde{G}$ *is Birkhoff integrable on* Ω *and for each* $A \in \Sigma$

$$(B)\int_A (\tilde{F} + \tilde{G}) d\mu = (B)\int_A \tilde{F} d\mu + (B)\int_A \tilde{G} d\mu,$$

(2) $\lambda\tilde{F}$ *is Birkhoff integrable on* Ω *and for each* $A \in \Sigma$

$$(B)\int_A \lambda\tilde{F} d\mu = \lambda(B)\int_A \tilde{F} d\mu.$$

Proof. (1) Let $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ be Birkhoff integrable on Ω. Then for each $A \in \Sigma$ there exist $u_A, v_A \in \mathcal{F}(X)$ such that $[u_A]^r = (B)\int_A \tilde{F}^r d\mu$, $[v_A]^r = (B)\int_A \tilde{G}^r d\mu$ for each $r \in (0,1]$. Thus $j \circ \tilde{F}^r$ and $j \circ \tilde{G}^r$ are Birkhoff integrable on Ω and $j([u_A]^r) = j((B)\int_A \tilde{F}^r d\mu) = \int_A j \circ \tilde{F}^r d\mu$, $j([v_A]^r) = j((B)\int_A \tilde{G}^r d\mu) = \int_A j \circ \tilde{G}^r d\mu$ for each $r \in (0,1]$ and $A \in \Sigma$. Hence $j \circ (\tilde{F} + \tilde{G})^r = j \circ (\tilde{F}^r + \tilde{G}^r)$ is Birkhoff integrable on Ω and

$$\begin{aligned}
[j([u_A + v_A]^r)](x^*) &= [j([u_A]^r) + j([v_A]^r)](x^*) \\
&= [j([u_A]^r)](x^*) + [j([v_A]^r)](x^*) \\
&= [j((B)\int_A \tilde{F}^r d\mu)](x^*) + [j((B)\int_A \tilde{G}^r d\mu)](x^*) \\
&= [(B)\int_A j \circ \tilde{F}^r d\mu)](x^*) + [(B)\int_A j \circ \tilde{G}^r d\mu](x^*) \\
&= [(B)\int_A j \circ (\tilde{F}^r + \tilde{G}^r) d\mu](x^*) \\
&= [(B)\int_A j \circ (\tilde{F} + \tilde{G})^r d\mu](x^*)
\end{aligned}$$

for each $x^* \in B_{X^*}$, $r \in (0,1]$ and $A \in \Sigma$. Hence j $([u_A + v_A]^r) = \int_A j \circ (\tilde{F} + \tilde{G})^r d\mu$ for each $r \in (0,1]$ and $A \in \Sigma$. Thus $[u_A + v_A]^r = (B)\int_A (\tilde{F}+\tilde{G})^r d\mu$ for each $r \in (0,1]$ and $A \in \Sigma$. Hence $\tilde{F}+\tilde{G}$ is Birkhoff integrable on Ω and for each $A \in \Sigma$

$$(B)\int_A (\tilde{F} + \tilde{G}) d\mu = u_A + v_A = (B)\int_A \tilde{F} d\mu + (B)\int_A \tilde{G} d\mu.$$

(2) Let $\tilde{F} : \Omega \to \mathcal{F}(X)$ be Birkhoff integrable on Ω and $\lambda \geq 0$. Then there exists $u_A \in \mathcal{F}(X)$ such that $[u_A]^r = (B)\int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. Since $j([\lambda u_A]^r) = \lambda j([u_A]^r)$ for each $r \in (0,1]$ and $A \in \Sigma$, using the same method as (1) we obtain that $\lambda \tilde{F}$ is Birkhoff integrable on Ω and for each $A \in \Sigma$

$$(B)\int_A \lambda\tilde{F}d\mu = \lambda(B)\int_A \tilde{F}d\mu.$$

□

LEMMA 3.3. *Let $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ be Birkhoff integrable set-valued mappings. Then*

(1) *if $F(\omega) = G(\omega)$ μ-a.e., then $(B)\int_A Fd\mu = (B)\int_A Gd\mu$ for each $A \in \Sigma$;*

(2) *if X is separable and $(B)\int_A Fd\mu = (B)\int_A Gd\mu$ for each $A \in \Sigma$, then $F(\omega) = G(\omega)$ μ-a.e.*

Proof. (1) Since $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are Birkhoff integrable on Ω, $j \circ F$ and $j \circ G$ are Birkhoff integrable on Ω and there exist $(B)\int_A Fd\mu, (B)\int_A Gd\mu \in CWK(X)$ such that $j((B)\int_A Fd\mu) = (B)\int_A j \circ Fd\mu$, $j((B)\int_A Gd\mu) = (B)\int_A j \circ Gd\mu$ for each $A \in \Sigma$.

If $F(\omega) = G(\omega)$ μ-a.e., then $(j \circ F)(\omega) = (j \circ G)(\omega)$ μ-a.e. Hence

$$j((B)\int_A Fd\mu) = (B)\int_A j \circ Fd\mu = (B)\int_A j \circ Gd\mu = j((B)\int_A Gd\mu)$$

for each $A \in \Sigma$. Thus

$$\begin{aligned} s(x^*, (B)\int_A Fd\mu) &= [j((B)\int_A Fd\mu)](x^*) \\ &= [j((B)\int_A Gd\mu)](x^*) \\ &= s(x^*, (B)\int_A Gd\mu) \end{aligned}$$

for each $x^* \in B_{X^*}$ and $A \in \Sigma$. Since $(B)\int_A Fd\mu, (B)\int_A Gd\mu \in CWK(X)$ for each $A \in \Sigma$, by the separation theorem $(B)\int_A Fd\mu = (B)\int_A Gd\mu$ for each $A \in \Sigma$.

(2) If $(B)\int_A Fd\mu = (B)\int_A Gd\mu$ for each $A \in \Sigma$, then

$$(B)\int_A j\circ Fd\mu = j((B)\int_A Fd\mu) = j((B)\int_A Gd\mu) = (B)\int_A j\circ Gd\mu$$

for each $A \in \Sigma$. Since X is a separable Banach space, by [2, Theorem 24] $(j\circ F)(\omega) = (j\circ G)(\omega)$ μ-a.e. and so $H(F(\omega), G(\omega)) = \|(j\circ F)(\omega) - (j\circ G)(\omega)\|_\infty = 0$ μ-a.e. Hence $F(\omega) = G(\omega)$ μ-a.e.

□

THEOREM 3.4. *Let $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ be Birkhoff integrable on Ω. If $\tilde{F}(\omega) = \tilde{G}(\omega)$ μ-a.e., then $(B)\int_A \tilde{F}d\mu = (B)\int_A \tilde{G}d\mu$ for each $A \in \Sigma$.*

Proof. Since $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ are Birkhoff integrable on Ω, for each $A \in \Sigma$ there exist $u_A, v_A \in \mathcal{F}(X)$ such that $[u_A]^r = (B)\int_A \tilde{F}^r d\mu$, $[v_A]^r = (B)\int_A \tilde{G}^r d\mu$ for each $r \in (0,1]$. If $\tilde{F}(\omega) = \tilde{G}(\omega)$ μ-a.e., then $\tilde{F}^r(\omega) = \tilde{G}^r(\omega)$ μ-a.e. for each $r \in (0,1]$. By Lemma 3.3 $[u_A]^r = (B)\int_A \tilde{F}^r d\mu = (B)\int_A \tilde{G}^r d\mu = [v_A]^r$ for each $r \in (0,1]$ and $A \in \Sigma$ and so $(B)\int_A \tilde{F}d\mu = u_A = v_A = (B)\int_A \tilde{G}d\mu$ for each $A \in \Sigma$. □

If X is separable and $F : \Omega \to CWK(X)$ is Birkhoff integrable on Ω, then

$$(B)\int_A Fd\mu = \left\{(B)\int_A fd\mu : f \text{ is a Birkhoff integrable selector of } F\right\}$$

for each $A \in \Sigma$ [3].

LEMMA 3.5. *Let X be separable and let $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ be Birkhoff integrable set-valued mappings. If $F(\omega) \subseteq G(\omega)$ on Ω, then $(B)\int_A Fd\mu \subseteq (B)\int_A Gd\mu$ for each $A \in \Sigma$.*

Proof. Since $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are Birkhoff integrable on Ω and $F(\omega) \subseteq G(\omega)$ on Ω, for each $A \in \Sigma$

$$
\begin{aligned}
(B)\int_A F d\mu &= \left\{(B)\int_A f d\mu : f \text{ is a Birkhoff integrable selector of } F\right\} \\
&\subseteq \left\{(B)\int_A g d\mu : g \text{ is a Birkhoff integrable selector of } G\right\} \\
&= (B)\int_A G d\mu.
\end{aligned}
$$

□

THEOREM 3.6. *Let X be separable and let $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ be Birkhoff integrable on Ω. If $\tilde{F}(\omega) \le \tilde{G}(\omega)$ on Ω, then $(B)\int_A \tilde{F} d\mu \le (B)\int_A \tilde{G} d\mu$ for each $A \in \Sigma$.*

Proof. (1) Since $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ are Birkhoff integrable on Ω, for each $A \in \Sigma$ there exist $u_A, v_A \in \mathcal{F}(X)$ such that $[u_A]^r = (B)\int_A \tilde{F}^r d\mu$, $[v_A]^r = (B)\int_A \tilde{G}^r d\mu$ for each $r \in (0,1]$. If $\tilde{F}(\omega) \le \tilde{G}(\omega)$ on Ω, then $\tilde{F}^r(\omega) \subseteq \tilde{G}^r(\omega)$ on Ω for each $r \in (0,1]$. By Lemma 3.5 $[u_A]^r = (B)\int_A \tilde{F}^r d\mu \subseteq (B)\int_A \tilde{G}^r d\mu = [v_A]^r$ for each $r \in (0,1]$ and so $(B)\int_A \tilde{F} d\mu = u_A \le v_A = (B)\int_A \tilde{G} d\mu$ for each $A \in \Sigma$.

□

LEMMA 3.7. *Let X be separable. If $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are measurable, integrably bounded and Birkhoff integrable set-valued mappings, then $H(F,G)$ is integrable on Ω and*

$$
H\left((B)\int_\Omega F d\mu, (B)\int_\Omega G d\mu\right) \le \int_\Omega H(F,G) d\mu.
$$

Proof. Since $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are measurable, there exist Castaing representations (f_n) and (g_n) for F and G. Since f_n and g_n are measurable for all $n \in \mathbb{N}$,

$$
H(F(\omega), G(\omega)) = \max\left(\sup_{n\ge1}\inf_{k\ge1}\|f_n(\omega) - g_k(\omega)\|, \sup_{n\ge1}\inf_{k\ge1}\|g_n(\omega) - f_k(\omega)\|\right)
$$

is measurable. Since $F:\Omega\to CWK(X)$ and $G:\Omega\to CWK(X)$ are integrably bounded, there exist integrable real-valued functions h_1 and h_2 on Ω such that for each $\omega\in\Omega$, $\|x\|\leq h_1(\omega)$ for all $x\in F(\omega)$ and $\|x\|\leq h_2(\omega)$ for all $x\in G(\omega)$. Hence

$$H(F(\omega),G(\omega))\leq H(F(\omega),\{0\})+H(G(\omega),\{0\})\leq h_1(\omega)+h_2(\omega)$$

for each $\omega\in\Omega$. Therefore $H(F,G)$ is integrable on Ω. Since $F:\Omega\to CWK(X)$ and $G:\Omega\to CWK(X)$ are Birkhoff integrable on Ω, $j\circ F$ and $j\circ G$ are Birkhoff integrable on Ω and there exist $(B)\int_\Omega Fd\mu$, $(B)\int_\Omega Gd\mu$ $\in CWK(X)$ such that $j((B)\int_\Omega Fd\mu)=(B)\int_\Omega j\circ Fd\mu$ and $j((B)\int_\Omega Gd\mu)$ $=(B)\int_\Omega j\circ Gd\mu$. Since X is separable, by [3, Proposition 3.2] $(B)\int_\Omega Fd\mu$ $=(P)\int_\Omega Fd\mu$ and $(B)\int_\Omega Gd\mu=(P)\int_\Omega Gd\mu$. Hence

$$\begin{aligned}
H\left((B)\int_\Omega Fd\mu,(B)\int_\Omega Gd\mu\right) &= \left\|j((B)\int_\Omega Fd\mu)-j((B)\int_\Omega Gd\mu)\right\|_\infty\\
&=\sup_{x^*\in B_{X^*}}\left|[j((B)\int_\Omega Fd\mu)](x^*)-[j((B)\int_\Omega Gd\mu)](x^*)\right|\\
&=\sup_{x^*\in B_{X^*}}\left|s(x^*,(B)\int_\Omega Fd\mu)-s(x^*,(B)\int_\Omega Gd\mu)\right|\\
&=\sup_{x^*\in B_{X^*}}\left|s(x^*,(P)\int_\Omega Fd\mu)-s(x^*,(P)\int_\Omega Gd\mu)\right|\\
&=\sup_{x^*\in B_{X^*}}\left|\int_\Omega s(x^*,F)d\mu-\int_\Omega s(x^*,G)d\mu\right|\\
&\leq\sup_{x^*\in B_{X^*}}\int_\Omega |s(x^*,F)-s(x^*,G)|\,d\mu\\
&\leq\int_\Omega\sup_{x^*\in B_{X^*}}|s(x^*,F)-s(x^*,G)|\,d\mu\\
&=\int_\Omega H(F,G)d\mu
\end{aligned}$$

□

A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *integrably bounded* if there exists an integrable real-valued function h on Ω such that for each $\omega \in \Omega$, $\|x\| \leq h(\omega)$ for all $x \in \tilde{F}^0(\omega)$, where $\tilde{F}^0(\omega) = cl\left(\cup_{0<r\leq 1}\tilde{F}^r(\omega)\right)$.

THEOREM 3.8. *Let X be separable. If $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ are measurable, integrably bounded and Birkhoff integrable fuzzy mappings, then $D(\tilde{F}, \tilde{G})$ is integrable on Ω and*

$$D\left((B)\int_\Omega \tilde{F}d\mu, (B)\int_\Omega \tilde{G}d\mu\right) \leq \int_\Omega D(\tilde{F}, \tilde{G})d\mu.$$

Proof. Since $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ are measurable, there exist Castaing representations (f_n^r) and (g_n^r) for $\tilde{F}^r$ and $\tilde{G}^r$ for each $r \in (0, 1]$. Since f_n^r and g_n^r are measurable for all $n \in \mathbb{N}$,

$$\begin{aligned} &H(\tilde{F}^r(\omega), \tilde{G}^r(\omega)) \\ &\qquad = \max\left(\sup_{n\geq 1}\inf_{k\geq 1}\|f_n^r(\omega) - g_k^r(\omega)\|, \sup_{n\geq 1}\inf_{k\geq 1}\|g_n^r(\omega) - f_k^r(\omega)\|\right) \end{aligned}$$

is measurable for each $r \in (0, 1]$. Hence $D(\tilde{F}(\omega), \tilde{G}(\omega)) = \sup_{k\geq 1} H(\tilde{F}^{r_k}(\omega), \tilde{G}^{r_k}(\omega))$ is measurable, where $\{r_k : k \in \mathbb{N}\}$ is dense in $(0, 1]$. Since $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ are integrably bounded, there exist integrable real-valued functions h_1 and h_2 on Ω such that for each $\omega \in \Omega$, $\|x\| \leq h_1(\omega)$ for all $x \in \tilde{F}^0(\omega)$ and $\|x\| \leq h_2(\omega)$ for all $x \in \tilde{G}^0(\omega)$. Hence

$$D(\tilde{F}(\omega), \tilde{G}(\omega)) \leq D(\tilde{F}(\omega), \tilde{0}) + D(\tilde{G}(\omega), \tilde{0}) \leq h_1(\omega) + h_2(\omega)$$

for each $\omega \in \Omega$. Therefore $D(\tilde{F}, \tilde{G})$ is integrable on Ω. By Lemma 3.7

$$H\left((B)\int_\Omega \tilde{F}^r d\mu, (B)\int_\Omega \tilde{G}^r d\mu\right) \leq \int_\Omega H(\tilde{F}^r, \tilde{G}^r)d\mu$$

for each $r \in (0,1]$. Hence

$$\begin{aligned} D\left((B)\int_\Omega \tilde{F}d\mu, (B)\int_\Omega \tilde{G}d\mu\right) &= \sup_{r\in(0,1]} H\left(\left[(B)\int_\Omega \tilde{F}d\mu\right]^r, \left[(B)\int_\Omega \tilde{G}d\mu\right]^r\right) \\ &= \sup_{r\in(0,1]} H\left((B)\int_\Omega \tilde{F}^r d\mu, (B)\int_\Omega \tilde{G}^r d\mu\right) \\ &\le \sup_{r\in(0,1]} \int_\Omega H(\tilde{F}^r, \tilde{G}^r)d\mu \\ &\le \int_\Omega \sup_{r\in(0,1]} H(\tilde{F}^r, \tilde{G}^r)d\mu \\ &= \int_\Omega D(\tilde{F}, \tilde{G})d\mu. \end{aligned}$$

□

THEOREM 3.9. *Let $F_n : \Omega \to CWK(X)$ be a Birkhoff integrable set-valued mapping for each $n \in \mathbb{N}$ and let $F : \Omega \to CWK(X)$. If (F_n) converges uniformly to F on Ω, then $F : \Omega \to CWK(X)$ is Birkhoff integrable on Ω and*

$$\lim_{n\to\infty} (B)\int_\Omega F_n d\mu = (B)\int_\Omega F d\mu.$$

Proof. Since $F_n : \Omega \to CWK(X)$ is Birkhoff integrable on Ω for each $n \in \mathbb{N}$, $j \circ F_n$ is Birkhoff integrable on Ω and there exists $(B)\int_\Omega F_n d\mu \in CWK(X)$ such that $j((B)\int_\Omega F_n d\mu) = (B)\int_\Omega j \circ F_n d\mu$ for each $n \in \mathbb{N}$. Since (F_n) converges uniformly to F on Ω, $(j \circ F_n)$ also converges uniformly to $j \circ F$ on Ω. By [1, Theorem 4] $j \circ F$ is Birkhoff integrable on Ω and $\lim_{n\to\infty} (B)\int_\Omega j \circ F_n d\mu = (B)\int_\Omega j \circ F d\mu$. Hence $F : \Omega \to CWK(X)$ is Birkhoff integrable on Ω and

$$\lim_{n\to\infty} H\left((B)\int_\Omega F_n d\mu, (B)\int_\Omega F d\mu\right)$$
$$= \lim_{n\to\infty}\left\| j\left((B)\int_\Omega F_n d\mu\right) - j\left((B)\int_\Omega F d\mu\right)\right\|_\infty$$
$$= \lim_{n\to\infty}\left\|\int_\Omega j\circ F_n d\mu - \int_\Omega j\circ F d\mu\right\|_\infty = 0.$$

Thus $\lim_{n\to\infty}(B)\int_\Omega F_n d\mu = (B)\int_\Omega F d\mu$.

□

A set-valued mapping $F : \Omega \to CWK(X)$ is said to be *absolutely Birkhoff integrable* on Ω if the composition $j\circ F : \Omega \to \ell_\infty(B_{X^*})$ is absolutely Birkhoff integrable on Ω.

From [1, Theorem 7] and [1, Corollary 8], we can obtain the following two theorems using the same method in the Theorem 3.9.

Theorem 3.10. *Let $F_n : \Omega \to CWK(X)$ be a Birkhoff integrable set-valued mapping for each $n \in \mathbb{N}$ and let $F : \Omega \to CWK(X)$ be a set-valued mapping such that (F_n) converges to F almost uniformly on Ω. If there exists an integrable real-valued function h on Ω such that $\|F_n(\omega)\| \le h(\omega)$ for all $n \in \mathbb{N}$ and almost all $\omega \in \Omega$, then $F : \Omega \to CWK(X)$ is absolutely Birkhoff integrable on Ω and*

$$\lim_{n\to\infty}(B)\int_\Omega F_n d\mu = (B)\int_\Omega F d\mu.$$

Theorem 3.11. *Let $F_n : \Omega \to CWK(X)$ be a Birkhoff integrable set-valued mapping such that $j\circ F_n$ is measurable for each $n \in \mathbb{N}$ and let $F : \Omega \to CWK(X)$ be a set-valued mapping such that (F_n) converges to F almost everywhere on Ω. If there exists an integrable real-valued function h on Ω such that $\|F_n(\omega)\| \le h(\omega)$ for all $n \in \mathbb{N}$ and almost all $\omega \in \Omega$, then $F : \Omega \to CWK(X)$ is absolutely Birkhoff integrable on Ω and*

$$\lim_{n\to\infty}(B)\int_\Omega F_n d\mu = (B)\int_\Omega F d\mu.$$

$\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *j-measurable* if $j\circ \tilde{F}^r : \Omega \to \ell_\infty(B_{X^*})$ is measurable for each $r \in (0, 1]$.

THEOREM 3.12. *Let X be separable and let $\tilde{F}_n : \Omega \to \mathcal{F}(X)$ be a j-measurable and Birkhoff integrable fuzzy mapping for each $n \in \mathbb{N}$. If $(\tilde{F}_n)$ converges to $\tilde{F} : \Omega \to \mathcal{F}(X)$ on Ω and there exists an integrable real-valued function h on Ω such that $\|\tilde{F}_n^0(\omega)\| \leq h(\omega)$ on Ω for all $n \in \mathbb{N}$, then $\tilde{F} : \Omega \to \mathcal{F}(X)$ is Birkhoff integrable on Ω and*

$$\lim_{n\to\infty} (B)\int_\Omega \tilde{F}_n d\mu = (B)\int_\Omega \tilde{F} d\mu.$$

Proof. Since $(\tilde{F}_n)$ converges to $\tilde{F}$ on Ω, for each $\epsilon > 0$ and $\omega \in \Omega$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow D(\tilde{F}_n(\omega), \tilde{F}(\omega)) < \epsilon$. Hence

$$\begin{aligned}\|\tilde{F}^0(\omega)\| &= D(\tilde{F}(\omega), \tilde{0}) \leq D(\tilde{F}(\omega), \tilde{F}_N(\omega)) + D(\tilde{F}_N(\omega), \tilde{0}) \\ &< \|\tilde{F_N}^0(\omega)\| + \epsilon \leq h(\omega) + \epsilon\end{aligned}$$

for each $\omega \in \Omega$. Since $\epsilon > 0$ is arbitrary, $\|\tilde{F}^0(\omega)\| \leq h(\omega)$ on Ω. Thus $\tilde{F} : \Omega \to \mathcal{F}(X)$ is integrably bounded. Since $\tilde{F}_n : \Omega \to \mathcal{F}(X)$ is Birkhoff integrable on Ω for each $n \in \mathbb{N}$, $\tilde{F}_n^r : \Omega \to CWK(X)$ is Birkhoff integrable on Ω for each $n \in \mathbb{N}$ and $r \in (0,1]$. Since $\tilde{F}_n : \Omega \to \mathcal{F}(X)$ is j-measurable for each $n \in \mathbb{N}$, $j \circ \tilde{F}_n^r : \Omega \to \ell_\infty(B_{X^*})$ is measurable for each $n \in \mathbb{N}$ and $r \in (0,1]$. Since $(\tilde{F}_n)$ converges to $\tilde{F}$ on Ω, $(\tilde{F}_n^r)$ converges to $\tilde{F}^r$ on Ω for each $r \in (0,1]$. Since $\|\tilde{F_n}^0(\omega)\| \leq h(\omega)$ on Ω for each $n \in \mathbb{N}$, $\|\tilde{F_n}^r(\omega)\| \leq h(\omega)$ on Ω for each $r \in (0,1]$ and $n \in \mathbb{N}$. By Theorem 3.11, $\tilde{F}^r : \Omega \to CWK(X)$ is Birkhoff integrable on Ω for each $r \in (0,1]$. Let $A \in \Sigma$. Then there exists $M_r \in CWK(X)$ such that $M_r = (B)\int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. For $r_1, r_2 \in (0,1]$ with $r_1 < r_2$, $\tilde{F}^{r_1}(\omega) \supseteq \tilde{F}^{r_2}(\omega)$ for each $\omega \in \Omega$. By Lemma 3.5 $M_{r_1} = (B)\int_A \tilde{F}^{r_1} d\mu \supseteq (B)\int_A \tilde{F}^{r_2} d\mu = M_{r_2}$. Let $r \in (0,1]$ and (r_n) be a sequence in $(0,1]$ such that $r_1 \leq r_2 \leq r_3 \leq \cdots$ and $\lim_{n\to\infty} r_n = r$. Then $\tilde{F}^r(\omega) = \cap_{n=1}^\infty \tilde{F}^{r_n}(\omega)$ on Ω. By [10, Lemma 4.2], $\lim_{n\to\infty} s(x^*, \tilde{F}^{r_n}(\omega)) = s(x^*, \tilde{F}^r(\omega))$ on Ω for each $x^* \in X^*$. Hence $\lim_{n\to\infty}(j \circ \tilde{F}^{r_n})(\omega) = (j \circ \tilde{F}^r)(\omega)$ on Ω. Since $\|(j \circ \tilde{F}^{r_n})(\omega)\|_\infty = \|\tilde{F}^{r_n}(\omega)\| \leq \|\tilde{F}^0(\omega)\| \leq h(\omega)$ on Ω for each $n \in \mathbb{N}$, by [1, Corollary 8] $j \circ \tilde{F}^r : \Omega \to \ell_\infty(B_{X^*})$ is Birkhoff integrable on Ω and

$\lim_{n\to\infty}(B)\int_A j\circ\tilde{F}^{r_n}d\mu=(B)\int_A j\circ\tilde{F}^r d\mu$. For each $x^*\in B_{X^*}$,

$$\begin{aligned}|s(x^*,M_{r_n})-s(x^*,M_r)| &= \left|s(x^*,(B)\int_A\tilde{F}^{r_n}d\mu)-s(x^*,(B)\int_A\tilde{F}^r d\mu)\right|\\ &= \left|[j((B)\int_A\tilde{F}^{r_n}d\mu)](x^*)-[j((B)\int_A\tilde{F}^r d\mu)](x^*)\right|\\ &= \left|[(B)\int_A j\circ\tilde{F}^{r_n}d\mu](x^*)-[(B)\int_A j\circ\tilde{F}^r d\mu](x^*)\right|\\ &\le \left\|(B)\int_A j\circ\tilde{F}^{r_n}d\mu-(B)\int_A j\circ\tilde{F}^r d\mu\right\|_\infty\to 0 \text{ as } n\to\infty.\end{aligned}$$

Thus $\lim_{n\to\infty}s(x^*,M_{r_n})=s(x^*,M_r)$ for each $x^*\in B_{X^*}$ and so $\lim_{n\to\infty}s(x^*,M_{r_n})=s(x^*,M_r)$ for each $x^*\in X^*$. By [10, Lemma 4.2], $M_r=\cap_{n=1}^\infty M_{r_n}$. Let $M_0=X$. By [10, Lemma 4.1], there exists $u_A\in\mathcal{F}(X)$ such that $[u_A]^r=M_r=(B)\int\tilde{F}^r d\mu$ for each $r\in(0,1]$. Hence $\tilde{F}:\Omega\to\mathcal{F}(X)$ is Birkhoff integrable on Ω. By Theorem 3.8 and the Lebesgue Convergence Theorem,

$$D\left((B)\int_\Omega\tilde{F}_n d\mu,(B)\int_\Omega\tilde{F}d\mu\right)\le\int_\Omega D(\tilde{F}_n,\tilde{F})d\mu\to 0 \text{ as } n\to\infty.$$

Thus $\lim_{n\to\infty}(B)\int_\Omega\tilde{F}_n d\mu=(B)\int_\Omega\tilde{F}d\mu$.

□

References

[1] M. Balcerzak and M. Potyrala, *Convergence theorems for the Birkhoff integral*, Czech. Math. J. **58** (2008), 1207-1219.

[2] G. Birkhoff, *Integration of functions with values in a Banach space* Trans. Amer. Math. Soc. **38** (1935), 357–378.

[3] B. Cascales and J. Rodriguez, *Birkhoff integral for multi-valued functions*, J. Math. Anal. Appl. **297** (2004), 540-560.

[4] B. Cascales and J. Rodriguez, *The Birkhoff integral and the property of Bourgain*, Math. Ann. **331** (2005), 259–279.

[5] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math., Vol. 580 Springer-Verlag Berlin, 1977.

[6] K. El Amri and C. Hess, *On the Pettis integral of closed valued multifunctions*, Set-Valued Anal. **8** (2000), 329–360.

[7] V. Marraffa, *A charactirization of absolutely summing operators by means of McShane integrable functions*, J. Math. Anal. Appl. **293** (2004), 71–78.
[8] B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. **44** (1938), 277–304.
[9] J. Rodriguez, *On the existence of Pettis integrable functions which are not Birkhoff integrable*, Proc. Amer. Math. Soc. **133** (2005), 1157–1163.
[10] X. Xue, M. Ha and M. Ma, *Random fuzzy number integrals in Banach spaces*, Fuzzy Sets and Systems **66** (1994), 97–111.
[11] X. Xue, X. Wang and L. Wu , *On the convergence and representation of random fuzzy number integrals*, Fuzzy Sets and Systems **103** (1999), 115–125.

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