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ON RIGHT S-$(1,\mathcal{P})$-ABSORBING IDEALS IN NONCOMMUTATIVE RINGS

Alaa Abouhalaka* and Nico Groenewald

Abstract. We introduce and investigate the class of right S-$(1,\mathcal{P})$-absorbing ideals in noncommutative rings, which extends the notion of strongly S-1-absorbing primary ideals from the commutative setting. An ideal K of a ring R disjoint from an m-system S is called right S-$(1,\mathcal{P})$-absorbing if, whenever $a,b,c \in R$ are nonunits with $aRbRc \subseteq K$, then either $ab\langle s\rangle \subseteq K$ or $c\langle s\rangle \subseteq \mathcal{P}(R)$ for some $s \in S$. We establish fundamental properties of these ideals, explore their connections with right S-prime and S-$\mathcal{P}$-ideals, and present illustrative examples. Structural results concerning localization, homomorphisms, and trivial ring extensions are obtained. In particular, we show conditions under which right S-$(1,\mathcal{P})$-absorbing ideals coincide with S-primary ideals, and we characterize their behavior in local rings. These results demonstrate how right S-$(1,\mathcal{P})$-absorbing ideals provide a natural framework unifying several prime-like generalizations.

1. Introduction

Let R be a commutative ring with unity $(1 \neq 0)$, and let S be a multiplicative subset of R. According to [13, 16], a proper ideal I of R with $I \cap S = \emptyset$ is called an S-1-*absorbing primary ideal* of R associated to $s \in S$ if, whenever $abc \in I$ for some nonunit elements $a,b,c \in R$, one has either $sab \in I$ or $sc \in \sqrt{I}$. If the same condition holds with $sc \in \sqrt{(0)}$, then I is called a *strongly* S-1-*absorbing primary ideal* associated to s. These notions extend the ideas of S-prime and S-primary ideals that were studied in [10, 15]. For instance, in [15], a proper ideal P disjoint from S is called S-*primary* if there exists $s \in S$ such that for all $a,b \in R$, the condition $ab \in P$ implies $sa \in P$ or $sb \in \sqrt{P}$. Moreover, the strongly S-1-absorbing primary ideal generalizes the concept of strongly 1-absorbing primary ideals introduced in [7].

In the noncommutative setting, analogous generalizations have been investigated. In [11], the notion of *1-absorbing prime ideals* was introduced: a proper ideal K of a noncommutative ring with identity R is called a 1-absorbing prime ideal if, for all nonunit elements $a,b,c \in R$, the condition $aRbRc \subseteq K$ implies either $ab \in K$ or $c \in K$. Further generalizations appear in [12], where the concept of n-ideals was extended to any special radical ρ. In particular, a proper ideal K is called a ρ-*ideal*

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∗ Corresponding author.

if for all $a, b \in R$, the condition $aRb \subseteq K$ forces $a \in K$ or $b \in \rho(R)$. Specializing ρ to the prime radical $\mathcal{P}(R)$ yields the definition of a $(1,\mathcal{P})$-*absorbing ideal*: whenever $a, b, c \in R$ are nonunits with $aRbRc \subseteq K$, then $ab \in K$ or $c \in \mathcal{P}(R)$. Thus $(1,\mathcal{P})$-absorbing ideals unify the notions of 1-absorbing and $\mathcal{P}$-ideals.

The introduction of m-systems enriches this theory further. Given an m-system $S \subseteq R$, new classes of generalized prime-like ideals naturally arise. For example, [1,2] define a right S-prime ideal P corresponding to $s \in S$ as an ideal disjoint from S such that, for all ideals A, B with $AB \subseteq P$, either $A\langle s\rangle \subseteq P$ or $B\langle s\rangle \subseteq P$. In [9], the concept of an S-$\mathcal{P}$-ideal was introduced: a proper ideal K disjoint from S is called an S-$\mathcal{P}$-ideal if, whenever $AB \subseteq K$ for ideals A, B of R, one has either $A\langle s\rangle \subseteq K$ or $B\langle s\rangle \subseteq \mathcal{P}(R)$ for some fixed $s \in S$. These developments highlight the active progress in extending commutative prime-like behavior to the noncommutative framework [3,4,9].

In this paper, we introduce and investigate the class of *right* S-$(1,\mathcal{P})$-*absorbing ideals* in noncommutative rings. This concept generalizes strongly S-1-absorbing primary ideals from the commutative case. Explicitly, a proper ideal K of R disjoint from S is called right S-$(1,\mathcal{P})$-absorbing if, for all nonunit elements $a, b, c \in R$, the inclusion $aRbRc \subseteq K$ implies either $ab\langle s\rangle \subseteq K$ or $c\langle s\rangle \subseteq \mathcal{P}(R)$ for some $s \in S$.

Our study proceeds as follows. In Section 2 we establish fundamental properties of right S-$(1,\mathcal{P})$-absorbing ideals and clarify their relationship with other classes of ideals through examples. We prove in Theorem 2.11 that if a ring R admits a right S-$(1,\mathcal{P})$-absorbing ideal I that is not a right S-rad(I)-ideal, then R is local. Moreover, Lemma 2.12 shows that if I is an ideal of R with $\mathcal{P}(R) \subseteq I$ and R is a local ring with unique maximal ideal M, then:

1. if I is a right S-$(1,\mathcal{P})$-absorbing ideal associated to some $s \in S$ that is not a right S-rad(I)-ideal, then $M^2 \subseteq I \subsetneq M$;
2. if $M^2 \subseteq I \subsetneq M$, then I is a right S-$(1,\mathcal{P})$-absorbing ideal associated to some $s \in S$.

Furthermore, we establish structural results on how the right S-$(1,\mathcal{P})$-absorbing property behaves under ring homomorphisms and trivial ring extensions. In Theorem 2.26 we prove that if I is an ideal of R disjoint from S with $\mathcal{P}(R) \subseteq I$, and $s \in S$ is central, then I is a right S-$(1,\mathcal{P})$-absorbing ideal associated to s if and only if

1. I is a right S-$(1, \mathrm{rad}(I))$-absorbing ideal associated to s, and
2. $\mathrm{rad}(I : \langle s\rangle)^* = (\mathcal{P}(R) : \langle s\rangle)$.

As an immediate consequence (Corollary 2.27), if I is a right S-prime ideal corresponding to $s \in S$ and satisfies the above conditions, then I is right S-$(1,\mathcal{P})$-absorbing associated to s precisely when

$$(I : \langle s\rangle) = (\mathcal{P}(R) : \langle s\rangle).$$

Throughout the paper all rings are assumed to be noncommutative and unital. In addition, by "ideal" we mean a two-sided proper ideal unless otherwise stated. For an ideal I of R, we denote by $\mathrm{rad}(I)$ the prime radical of I, that is, the intersection of all prime ideals of R containing I. Moreover, $\mathcal{P}(R)$ denotes the prime radical of R, namely, the intersection of all prime ideals of R. It is well known that radicalization is idempotent; in particular, $\mathcal{P}(\mathcal{P}(R)) = \mathcal{P}(R)$.

2. Right S-$(1,\mathcal{P})$-absorbing ideals and their properties

In this section we establish the basic properties of right S-$(1,P)$-absorbing ideals. Our aim is to position them among other generalizations of prime ideals, and to prepare the ground for the structural results in later sections.

Before presenting our main definitions, let us recall that prime-like conditions in noncommutative rings often require the intervention of auxiliary structures such as m-systems. These provide the flexibility to adapt commutative notions to the richer, noncommutative framework. In particular, the study of right S-prime ideals in [1,4,6] illustrates the necessity of employing m-systems instead of multiplicatively closed sets in the noncommutative setting. Motivated by these observations, we now extend the concept by incorporating m-systems, which leads us to the following definition.

Definition 2.1. Let R be a ring and let $S \subseteq R$ be an m-system of R.

1. An ideal K of R with $K \cap S = \varnothing$ is called *right S-$(1,\mathcal{P})$-absorbing associated to $s \in S$* if for all nonunit elements $a,b,c \in R$, the inclusion $aRbRc \subseteq K$ implies that either $ab\langle s\rangle \subseteq K$ or $c\langle s\rangle \subseteq \mathcal{P}(R)$.
2. An ideal K of R with $K \cap S = \varnothing$ is called *right S-$(1,\operatorname{rad}(K))$-absorbing associated to $s \in S$* if for all nonunit elements $a,b,c \in R$, the inclusion $aRbRc \subseteq K$ implies that either $ab\langle s\rangle \subseteq K$ or $c\langle s\rangle \subseteq \operatorname{rad}(K)$.

A more convenient formulation of the right S-$(1,P)$-absorbing property can be given in terms of products of ideals.

Proposition 2.2. *Let R be a ring with identity and let K be an ideal of R disjoint from S. Then K is right S-$(1,\mathcal{P})$-absorbing if and only if for all proper ideals A,B,C of R with $ABC \subseteq K$, we have either $AB\langle s\rangle \subseteq K$ or $C\langle s\rangle \subseteq \mathcal{P}(R)$ for some $s \in S$.*

Proof. Assume first that K is right S-$(1,\mathcal{P})$-absorbing. Let A,B,C be ideals of R with $ABC \subseteq K$, and suppose that $C\langle s\rangle \nsubseteq \mathcal{P}(R)$ for all $s \in S$. Then there exists $c \in C \setminus \mathcal{P}(R)$ such that $c\langle s\rangle \nsubseteq \mathcal{P}(R)$. For any $a \in A$ and $b \in B$, we have $aRbRc \subseteq K$. Since K is right S-$(1,\mathcal{P})$-absorbing and a,b,c are nonunits, it follows that either $c\langle s\rangle \subseteq \mathcal{P}(R)$ (contradicting the choice of c) or $ab\langle s\rangle \subseteq K$. As AB is generated by such products ab, it follows that $AB\langle s\rangle \subseteq K$.

Conversely, suppose that for all proper ideals A,B,C with $ABC \subseteq K$, either $C\langle s\rangle \subseteq \mathcal{P}(R)$ or $AB\langle s\rangle \subseteq K$. Take nonunit elements $a,b,c \in R$ with $aRbRc \subseteq K$. Then $\langle a\rangle\langle b\rangle\langle c\rangle \subseteq K$, so by assumption either $c\langle s\rangle \subseteq \mathcal{P}(R)$ or $ab\langle s\rangle \subseteq K$. Thus K is right S-$(1,\mathcal{P})$-absorbing. □

The next remark clarifies the relationship between our new notion and existing prime-like generalizations in the literature.

Remark 2.3. Let P be an ideal of a ring R, and let S be an m-system of R such that $P \cap S = \emptyset$.

1. P is called $(1,\mathcal{P})$-absorbing if whenever nonunit elements $a,b,c \in R$ and $aRbRc \subseteq P$, then $ab \in P$ or $c \in \mathcal{P}(R)$. It is clear that every $(1,\mathcal{P})$-absorbing ideal is a right S-$(1,\mathcal{P})$-absorbing ideal. However, the converse is not true as we show in Example 2.5.
2. According to Definition 2.2 of [1], the ideal P is called a *right S-prime ideal* if for all ideals A,B of R with $AB \subseteq P$, either $A\langle s\rangle \subseteq P$ or $B\langle s\rangle \subseteq P$. It is straightforward to verify that any right S-prime ideal contained in the prime

radical is a right S-$(1,\mathcal{P})$-absorbing ideal. However, if a right S-prime ideal is not contained in the prime radical, then it need not be right S-$(1,\mathcal{P})$-absorbing.

For instance, take

$$R_0 = \mathbb{Z}_{48}[x], \qquad R = M_n(R_0).$$

Let

$$S = \{s^{2^k} \mid k \geq 0\}, \qquad s = 8(e_{11} + \cdots + e_{nn}) \in R,$$

and consider the ideal

$$I = M_n(\langle 8x \rangle) \subseteq R.$$

By Theorem 10.21 of [14], we have

$$\mathcal{P}(R) = M_n(\mathcal{P}(R_0)),$$

and by Theorem 3 of [8],

$$\mathcal{P}(R_0) = \mathcal{P}(\mathbb{Z}_{48})[x].$$

Since the prime radical of $\mathbb{Z}_{48}$ is $\langle 6 \rangle$, it follows that

$$\mathcal{P}(R_0) = \langle 6 \rangle[x], \qquad \mathcal{P}(R) = M_n(\langle 6 \rangle[x]).$$

Example 1 of [9] shows that I is a right S-prime ideal of R.

It is clear that $8x \notin \langle 6 \rangle[x]$. Hence $\langle 8x \rangle \nsubseteq \langle 6 \rangle[x]$, and therefore

$$I = M_n(\langle 8x \rangle) \nsubseteq M_n(\langle 6 \rangle[x]) = \mathcal{P}(R).$$

Now let

$$A_1 = A_2 = M_n(\langle 2 \rangle), \qquad A_3 = M_n(\langle 4x \rangle).$$

Since $2 \cdot 2 \cdot 4x = 16x \in \langle 8x \rangle$, we obtain

$$A_1 A_2 A_3 \subseteq I.$$

However, for any $s' \in S$, we have

$$A_1 A_2 \langle s' \rangle = M_n(\langle 4 \rangle)\langle s' \rangle \nsubseteq M_n(\langle 8x \rangle) = I,$$

and

$$A_3 \langle s' \rangle = M_n(\langle 4x \rangle)\langle s' \rangle \nsubseteq M_n(\langle 6 \rangle[x]) = \mathcal{P}(R).$$

Thus I fails the right S-$(1,\mathcal{P})$-absorbing condition.

3. According to Definition 2 of [9], the ideal P is called an S-$\mathcal{P}$-ideal if and only if whenever $AB \subseteq P$, either $A\langle s \rangle \subseteq P$ or $B\langle s \rangle \subseteq \mathcal{P}(R)$ for all ideals A, B of R, and for some $s \in S$. Clearly, every S-$\mathcal{P}$-ideal is right S-$(1,\mathcal{P})$-absorbing.
4. According to Definition 3.1 of [5], the ideal K is said to be *right S-primary* if for any pair of ideals I, J of R with $IJ \subseteq K$, it follows that either $I\langle s \rangle \subseteq K$ or $J^n\langle s \rangle \subseteq K$ for some $n \in \mathbb{N}$ and a (fixed) $s \in S$. Let $R = M_2(\mathbb{Z}[x])$, and $K = M_2(\langle 9x \rangle)$. It is shown in Example 3.1 of [5], that K is a right S-primary ideal, where

$$S = \{s^{2^k} : k \geq 0\} = \{s, s^2, s^4, s^8, \ldots\},$$

and $s = 3(e_{11} + e_{22})$. We now show that K is not a right S-$(1,\mathcal{P})$-absorbing ideal. Consider the nonunit elements:

$$a = 3e_{11}, \quad b = 3e_{11}, \quad c = xe_{11}.$$

Then, $aRbRc = 9xfe_{11} \subseteq K$ for some $f \in \mathbb{Z}[x]$.

However,

$$ab\langle s \rangle \nsubseteq K \text{ and } c\langle s \rangle \nsubseteq \mathcal{P}(R) = 0 \quad \text{for all} \quad s \in S.$$

Proposition 2.4. *Let R be a ring, $S \subseteq R$ an m-system, and $I \trianglelefteq R$ an ideal disjoint from S. Define*

$$S_M := \{\, sI_n : s \in S \,\} \subseteq M_n(R), \qquad J := M_n(I) \trianglelefteq M_n(R).$$

If I is right S-$(1,\mathcal{P})$-absorbing in R (associated to some $s \in S$), then J is right S_M-$(1,\mathcal{P})$-absorbing in $M_n(R)$ (associated to sI_n).

Proof. We use the ideal-theoretic characterization of right S-$(1,\mathcal{P})$-absorbing ideals.

Let $A, B, C \trianglelefteq M_n(R)$ be ideals such that $ABC \subseteq J$. Since every ideal of $M_n(R)$ is of the form $M_n(A')$ for some $A' \trianglelefteq R$, there exist ideals $A', B', C' \trianglelefteq R$ such that $A = M_n(A')$, $B = M_n(B')$, $C = M_n(C')$. We have

$$ABC = M_n(A')\,M_n(B')\,M_n(C') = M_n(A'B'C') \subseteq M_n(I),$$

which implies $A'B'C' \subseteq I$ in R.

Since I is right S-$(1,\mathcal{P})$-absorbing in R (associated to $s \in S$), it follows by Proposition 2.2 that either

$$A'B'\langle s\rangle \subseteq I \quad \text{or} \quad C'\langle s\rangle \subseteq \mathcal{P}(R).$$

If $A'B'\langle s\rangle \subseteq I$, then

$$AB\langle sI_n\rangle = M_n(A')\,M_n(B')\,\langle sI_n\rangle = M_n(A'B'\langle s\rangle) \subseteq M_n(I) = J.$$

If $C'\langle s\rangle \subseteq \mathcal{P}(R)$, then

$$C\langle sI_n\rangle = M_n(C')\,\langle sI_n\rangle = M_n(C'\langle s\rangle) \subseteq M_n(\mathcal{P}(R)) = \mathcal{P}(M_n(R)).$$

The equality $\mathcal{P}(M_n(R)) = M_n(\mathcal{P}(R))$ follows from Theorem 10.21 of [14].

Thus J is right S_M-$(1,\mathcal{P})$-absorbing in $M_n(R)$ associated to sI_n. $\square$

To illustrate the distinction between right S-$(1,P)$-absorbing ideals and classical $(1,P)$-absorbing ideals, we present the following example.

Example 2.5. Let $R_0 = \mathbb{Z}_{24}$ and $I_0 = 12\mathbb{Z}_{24}$. Recall that $\mathcal{P}(R_0) = 6\mathbb{Z}_{24}$. Let $S = \{3^n : n \geq 0\}$ be an m-system of R_0 with distinguished element $s = 3$.

Set

$$R = M_2(R_0), \qquad J = M_2(I_0) = M_2(12\mathbb{Z}_{24}) \trianglelefteq R, \qquad S_R = \{3^n I_2 : n \geq 0\} \subseteq R.$$

It is shown in Example 2.2 of [13] that I_0 is strongly S-1-absorbing primary in $\mathbb{Z}_{24}$ (associated to $s = 3$). Hence, by Proposition 2.4, J is a right S_R-$(1,\mathcal{P})$-absorbing ideal of R.

We show that J is not $(1,\mathcal{P})$-absorbing. Consider the nonunit elements

$$A = 2I_2, \qquad B = 2I_2, \qquad C = 3I_2 \in R.$$

Since

$$ARBRC = (2I_2)R(2I_2)R(3I_2) = 12R = M_2(12\mathbb{Z}_{24}) = J,$$

we have $ARBRC \subseteq J$.

However,

$$AB = (2I_2)(2I_2) = 4I_2 \notin J,$$

and

$$C = 3I_2 \notin \mathcal{P}(R) = M_2(\mathcal{P}(R_0)) = M_2(6\mathbb{Z}_{24}).$$

Thus J fails to be $(1,\mathcal{P})$-absorbing.

Recall that for ideals I, J of R,

$$(I : J)^* = \{\, x \in R \mid Jx \subseteq I \,\} \quad \text{and} \quad (I : J) = \{\, x \in R \mid xJ \subseteq I \,\}.$$

PROPOSITION 2.6. *Let I be an ideal of a ring R disjoint from S. Suppose that*

$$(\mathcal{P}(R) : \langle s\rangle) = \mathcal{P}(R)$$

for some $s \in S$. Then the following statements are equivalent:

1. *I is a right S-$(1, \mathcal{P})$-absorbing ideal associated to $s \in S$.*
2. *$(I : \langle s\rangle)$ is a $(1, \mathcal{P})$-absorbing ideal.*

Proof. (2) $\Rightarrow$ (1): Let $a, b, c \in R$ be nonunits with $aRbRc \subseteq I$. Then $aRbRc \subseteq (I : \langle s\rangle)$, and since $(I : \langle s\rangle)$ is $(1, \mathcal{P})$-absorbing, either

$$ab \in (I : \langle s\rangle) \quad \text{or} \quad c \in \mathcal{P}(R).$$

In the first case, $ab\langle s\rangle \subseteq I$. In the second case, $c\langle s\rangle \subseteq \mathcal{P}(R)$. Thus I is right S-$(1, \mathcal{P})$-absorbing.
(1) $\Rightarrow$ (2): Conversely, suppose $a, b, c \in R$ satisfy $aRbRc \subseteq (I : \langle s\rangle)$. Then

$$\langle a\rangle\langle b\rangle\big(\langle c\rangle\langle s\rangle\big) \subseteq I.$$

Since I is right S-$(1, \mathcal{P})$-absorbing, either

$$\langle a\rangle\langle b\rangle\langle s\rangle \subseteq I \quad \text{or} \quad \langle c\rangle\langle s\rangle\langle s\rangle = \langle c\rangle\langle s\rangle^2 \subseteq \mathcal{P}(R).$$

In the first case, $\langle a\rangle\langle b\rangle \subseteq (I : \langle s\rangle)$. In the second case,

$$(\langle c\rangle\langle s\rangle)^2 = \langle c\rangle\langle s\rangle\langle c\rangle\langle s\rangle \subseteq \langle c\rangle\langle s\rangle^2 \subseteq P(R).$$

Since $\mathcal{P}(R)$ is a semiprime ideal, we get $\langle c\rangle\langle s\rangle \subseteq P(R)$, and hence $c \in (\mathcal{P}(R) : \langle s\rangle) = \mathcal{P}(R)$. Thus, $(I : \langle s\rangle)$ is a $(1, \mathcal{P})$-absorbing ideal. □

PROPOSITION 2.7. *Let I be an ideal of a ring R with $I \cap S = \varnothing$, and suppose there exists $s \in S$ such that*

$$(\mathcal{P}(R) : \langle s\rangle) = \mathcal{P}(R).$$

Then the following are equivalent:

(1) I is a right S-$(1, \mathcal{P})$-absorbing ideal of R associated to s.
(2) The following conditions hold:
  (a) $I \subseteq (\mathcal{P}(R) : \langle s\rangle)$.
  (b) For every choice of nonunits $a, b, c \in R$ satisfying $aRbRc \subseteq I$, we have

$$ab \in (I : \langle s\rangle) \quad \textit{or} \quad c \in (\mathrm{rad}(I) : \langle s\rangle).$$

Proof. (1) $\Rightarrow$ (2): (a) Assume, to the contrary, that $I \nsubseteq (\mathcal{P}(R) : \langle s\rangle)$. Then there exists $a \in I \setminus (\mathcal{P}(R) : \langle s\rangle)$. We have

$$\langle s\rangle\langle s\rangle\langle a\rangle \subseteq I.$$

Applying the right S-$(1, \mathcal{P})$-absorbing property, either

$$\langle s\rangle^3 \subseteq I \quad \text{or} \quad \langle a\rangle\langle s\rangle \subseteq \mathcal{P}(R).$$

The first alternative is impossible because $I \cap S = \varnothing$. The second alternative gives $a \in (\mathcal{P}(R) : \langle s\rangle)$, contradicting our choice of a. Therefore, $I \subseteq (\mathcal{P}(R) : \langle s\rangle)$.

(b) Now suppose $a, b, c \in R$ are nonunits with $aRbRc \subseteq I$. By the right S-$(1, \mathcal{P})$-absorbing property, either

$$ab \in (I : \langle s\rangle) \quad \text{or} \quad c \in (\mathcal{P}(R) : \langle s\rangle).$$

Since $I \subseteq (\mathcal{P}(R) : \langle s\rangle)$ from part (a), and since $\mathcal{P}(R) \subseteq \mathrm{rad}(I)$, we have $(\mathcal{P}(R) : \langle s\rangle) \subseteq (\mathrm{rad}(I) : \langle s\rangle)$, so the second condition in (b) follows.

(2) $\Rightarrow$ (1): Assume A, B, C are ideals of R with $ABC \subseteq I$. Suppose, for contradiction, that $C\langle s\rangle \nsubseteq \mathcal{P}(R)$ for every $s \in S$. Then there exists $c \in C \setminus (\mathcal{P}(R) : \langle s\rangle)$. For all $a \in A$ and $b \in B$, we have

$$aRbRc \subseteq ABC \subseteq I.$$

By hypothesis (2b), either

$$ab \in (I : \langle s\rangle) \quad \text{or} \quad c \in (\operatorname{rad}(I) : \langle s\rangle).$$

Since $I \subseteq (\mathcal{P}(R) : \langle s\rangle) = \mathcal{P}(R)$, then $\operatorname{rad}(I) \subseteq \mathcal{P}(\mathcal{P}(R)) = \mathcal{P}(R)$. Because $\mathcal{P}(R) \subseteq \operatorname{rad}(I)$, we obtain $\operatorname{rad}(I) = \mathcal{P}(R)$. The latter alternative therefore implies $c \in \mathcal{P}(R)$, contradicting our choice of c. Hence, the first alternative must hold for all $a \in A$ and $b \in B$, which means $AB\langle s\rangle \subseteq I$. Thus I is right S-$(1,\mathcal{P})$-absorbing. □

Now consider the following condition:

($\star$) For all nonunits $a, b, c \in R$ with $aRbRc \subseteq I \Longrightarrow ab\langle s\rangle \subseteq I$ or $c \in rad(I)$.

Theorem 2.8. *Let K be an ideal of R disjoint from S, and assume there exists $s \in S$ such that*

$$(\mathcal{P}(R) : \langle s\rangle) = \mathcal{P}(R).$$

Then K is a right S-$(1,\mathcal{P})$-absorbing ideal associated with s if and only if one of the following holds:

1. *Condition ($\star$) is satisfied for K and*

$$\operatorname{rad}(K) = (\mathcal{P}(R) : \langle s\rangle),$$

2. *R has a unique maximal ideal M such that*

$$M^2 \subseteq (K : \langle s\rangle).$$

Proof. Suppose K is a right S-$(1,\mathcal{P})$-absorbing ideal. For all nonunits $a, b, c \in R$ with $aRbRc \subseteq K$, either $ab\langle s\rangle \subseteq K$ or $c\langle s\rangle \subseteq \mathcal{P}(R)$. The latter case implies $c \in (\mathcal{P}(R) : \langle s\rangle) = \mathcal{P}(R) \subseteq \operatorname{rad}(K)$. Thus ($\star$) holds. Now if $\operatorname{rad}(K) \neq (\mathcal{P}(R) : \langle s\rangle)$, then we have to show that (2) holds. Since $\mathcal{P}(R) = (\mathcal{P}(R) : \langle s\rangle) \subsetneq \operatorname{rad}(K)$, assume that for some nonunit elements a, b of R, $ab\langle s\rangle \nsubseteq K$ for all $s \in S$. Then for all $x \in K$, we have $aRbRx \subseteq K$. By assumption, $x\langle s\rangle \subseteq \mathcal{P}(R)$, and hence, $K \subseteq (\mathcal{P}(R) : \langle s\rangle) = \mathcal{P}(R)$. Consequently, $\operatorname{rad}(K) \subseteq \mathcal{P}(\mathcal{P}(R)) = \mathcal{P}(R) \subset \operatorname{rad}(K)$, which is a contradiction. Thus, $ab\langle s\rangle \subseteq K$ for all nonunit elements $a, b \in R$. Hence $ab \subseteq (K : \langle s\rangle)$ for all nonunit elements $a, b \in R$. Let M be a maximal ideal of R. We have $M^2 \subseteq (K : \langle s\rangle)$, and hence $M = \operatorname{rad}(M^2) \subseteq \operatorname{rad}(K : \langle s\rangle)$. Thus, $M = \operatorname{rad}(K : \langle s\rangle)$ for each maximal ideal M of R. Consequently, R has a unique maximal ideal $M = \operatorname{rad}(K : \langle s\rangle)$ and $M^2 \subseteq (K : \langle s\rangle)$.

Conversely, let $aRbRc \subseteq K$ for some nonunits $a, b, c \in R$. Suppose $ab\langle s\rangle \nsubseteq K$. If (1) holds, then $c \in \operatorname{rad}(K) = (\mathcal{P}(R) : \langle s\rangle)$, hence $c\langle s\rangle \subseteq \mathcal{P}(R)$. Thus K is a right S-$(1,\mathcal{P})$-absorbing ideal of R. Now, suppose that (2) holds (i.e., R has exactly one maximal ideal M such that $M^2 \subseteq (K : \langle s\rangle)$). Then, for each pair of nonunit elements $x, y \in R$, we have, by Zorn's Lemma, $xy \in \langle x\rangle\langle y\rangle \subseteq M^2 \subseteq (K : \langle s\rangle)$. Thus, K is trivially a right S-$(1,\mathcal{P})$-absorbing ideal of R. □

We consider the idealization $R \boxplus M$ of an R-R-bimodule M, where $(R \boxplus M, +) = (R, +) \oplus (M, +)$ and multiplication is defined by

$$(r, m)(s, n) = (rs, rn + ms).$$

The units of $R \boxplus M$ are precisely the elements of the form (r, m) with $r \in U(R)$. Moreover, if S is an m-system of R, then both $S \boxplus M$ and $S \boxplus \{0\}$ are m-systems of $R \boxplus M$.

THEOREM 2.9. *Let R be a ring, M an R–R-bimodule, and I a proper ideal of R. Suppose S is an m-system of R with $I \cap S = \emptyset$. Then the following statements are equivalent:*

1. *I is a right S-$(1, \mathcal{P})$-*absorbing *ideal of R;*
2. *$I \boxplus M$ is a right $(S \boxplus \{0\})$-$(1, \mathcal{P})$-*absorbing *ideal of $R \boxplus M$;*
3. *$I \boxplus M$ is a right $(S \boxplus M)$-$(1, \mathcal{P})$-*absorbing *ideal of $R \boxplus M$.*

Proof. $(1 \Rightarrow 2)$ Assume I is a right S-$(1, \mathcal{P})$-absorbing ideal of R. Let

$$(a, m)(R \boxplus M)(b, n)(R \boxplus M)(c, p) \subseteq I \boxplus M,$$

for some nonunits $a, b, c \in R$ and $m, n, p \in M$. Then $aRbRc \subseteq I$.

Since I is right S-$(1, \mathcal{P})$-absorbing, there exists $s \in S$ such that either

$$ab\langle s\rangle \subseteq I \quad \text{or} \quad c\langle s\rangle \subseteq \mathcal{P}(R).$$

Case 1: If $ab\langle s\rangle \subseteq I$, then

$$\begin{aligned}(a, m)(b, n)\langle(s, 0)\rangle &= (a, m)(b, n)(RsR,\ MsR + RsM)\\ &= (ab,\ an + mb)(RsR,\ MsR + RsM)\\ &\subseteq (ab\langle s\rangle, M) \subseteq I \boxplus M.\end{aligned}$$

Case 2: If $c\langle s\rangle \subseteq \mathcal{P}(R)$, then

$$(c, p)(RsR,\ MsR + RsM) \subseteq (c\langle s\rangle, M) \subseteq \mathcal{P}(R) \boxplus M = \mathcal{P}(R \boxplus M).$$

Thus, $I \boxplus M$ is a right $(S \boxplus \{0\})$-$(1, \mathcal{P})$-absorbing ideal of $R \boxplus M$.

$(2 \Rightarrow 3)$ This follows immediately since $S \boxplus \{0\} \subseteq S \boxplus M$.

$(3 \Rightarrow 1)$ Suppose $aRbRc \subseteq I$ for some nonunits $a, b, c \in R$. Then

$$(a, 0)(R \boxplus M)(b, 0)(R \boxplus M)(c, 0) \subseteq I \boxplus M.$$

Since $I \boxplus M$ is a right $(S \boxplus M)$-$(1, \mathcal{P})$-absorbing ideal of $R \boxplus M$ and $(a, 0), (b, 0), (c, 0)$ are nonunits, there exists $(s, m) \in S \boxplus M$ such that

$$(a, 0)(b, 0)\langle(s, m)\rangle \subseteq I \boxplus M \quad \text{or} \quad (c, 0)\langle(s, m)\rangle \subseteq \mathcal{P}(R \boxplus M) = \mathcal{P}(R) \boxplus M.$$

Case 1:

$$\begin{aligned}(a, 0)(b, 0)\langle(s, m)\rangle &= (ab, 0)(R \boxplus M)(s, m)(R \boxplus M)\\ &= (ab, 0)(RsR,\ RsM + RmR + MsR)\\ &= (ab\langle s\rangle,\ RsM + RmR + MsR) \subseteq I \boxplus M,\end{aligned}$$

so $ab\langle s\rangle \subseteq I$.

Case 2:

$$(c, 0)\langle(s, m)\rangle \subseteq (c\langle s\rangle, M) \subseteq \mathcal{P}(R \boxplus M) = \mathcal{P}(R) \boxplus M,$$

so $c\langle s\rangle \subseteq \mathcal{P}(R)$.

Therefore, either $ab\langle s\rangle \subseteq I$ or $c\langle s\rangle \subseteq \mathcal{P}(R)$, and hence I is a right S-$(1, \mathcal{P})$-absorbing ideal of R. □

Definition 2.10. Let R be a ring and let $S \subseteq R$ be an m-system of R. An ideal K of R disjoint from S is called a right S-rad(K)-ideal if for all $a, b \in R$, the condition $aRb \subseteq K$ implies that either

$$a\langle s\rangle \subseteq K \quad \text{or} \quad b\langle s\rangle \subseteq \operatorname{rad}(K),$$

for some $s \in S$.

Theorem 2.11. *Let I be a right S-$(1,\mathcal{P})$-absorbing ideal of R associated to some $s \in S$. If I is not a right S-rad(I)-ideal, then R is a local ring.*

Proof. Since I is not a right S-rad(I)-ideal of R associated to s, there exist $a, b \in R$ such that $aRb \subseteq I$ with $a\langle s\rangle \nsubseteq I$ and $b\langle s\rangle \nsubseteq \operatorname{rad}(I)$. Hence $b\langle s\rangle \nsubseteq \mathcal{P}(R)$. It follows that a and b are nonunit elements in R. Suppose that x is a nonunit element and y is a unit element of R. We claim that $x+y$ is a unit element of R, and so the proof follows from [11, Lemma 4.1]. Assume, to the contrary, that $x+y$ is a nonunit element of R. Since I is a right S-$(1,\mathcal{P})$-absorbing ideal and $xRaRb \subseteq I$ with $b\langle s\rangle \nsubseteq \mathcal{P}(R)$, we obtain $xa\langle s\rangle \subseteq I$. But also $(x+y)RaRb \subseteq I$, which implies $(x+y)a\langle s\rangle \subseteq I$. Hence $ya\langle s\rangle \subseteq I$, and since y is a unit, it follows that $a\langle s\rangle \subseteq I$, a contradiction. Therefore $x+y$ is a unit, and the proof follows from [11, Lemma 4.1]. □

The next lemma refines this local characterization by identifying explicit conditions under which an ideal in a local ring is right S-$(1,P)$-absorbing.

Lemma 2.12. *Let R be a ring, and let I be a proper ideal of R such that $I \cap S = \varnothing$ and $\mathcal{P}(R) \subseteq I$. If R is local with unique maximal ideal M, then*

1. If I is a right S-$(1,\mathcal{P})$-absorbing ideal associated to some $s \in S$ of R which is not a right S-rad(I)-ideal, then $M^2 \subseteq I \subsetneq M$.
2. If $M^2 \subseteq I \subsetneq M$, then I is a right S-$(1,\mathcal{P})$-absorbing ideal associated to some $s \in S$ of R.

Proof.

1. Suppose I is a right S-$(1,\mathcal{P})$-absorbing ideal but not a right S-rad(I)-ideal. Since I is not a right S-rad(I)-ideal, we have $I \subsetneq M$ and there exist $a, b \in R$ such that $aRb \subseteq I$ with $a\langle s\rangle \nsubseteq I$ and $b\langle s\rangle \nsubseteq \operatorname{rad}(I)$ for every $s \in S$. If a is a unit in R, then $aRb\langle s\rangle \subseteq I$ forces $b\langle s\rangle \subseteq I \subseteq \operatorname{rad}(I)$, a contradiction. Similarly, if b is a unit, then $a\langle s\rangle = ab^{-1}b\langle s\rangle \subseteq aRb\langle s\rangle \subseteq I$, again a contradiction. Hence, a and b must be nonunits in R. To show $M^2 \subseteq I$, it suffices to prove $xy \in I$ for all $x, y \in M$. Take $x, y \in M$. Then $(xRyR)RaRb \subseteq I$. Since $xRyR \subseteq M$, $a, b \in M$, and $b\langle s\rangle \nsubseteq \mathcal{P}(R)$ while I is right S-$(1,\mathcal{P})$-absorbing, it follows that $(xRyR)a\langle s\rangle \subseteq I$ for some $s \in S$. Again, as I is right S-$(1,\mathcal{P})$-absorbing, we obtain either $xy\langle s\rangle \subseteq I$ or $a\langle s\rangle^2 \subseteq \mathcal{P}(R)$. Suppose $a\langle s\rangle^2 \subseteq \mathcal{P}(R)$. Since S is an m-system, we have $\langle s\rangle^2 \cap S \neq \varnothing$. Take $s_1 \in \langle s\rangle\langle s\rangle \cap S$. Then $a\langle s_1\rangle \subseteq \mathcal{P}(R) \subseteq I$, a contradiction. Hence $xy\langle s\rangle \subseteq I$, giving $M^2\langle s\rangle \subseteq I$. If s is not a unit, then $\langle s\rangle^3 \subseteq M^2\langle s\rangle \subseteq I$. But since $I \cap S = \varnothing$ while $\langle s\rangle^3 \cap S \neq \varnothing$, this is impossible. Thus s must be a unit, and therefore $M^2 \subseteq I$.
2. Suppose $M^2 \subseteq I \subsetneq M$. Then I is proper. Take $a, b, c \in M$ with $aRbRc \subseteq I$. Then $ab\langle s\rangle \subseteq M^2\langle s\rangle \subseteq I$. Thus I is a right S-$(1,\mathcal{P})$-absorbing ideal.

□

The next theorem summarizes several equivalent formulations of the right S-$(1,\mathcal{P})$-absorbing property.

Theorem 2.13. *Let I be a proper ideal of R and disjoint from S. Then the following statements are equivalent.*

(1) I is a right S-$(1,\mathcal{P})$-absorbing ideal of R associated to $s \in S$.
(2) For all nonunit elements $a, b \in R$ such that $ab \notin (I : \langle s\rangle)$, $(I : aRbR)^* \subseteq (\mathcal{P}(R) : \langle s\rangle)$.
(3) For every proper ideal J of R such that $aRbJ \subseteq I$, either $ab\langle s\rangle \subseteq I$ or $J\langle s\rangle \subseteq \mathcal{P}(R)$.
(4) For all proper ideals J, K of R such that $aJK \subseteq I$, either $aJ\langle s\rangle \subseteq I$ or $K\langle s\rangle \subseteq \mathcal{P}(R)$.
(5) For all proper ideals J, K, L of R such that $JKL \subseteq I$, either $JK\langle s\rangle \subseteq I$ or $L\langle s\rangle \subseteq \mathcal{P}(R)$.

Proof. $(1 \Rightarrow 2)$: Let a, b be nonunit elements in R such that $ab \notin (I : \langle s\rangle)$, then $ab\langle s\rangle \nsubseteq I$. Let $c \in (I : aRbR)^*$, then $aRbRc \subseteq I$ implies that $c\langle s\rangle \subseteq \mathcal{P}(R)$ since $ab\langle s\rangle \nsubseteq I$. (note that c is not a unit in R, since if c is a unit, then since $aRbRc \subseteq I$, it implies that $ab\langle s\rangle \subseteq I$, which is a contradiction) Hence $(I : aRbR)^* \subseteq (\mathcal{P}(R) : \langle s\rangle)$.
$(2 \Rightarrow 3)$: Suppose that $aRbJ \subseteq I$. If $ab \notin (I : \langle s\rangle)$, then $(I : aRbR)^* \subseteq (\mathcal{P}(R) : \langle s\rangle)$. Since $J \subseteq (I : aRbR)^*$, we have $J \subseteq (\mathcal{P}(R) : \langle s\rangle)$. Hence $J\langle s\rangle \subseteq \mathcal{P}(R)$.
$(3 \Rightarrow 4)$: Let a be a nonunit element in R such that $aJK \subseteq I$. Suppose $K\langle s\rangle \nsubseteq \mathcal{P}(R)$. We show that $aJ\langle s\rangle \subseteq I$. Let $y \in J$. Hence $aRyK \subseteq I$. If $ay \notin (I : \langle s\rangle)$, then from (3) we have $K\langle s\rangle \subseteq \mathcal{P}(R)$, a contradiction. Thus, $ay\langle s\rangle \subseteq I$. Consequently, $aJ\langle s\rangle \subseteq I$.
$(4 \Rightarrow 5)$: Let J, K, L be proper ideals of R such that $JKL \subseteq I$. Suppose $L\langle s\rangle \nsubseteq \mathcal{P}(R)$. We show that $JK\langle s\rangle \subseteq I$. Let $a \in J$, then $aKL \subseteq I$. Now, by (4) $aK\langle s\rangle \subseteq I$ since $L\langle s\rangle \nsubseteq \mathcal{P}(R)$. Hence, we conclude that $JK\langle s\rangle \subseteq I$.
$(5 \Rightarrow 1)$: By Proposition 2.2. □

Proposition 2.14. *If I and J are right S-$(1,\mathcal{P})$-absorbing ideals of R associated to $s \in S$, then $I \cap J$ is a right S-$(1,\mathcal{P})$-absorbing ideal of R associated to s.*

Proof. The proof is straightforward. □

Proposition 2.15. *Let S be an m-system of a ring R and P be a right S-$(1,\mathcal{P})$-absorbing ideal of R associated to $s \in S$. Let I be an ideal of R meets S, then we have the following:*

(1) PI is a right S-$(1,\mathcal{P})$-absorbing ideal of R.
(2) $I \cap P$ is a right S-$(1,\mathcal{P})$-absorbing ideal of R.

Proof. (1) Let A, B, C be some proper ideals of R such that $ABC \subseteq PI$. Then $ABC \subseteq P$, thus either $AB\langle s\rangle \subseteq P$ or $C\langle s\rangle \subseteq P$. If $AB\langle s\rangle \subseteq P$, then for some $x \in I \cap S$, $ABRsRxR \subseteq PI$. Now, because s and x are in the m-system S, there exists $r \in R$ such that $s_1 = srx$, and hence $AB\langle s_1\rangle \subseteq PI$. If $C\langle s\rangle \subseteq P$, then similarly, we obtain $C\langle s_1\rangle \subseteq PI$. Hence, PI is a right S-$(1,\mathcal{P})$-absorbing ideal of R, since $PI \cap S = \emptyset$.
(2) Similar to (1). □

Lemma 2.16. *If I is a right S-$(1, rad(I))$-absorbing ideal of R associated to $s \in S$ and J is a proper ideal of R such that $J \nsubseteq I$ with $(I : J) \cap S = \emptyset$, then $(I : J)$ is a right S-$\mathrm{rad}(I)$-ideal of R associated to s.*

Proof. $(I : J)$ is a proper ideal of R, since $J \nsubseteq I$. Let $a, b \in R$ such that $aRb \subseteq (I : J)$ with $a\langle s\rangle \nsubseteq (I : J)$. So, b is non-unit in R, since $a\langle s\rangle \nsubseteq (I : J)$. If a is a unit in R, then $b \in (I : J)$ implies $b\langle s\rangle \subseteq (I : J) \subseteq \mathrm{rad}((I : J))$. So, we may assume that a, b are non-units in R. Since $a\langle s\rangle \nsubseteq (I : J)$, there exists $c \in J$ such that $ca\langle s\rangle \nsubseteq I$. As a result, $cRaRb \subseteq I$ implies that $b\langle s\rangle \subseteq \mathrm{rad}(I) \subseteq \mathrm{rad}((I : J))$, since I is a right S-$(1, \mathrm{rad}(I))$-absorbing ideal of R associated to s and $ca\langle s\rangle \nsubseteq I$. Consequently, $(I : J)$ is a right S-$\mathrm{rad}(I)$-ideal of R associated to s. □

Proposition 2.17. *Let R be a ring. If there exists a right S-$(1,\mathcal{P})$-absorbing ideal I of R such that $\mathcal{P}(R) \subseteq I$ and I is not a right S-$\mathrm{rad}(I)$-ideal, then R is a local ring with maximal ideal M such that $M^2 \neq M$.*

Proof. Suppose that there exists a right S-$(1,\mathcal{P})$-absorbing ideal of R that is not a right S-$\mathrm{rad}(I)$-ideal. Then by Theorem 2.11, R is a local ring (with maximal ideal M). By Lemma 2.12, $M^2 \subseteq I \subsetneq M$. Therefore, we have the desired result. □

Theorem 2.18. *Let $f : R \to T$ be a ring epimorphism, $S \subseteq R$ an m-system, and K a right S-$(1,\mathcal{P})$-absorbing ideal of R disjoint from S, with* $\mathrm{Ker}(f) \subseteq K$*. Then $f(K)$ is a right $f(S)$-$(1,\mathcal{P})$-absorbing ideal of T.*

Proof. Since f is an epimorphism, $f(S)$ is an m-system in T. Also, $f(K) \cap f(S) = \emptyset$ because $K \cap S = \emptyset$. Now, let A_2, B_2, C_2 be ideals of T such that

$$A_2B_2C_2 \subseteq f(K).$$

Since f is surjective, there exist ideals A_1, B_1, C_1 of R such that $f(A_1) = A_2$, $f(B_1) = B_2$, and $f(C_1) = C_2$, and $\mathrm{Ker}(f) \subseteq A_1 \cap B_1 \cap C_1$. Then

$$f(A_1B_1C_1) = A_2B_2C_2 \subseteq f(K).$$

Hence,

$$A_1B_1C_1 \subseteq f^{-1}(f(A_1B_1C_1)) \subseteq f^{-1}(f(K)) = K + \mathrm{Ker}(f) = K,$$

since $\mathrm{Ker}(f) \subseteq K$. Because K is right S-$(1,\mathcal{P})$-absorbing, by Proposition 2.2, there exists $s \in S$ such that either

$$A_1B_1\langle s\rangle \subseteq K \quad \text{or} \quad C_1\langle s\rangle \subseteq \mathcal{P}(R).$$

If $A_1B_1\langle s\rangle \subseteq K$, then

$$A_2B_2\langle f(s)\rangle = f(A_1B_1\langle s\rangle) \subseteq f(K).$$

If $C_1\langle s\rangle \subseteq \mathcal{P}(R)$, then since $f(\mathcal{P}(R)) \subseteq \mathcal{P}(T)$ (as f is a surjective ring homomorphism),

$$C_2\langle f(s)\rangle = f(C_1\langle s\rangle) \subseteq f(\mathcal{P}(R)) \subseteq \mathcal{P}(T).$$

Therefore, $f(K)$ is right $f(S)$-$(1,\mathcal{P})$-absorbing in T associated to $f(s) \in f(S)$. □

Corollary 2.19. *Let $f : R \to T$ be a ring epimorphism, $S \subseteq R$ an m-system, and K an ideal of T such that $f^{-1}(K)$ is a right S-$(1,\mathcal{P})$-absorbing ideal of R. Then K is a right $f(S)$-$(1,\mathcal{P})$-absorbing ideal of T.*

Proof. Since $f^{-1}(K)$ is a right S-$(1,\mathcal{P})$-absorbing ideal of R and $\mathrm{Ker}(f) \subseteq f^{-1}(K)$, by Theorem 2.18, $f(f^{-1}(K)) = K$ is a right $f(S)$-$(1,\mathcal{P})$-absorbing ideal of T. □

Theorem 2.20. *Let R be a ring, $I \subseteq K$ be ideals of R, and $S \subseteq R$ be an m-system disjoint from K. Let $\pi : R \to R/I$ denote the canonical projection. If K is a right S-$(1,\mathcal{P})$-absorbing ideal of R, then K/I is a right $\pi(S)$-$(1,\overline{\mathcal{P}})$-absorbing ideal of R/I, where $\overline{\mathcal{P}}$ denotes the radical class in R/I corresponding to $\mathcal{P}$.*

Proof. Since $S \cap K = \emptyset$, we have $\pi(S) \cap (K/I) = \emptyset$. The map π is a surjective ring homomorphism with $\mathrm{Ker}(\pi) = I \subseteq K$. The result follows directly by applying Theorem 2.18 on the images of right S-$(1, \mathcal{P})$-absorbing ideals under epimorphisms. $\square$

THEOREM 2.21. *Let $f : R \to T$ be a surjective ring homomorphism. Let $S \subseteq R$ be an m-system and let K be an ideal of R such that $\mathrm{Ker}(f) \subseteq K \cap \mathcal{P}(R)$. If the image $f(K)$ is a right $f(S)$-$(1, \mathcal{P})$-absorbing ideal in T, then K is a right S-$(1, \mathcal{P})$-absorbing ideal in R.*

Proof. Assume that $f(K)$ is a right $f(S)$-$(1, \mathcal{P})$-absorbing ideal in T. To prove that K is right S-$(1, \mathcal{P})$-absorbing, we use the ideal-wise characterization (Proposition 2.2).

Let A, B, C be ideals of R such that $ABC \subseteq K$. Applying the surjective homomorphism f, we obtain

$$f(A)f(B)f(C) \;=\; f(ABC) \;\subseteq\; f(K).$$

Since $f(K)$ is right $f(S)$-$(1, \mathcal{P})$-absorbing in T, there exists an element $f(s) \in f(S)$ (for some $s \in S$) such that either

(i) $f(A)f(B)\langle f(s)\rangle \subseteq f(K)$, or
(ii) $f(C)\langle f(s)\rangle \subseteq \mathcal{P}(T)$.

Case (i). From $f(A)f(B)\langle f(s)\rangle \subseteq f(K)$ we get

$$f(AB\langle s\rangle) \;=\; f(A)f(B)\langle f(s)\rangle \;\subseteq\; f(K).$$

Hence

$$AB\langle s\rangle \;\subseteq\; f^{-1}(f(AB\langle s\rangle)) \;\subseteq\; f^{-1}(f(K)) \;=\; K + \mathrm{Ker}(f) \;=\; K,$$

since $\mathrm{Ker}(f) \subseteq K$. Therefore $AB\langle s\rangle \subseteq K$.

Case (ii). From $f(C)\langle f(s)\rangle \subseteq \mathcal{P}(T)$ we have

$$f(C\langle s\rangle) \;=\; f(C)\langle f(s)\rangle \;\subseteq\; \mathcal{P}(T).$$

Taking preimages and using surjectivity, we obtain

$$C\langle s\rangle \;\subseteq\; f^{-1}\big(\mathcal{P}(T)\big).$$

Identify $T \cong R/\mathrm{Ker}(f)$ via f. For the prime radical we have

$$\mathcal{P}(R/\mathrm{Ker}(f)) \;=\; \frac{\mathcal{P}(R) + \mathrm{Ker}(f)}{\mathrm{Ker}(f)},$$

hence

$$f^{-1}\big(\mathcal{P}(T)\big) \;=\; f^{-1}\big(\mathcal{P}(R/\mathrm{Ker}(f))\big) \;=\; \mathcal{P}(R) + \mathrm{Ker}(f).$$

Since $\mathrm{Ker}(f) \subseteq \mathcal{P}(R)$ by hypothesis, it follows that

$$f^{-1}\big(\mathcal{P}(T)\big) \;=\; \mathcal{P}(R).$$

Therefore $C\langle s\rangle \subseteq \mathcal{P}(R)$.

In either case, there exists $s \in S$ such that $AB\langle s\rangle \subseteq K$ or $C\langle s\rangle \subseteq \mathcal{P}(R)$. By Proposition 2.2, K is a right S-$(1, \mathcal{P})$-absorbing ideal of R. $\square$

COROLLARY 2.22. *Let $f : R \to T$ be a surjective ring homomorphism and let $S \subseteq R$ be an m-system. Suppose Q is a right $f(S)$-$(1, \mathcal{P})$-absorbing ideal of T. If $\mathrm{Ker}(f) \subseteq \mathcal{P}(R)$, then the preimage $f^{-1}(Q)$ is a right S-$(1, \mathcal{P})$-absorbing ideal of R.*

Proof. Set $K := f^{-1}(Q)$. Since f is surjective we have $f(K) = Q$, and clearly $\mathrm{Ker}(f) \subseteq K$. By hypothesis $\mathrm{Ker}(f) \subseteq \mathcal{P}(R)$, hence $\mathrm{Ker}(f) \subseteq K \cap \mathcal{P}(R)$. Therefore the hypotheses of Theorem 2.21 are satisfied for K, and that theorem yields that $K = f^{-1}(Q)$ is a right S-$(1,\mathcal{P})$-absorbing ideal of R. □

THEOREM 2.23. *Let $R = R_1 \times R_2$ be a direct product of unital rings, let $S = S_1 \times S_2$ with each $S_i \subseteq R_i$ an m-system, and let $K = K_1 \times K_2$ be a proper ideal of R with $K_1 \cap S_1 = \varnothing$. Assume additionally that $S_2 \cap \mathcal{P}(R_2) = \varnothing$. Then there is* no *right S-$(1,\mathcal{P})$-absorbing ideal of the form $K_1 \times K_2$ in R.*

Proof. Suppose, towards a contradiction, that $K = K_1 \times K_2$ is right S-$(1,\mathcal{P})$-absorbing. Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Set

$$a = e_1, \qquad b = e_1, \qquad c = e_2.$$

Then a, b, c are nonunits in R, and a direct calculation in the product shows

$$aRbRc = \{0\} \times \{0\} \subseteq K.$$

By the right S-$(1,\mathcal{P})$-absorbing property, there exists $s = (s_1, s_2) \in S_1 \times S_2$ such that either

$$ab\langle s\rangle \subseteq K \quad \text{or} \quad c\langle s\rangle \subseteq \mathcal{P}(R_1) \times \mathcal{P}(R_2).$$

Compute in components:

$$ab\langle s\rangle = (1,0)\langle (s_1,s_2)\rangle = (\langle s_1\rangle, 0).$$

Thus $ab\langle s\rangle \subseteq K$ would force $\langle s_1\rangle \subseteq K_1$, hence $s_1 \in K_1$, contradicting $K_1 \cap S_1 = \varnothing$.

On the other hand,

$$c\langle s\rangle = (0,1)\langle (s_1,s_2)\rangle = (0, \langle s_2\rangle).$$

So $c\langle s\rangle \subseteq \mathcal{P}(R_1) \times \mathcal{P}(R_2)$ would force $\langle s_2\rangle \subseteq \mathcal{P}(R_2)$, hence $s_2 \in \mathcal{P}(R_2)$, contradicting $S_2 \cap \mathcal{P}(R_2) = \varnothing$. Both alternatives are impossible, a contradiction. □

LEMMA 2.24. *Let R be a ring, K a semiprime ideal of R, and I any ideal of R. Then both $(K : I)$ and $(K : I)^*$ are semiprime ideals of R.*

Proof. Let $x \in R$ with $xRx \subseteq (K : I)$. Then $xRxI \subseteq K$. Take any $i \in I$. We have

$$(xi)R(xi) \subseteq xRxI \subseteq K.$$

Since K is semiprime, the condition $(xi)R(xi) \subseteq K$ implies $xi \in K$. As this holds for every $i \in I$, we obtain $xI \subseteq K$, and hence $x \in (K : I)$. Thus $(K : I)$ is semiprime.

A similar argument shows that $(K : I)^*$ is semiprime. □

COROLLARY 2.25. *[[13, Lemma 2.11]] Let S be an m-system of a ring R. Then for any $s \in S$,*

$$(\mathcal{P}(R) : \langle s\rangle) = \mathrm{rad}(\mathcal{P}(R) : \langle s\rangle).$$

Proof. Since the prime radical $\mathcal{P}(R)$ is semiprime, Lemma 2.24 implies that $(\mathcal{P}(R) : \langle s\rangle)$ is semiprime, and hence it is radical. □

THEOREM 2.26. *Let I be an ideal of R disjoint from S with $\mathcal{P}(R) \subseteq I$. Suppose further that there exists a central element $s \in S$. Then I is a right S-$(1,\mathcal{P})$-absorbing ideal associated to s if and only if*

1. *I is a right S-$(1, \mathrm{rad}(I))$-absorbing ideal associated to s, and*
2. $\mathrm{rad}(I : \langle s\rangle)^* = (\mathcal{P}(R) : \langle s\rangle)$.

Proof. Suppose first that I is a right S-$(1,\mathcal{P})$-absorbing ideal. Clearly, I is also right S-$(1,\operatorname{rad}(I))$-absorbing.

If $a \in (I : \langle s\rangle)^*$, then $\langle s\rangle a \subseteq I$ implies

$$\langle s\rangle\langle s\rangle\langle a\rangle \subseteq I.$$

Since $\langle s\rangle^3 \nsubseteq I$, we conclude that $a\langle s\rangle \subseteq \mathcal{P}(R)$, hence $a \in (\mathcal{P}(R) : \langle s\rangle)$. Thus,

$$(I : \langle s\rangle)^* \subseteq (\mathcal{P}(R) : \langle s\rangle).$$

By Corollary 2.25,

$$\operatorname{rad}((I : \langle s\rangle)^*) \subseteq \operatorname{rad}(\mathcal{P}(R) : \langle s\rangle) = (\mathcal{P}(R) : \langle s\rangle).$$

Conversely, for $x \in (\mathcal{P}(R) : \langle s\rangle)$ we have $x\langle s\rangle \subseteq \mathcal{P}(R) \subseteq I$, so $x \in (I : \langle s\rangle)$. Hence, again by Corollary 2.25,

$$(\mathcal{P}(R) : \langle s\rangle) = \operatorname{rad}(\mathcal{P}(R) : \langle s\rangle) \subseteq \operatorname{rad}(I : \langle s\rangle).$$

Since $s \in C(R)$, it follows that $(I : \langle s\rangle) = (I : \langle s\rangle)^*$. Therefore,

$$\operatorname{rad}(I : \langle s\rangle)^* = (\mathcal{P}(R) : \langle s\rangle).$$

Now suppose that I is a right S-$(1,\operatorname{rad}(I))$-absorbing ideal and

$$\operatorname{rad}(I : \langle s\rangle) = (\mathcal{P}(R) : \langle s\rangle).$$

Let $a,b,c \in R$ be nonunits such that $aRbRc \subseteq I$. Then either $ab\langle s\rangle \subseteq I$, or $c\langle s\rangle \subseteq \operatorname{rad}(I)$.

If $ab\langle s\rangle \subseteq I$, there is nothing to prove. If $c\langle s\rangle \subseteq \operatorname{rad}(I)$, then

$$c\langle s\rangle \subseteq \operatorname{rad}(I) \subseteq \operatorname{rad}(I : \langle s\rangle) = (\mathcal{P}(R) : \langle s\rangle).$$

Hence

$$csRcs \subseteq c\langle s\rangle\langle s\rangle \subseteq \mathcal{P}(R).$$

This implies $cs \in \mathcal{P}(R)$. Since $s \in C(R)$, we deduce $c\langle s\rangle \subseteq \mathcal{P}(R)$. Thus I is a right S-$(1,\mathcal{P})$-absorbing ideal. □

Corollary 2.27. *Let I be an ideal of R that satisfies the conditions in Theorem 2.26. If I is a right S-prime ideal associated to $s \in S$, then I is a right S-$(1,\mathcal{P})$-absorbing ideal associated to s if and only if*

$$(I : \langle s\rangle) = (\mathcal{P}(R) : \langle s\rangle).$$

Proof. Since I is a right S-prime ideal, then by Theorem 2.9 of [1], $(I : \langle s\rangle)$ is prime. Now if I is a right S-$(1,\mathcal{P})$-absorbing ideal associated to s, then by Theorem 2.26, we get:

$$(I : \langle s\rangle) = \operatorname{rad}(I : \langle s\rangle) = (\mathcal{P}(R) : \langle s\rangle).$$

Conversely, let $a,b,c \in R$ be nonunits such that $aRbRc \subseteq I$. Since I is right S-prime, then either $ab\langle s\rangle \subseteq I$, or $c\langle s\rangle \subseteq I$.

If $ab\langle s\rangle \subseteq I$, there is nothing to prove. If $c\langle s\rangle \subseteq I$, then

$$c \in (I : \langle s\rangle) = (\mathcal{P}(R) : \langle s\rangle).$$

Thus I is a right S-$(1,\mathcal{P})$-absorbing ideal. □

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Alaa Abouhalaka
Ministry of Education and Higher Education, Doha, Qatar
Department of Mathematics, Çukurova University, 01330 Balcalı, Adana, Turkey
E-mail: `alaa1aclids@gmail.com`

Nico Groenewald
Department of Mathematics, Nelson Mandela University, Gqeberha, South Africa
E-mail: `nico.groenewald@mandela.ac.za`