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REGULARITY OF CERTAIN SANDWICH SEMIGROUPS OF LINEAR TRANSFORMATIONS DEFINED BY THE NOTION OF CHARACTER

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ABSTRACT. In 2024, Tang-On and Rakbud introduced the concept of the character of a partial linear transformation between vector spaces. Moreover, the authors defined a class of sandwich semigroups of partial linear transformations, utilizing the notion of the character. In this paper, our primary objective is to investigate the regularity of elements in these sandwich semigroups.

1. Introduction

In this paper, the domain and range of a relation α are denoted by $\mathrm{Dom}(\alpha)$ and $\mathrm{Ran}(\alpha)$, respectively. For any nonempty sets X, Y and Z, the composition of a relation α from X into Y and a relation β from Y into Z, denoted by $\alpha \circ \beta$ (or simply $\alpha\beta$), is defined as follows:

$$\alpha \circ \beta = \{(x, z) \in X \times Z : \exists y \in Y, (x, y) \in \alpha \wedge (y, z) \in \beta\}.$$

According to the definition of the composition, for any function f and an element x of $\mathrm{Dom}(f)$, we denote the image of x under f by $(x)f$.

In semigroup theory, for any nonempty sets X and Y, a function whose domain is a subset of X and whose range is a subset of Y is called a *partial transformation from X into Y* (or simply *on* X if $X = Y$). Note that $\varnothing$, regarded as a function from $\varnothing$ into a nonempty subset of Y, is also a partial transformation from X into Y. A partial transformation from X into Y whose domain is exactly X is called a *full transformation from X into Y* (or simply *on* X if $X = Y$). A partial or full transformation from X into Y is commonly called a *transformation from X into Y*. Furthermore, in this paper, the sets of all partial transformations and full transformations from X into Y are denoted by $P(X, Y)$ and $T(X, Y)$, respectively. Notice that $T(X, Y) \subseteq P(X, Y)$. In the case where $X = Y$, the sets $P(X, Y)$ and $T(X, Y)$, which are semigroups under composition, are simply denoted by $P(X)$ and $T(X)$, respectively. For any $\mathcal{S} \subseteq P(X, Y)$ and $\alpha \in P(Y, X)\backslash\{\varnothing\}$, if $f \circ \alpha \circ g \in \mathcal{S}$ for all $f, g \in \mathcal{S}$, then $\mathcal{S}$ equipped with the binary operation $\bullet_\alpha$ defined by $f \bullet_\alpha g = f \circ \alpha \circ g$ is a semigroup, called a

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sandwich semigroup of transformations from X *into* Y (*with the sandwich transformation* α) and denoted by $\mathcal{S}_\alpha$. For any $f \in P(X,Y)$ and $\theta \in P(Y,X)$, if $f \circ \theta \circ f = f$, we call f a *θ-idempotent transformation.* It is clear that an element f of a sandwich semigroup of transformations $\mathcal{S}_\theta$ from X into Y is an idempotent element of $\mathcal{S}_\theta$ if and only if it is a θ-idempotent transformation.

For any $f \in P(X,Y)$, $A \subseteq \mathrm{Dom}(f)$ and $B \subseteq Y$, let $f|_A$ be a function from A into Y defined by $(x)f|_A = (x)f$ for all $x \in A$, called the *restriction of* f *to* A, and let

$$(A)f = \{y \in Y : \exists x \in A, (x,y) \in f\}$$

and

$$(B)f^{-1} = \{x \in X : \exists y \in B, (x,y) \in f\},$$

which is simply denoted by $(z)f^{-1}$ when $B = \{z\}$ for some $z \in Y$. From the definition of the composition, we have for all $g \in P(X,Y)$, $f \in P(Y,Z)$, $A \subseteq \mathrm{Dom}(g \circ f)$ and $B \subseteq X$ that $A \subseteq \mathrm{Dom}(g)$, $(A)g \subseteq \mathrm{Dom}(f)$, $(A)g \circ f = ((A)g)f$ and $(B \cap \mathrm{Dom}(g \circ f))g \circ f = ((B \cap \mathrm{Dom}(g))g \cap \mathrm{Dom}(f))f$.

Over the past half-century, the regularity of semigroups of transformations has been extensively studied by numerous researchers. Most of these studies have focused on the regularity of specific semigroups, although some general investigations have also been conducted (see [1, 2, 4–6]). A notable contribution in that direction was the work of K. D. Magill Jr. and S. Subbiah [2]. In that paper, the authors carried out a general study on the regularity of sandwich semigroups of transformations and applied their findings to particular cases. We summarize some of their key definitions and results below.

Definition 1.1. [2, Definition 2.1] Let X and Y be nonempty sets, and let $\mathcal{S}_\theta$ be a sandwich semigroup of transformations from X into Y. A subset A of Y is called an *$\mathcal{S}$-retract of* Y if there exists an idempotent element f of $\mathcal{S}_\theta$ such that $A = \mathrm{Ran}(f)$.

Definition 1.2. [2, Definition 2.2] Let X and Y be nonempty sets, and let $\mathcal{S}_\theta$ be a sandwich semigroup of transformations from X into Y. For any subsets A and B of Y, we say that A *is $\mathcal{S}$-isomorphic to* B if there exist $f, g \in \mathcal{S}$ such that $(A)\theta \circ f \subseteq B$, $(B)\theta \circ g \subseteq A$, $(\theta \circ f \circ \theta \circ g)\big|_A = \mathrm{id}_A$ and $(\theta \circ g \circ \theta \circ f)\big|_B = \mathrm{id}_B$. In this situation, the map $(\theta \circ f)\big|_A$ is called an *$\mathcal{S}$-isomorphism from* A *onto* B.

The following theorem is a general result established by Magill and Subbiah concerning the regularity of elements of sandwich semigroups of transformations between sets.

Theorem 1.3. [2, Theorem 3.2] *Let* X *and* Y *be nonempty sets. Also, let* $\mathcal{S}_\theta$ *be a sandwich semigroup of transformations from* X *into* Y, *and let* $f \in \mathcal{S}_\theta$. *If* f *is regular, then* $\mathrm{Ran}(f)$ *is an* $\mathcal{S}$*-retract of* Y. *Moreover, the following statements are equivalent.*

1. f *is regular.*
2. *There exists an* $\mathcal{S}$*-retract* A *of* Y *such that* $(\theta \circ f)\big|_A$ *is an* $\mathcal{S}$*-isomorphism from* A *onto* $\mathrm{Ran}(f)$.
3. *There exists* $A \subseteq Y$ *such that* $(\theta \circ f)\big|_A$ *is an* $\mathcal{S}$*-isomorphism from* A *onto* $\mathrm{Ran}(f)$.

In order to apply Theorem 1.3 to the sandwich semigroups $P(X,Y)_\theta$ and $T(X,Y)_\theta$ when $\theta \in T(Y,X)$, for all nonempty sets X and Y, the following lemma was provided.

LEMMA 1.4. [2, Lemma 5.1] *Let X and Y be nonempty sets, and let $\theta \in T(Y,X)$. Also, let $\mathcal{S}$ be either $P(X,Y)$ or $T(X,Y)$. Then, for any $A, B \subseteq Y$, the function $(\theta \circ f)\big|_A$ is an $\mathcal{S}$-isomorphism from A onto B if and only if $(\theta \circ f)\big|_A$ is a bijective function from A onto B and $\theta\big|_B$ is injective.*

By Theorem 1.3, together with Lemma 1.4, the following results were obtained.

THEOREM 1.5. [2, Theorem 5.3] *Let X and Y be nonempty sets, and let $\theta \in T(Y,X)$. Then, for any $f \in \mathcal{S}_\theta$, where $\mathcal{S}$ is either $P(X,Y)$ or $T(X,Y)$, the element f is regular if and only if $\theta\big|_{\mathrm{Ran}(f)}$ is injective and $(y)f^{-1} \cap \mathrm{Ran}(\theta) \neq \varnothing$ for all $y \in \mathrm{Ran}(f)$.*

THEOREM 1.6. [2, Theorem 5.4] *Let X and Y be nonempty sets with $|X| > 1$ and $|Y| > 1$, and let $\theta \in T(Y,X)$. Then the semigroup $\mathcal{S}_\theta$, where $\mathcal{S}$ is either $P(X,Y)$ or $T(X,Y)$, is regular if and only if θ is a bijective function from Y onto X.*

According to Definition 1.2, we observe that for any sandwich semigroup $\mathcal{S}_\theta$ of transformations from a set X into a set Y and any element f of $\mathcal{S}$, if $(\theta \circ f)\big|_A$ is an $\mathcal{S}$-isomorphism from a subset A of Y onto another subset B of Y, then $(\theta \circ f)\big|_A$ is bijective, and both $\theta\big|_A$ and $\theta\big|_B$ are injective. This observation, along with Theorem 1.3, implies that $\theta\big|_{\mathrm{Ran}(f)}$ is injective for all regular elements f of $\mathcal{S}_\theta$.

Recall that for any nonempty set X, a family $\mathscr{H} = \{H_i : i \in I\}$ of nonempty subsets of X such that $X = \bigcup_{i\in I} H_i$ and $H_i \cap H_j = \varnothing$ for all $i, j \in I$ with $i \neq j$, is called a *partition of* X. Now, we are going to refer to the work of Sanyatit et al. [6]. In that paper, the authors defined, for any partition $\mathscr{H} = \{H_i : i \in I\}$ of a nonempty set X and a subset A of X, the set

$$\mathrm{ind}(A) := \{i \in I : H_i \cap A \neq \varnothing\},$$

called the *index of* A. For any nonempty sets X with a partition $\mathscr{H} = \{H_i : i \in I\}$ and Y with a partition $\mathscr{K} = \{K_j : j \in J\}$, the following sets were defined:

$$P_{\mathscr{H},\mathscr{K}}(X,Y) := \{f \in P(X,Y) : \forall i \in I\, \exists j \in J,\ (\mathrm{Dom}(f) \cap H_i)f \subseteq K_j\},$$

$$Q_{\mathscr{H},\mathscr{K}}(X,Y) := \{f \in P_{\mathscr{H},\mathscr{K}}(X,Y) : \forall i \in \mathrm{ind}(\mathrm{Dom}(f)),\ H_i \subseteq \mathrm{Dom}(f)\}$$

and

$$T_{\mathscr{H},\mathscr{K}}(X,Y) := \{f \in T(X,Y) : \forall i \in I\, \exists j \in J,\ (H_i)f \subseteq K_j\}.$$

Extending the notion of the character introduced by Purisang and Rakbud [3], for every $f \in P_{\mathscr{H},\mathscr{H}}(X,Y)$, the *character of* f, denoted by $\chi^{(f)}$, was defined as the function from $\mathrm{ind}(\mathrm{Dom}(f))$ into J given by

$$\forall (i,j) \in \mathrm{ind}(\mathrm{Dom}(f)) \times J,\ (i)\,\chi^{(f)} = j \Leftrightarrow (H_i \cap \mathrm{Dom}(f))f \subseteq K_j.$$

It is worth noting that the notion of the character, as introduced in [3], has been adopted and applied in several works of M. Sarkar, and S.N. Singh [7–11]. The following results, concerning a key property of the character, were established in [6].

PROPOSITION 1.7. [6, Proposition 2.1] *Let X, Y and Z be nonempty sets with partitions $\mathscr{H} = \{H_i : i \in I\}$, $\mathscr{K} = \{K_j : j \in J\}$ and $\mathscr{Q} = \{Q_l : l \in E\}$, respectively. Also, let $g \in P_{\mathscr{H},\mathscr{K}}(X,Y)$ and $f \in P_{\mathscr{K},\mathscr{Q}}(Y,Z)$. Then the following statements hold.*

(1) *$g \circ f \in P_{\mathscr{H},\mathscr{Q}}(X,Z)$, in particular, $g \circ f \in Q_{\mathscr{H},\mathscr{Q}}(X,Z)$ if $g \in Q_{\mathscr{H},\mathscr{K}}(X,Y)$ and $f \in Q_{\mathscr{K},\mathscr{Q}}(Y,Z)$, and $g \circ f \in T_{\mathscr{H},\mathscr{Q}}(X,Z)$ if $g \in T_{\mathscr{H},\mathscr{K}}(X,Y)$ and $f \in T_{\mathscr{K},\mathscr{Q}}(Y,Z)$.*
(2) *$\chi^{(g\circ f)} \subseteq \chi^{(g)} \circ \chi^{(f)}$.*

(3) $\chi^{(g)} \circ \chi^{(f)} \subseteq \chi^{(g\circ f)}$ *if and only if* $(H_i \cap \mathrm{Dom}(g))g \cap \mathrm{Dom}(f) \neq \varnothing$ *for all* $i \in \mathrm{Dom}(\chi^{(g)} \circ \chi^{(f)})$.
(4) *If* $f \in Q_{\mathscr{K},\mathscr{Q}}(Y,Z)$, *then* $\chi^{(g\circ f)} = \chi^{(g)} \circ \chi^{(f)}$.

Extending the sets introduced by Rakbud [4], the following sets associated with each nonempty subset $\mathcal{S}$ of $P(I,J)$ were also defined in [6]:

$$Q^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(X,Y) = \{f \in Q_{\mathscr{H},\mathscr{K}}(X,Y) : \chi^{(f)} \in \mathcal{S}\}$$

and

$$T^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(X,Y) = \{f \in T_{\mathscr{H},\mathscr{K}}(X,Y) : \chi^{(f)} \in \mathcal{S}\}.$$

Note that $Q^{(P(I,J))}_{\mathscr{H},\mathscr{K}}(X,Y) = Q_{\mathscr{H},\mathscr{K}}(X,Y)$, and that if $\mathcal{S} \subseteq T(I,J)$, then

$$T^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(X,Y) = Q^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(X,Y),$$

in particular,

$$T^{(T(I,J))}_{\mathscr{H},\mathscr{K}}(X,Y) = Q^{(T(I,J))}_{\mathscr{H},\mathscr{K}}(X,Y) = T_{\mathscr{H},\mathscr{K}}(X,Y).$$

For each element α of $Q_{\mathscr{H},\mathscr{K}}(Y,X) \setminus \{\varnothing\}$, and for every sandwich semigroup $\mathcal{S}_{\chi^{(\alpha)}}$ of transformations from I into J, statement (4) of Proposition 2.1 induces a corresponding sandwich semigroup of transformations from X into Y, namely $Q^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(Y,X)_\alpha$. The main objective of [6] was to investigate the regularity of elements of $Q^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(Y,X)_\alpha$ with an application to $Q_{\mathscr{H},\mathscr{K}}(X,Y)$ and $P_{\mathscr{H},\mathscr{K}}(X,Y)$. This investigation was carried out using Theorem 1.3.

Motivated by the work of Sanyatit et al. [6], Tang-On and Rakbud [12] considered vector spaces expressed as internal direct sums of families of nonzero subspaces, instead of the setting of nonempty sets with partitions as in [6]. In place of general transformations, the authors focused on linear transformations. Based on this framework, they formulated the notion of character to align naturally with the structure and properties of vector spaces and linear mappings. Furthermore, they defined certain sets of linear transformations, analogous to those introduced in [6], using this redefined notion of character. In this paper, we aim to investigate the algebraic properties of these sets as semigroups, with particular emphasis on the regularity of their elements.

2. Preliminaries

In this section, we summarize and extend the results on the regularity of certain sandwich semigroups of transformations from the index set of one family of vector spaces into the index set of another, provided by Tang-On and Rakbud [12].

For any indexed families of vector spaces $\mathscr{U} = \{U_i : i \in I\}$ and $\mathscr{V} = \{V_j : j \in J\}$, let

$$P^*_{\mathscr{U},\mathscr{V}}(I,J) = \left\{f \in P(I,J) : \forall i \in \mathrm{Dom}(f), \dim(U_i) \leq \dim\left(V_{(i)f}\right)\right\},$$

$$\widehat{P}^*_{\mathscr{U},\mathscr{V}}(I,J) = \left\{f \in P^*_{\mathscr{U},\mathscr{V}}(I,J) : f \text{ is injective}\right\},$$

and

$$\begin{aligned} T^*_{\mathscr{U},\mathscr{V}}(I,J) &= P^*_{\mathscr{U},\mathscr{V}}(I,J) \cap T(I,J) \\ &= \left\{f \in T(I,J) : \forall i \in I, \dim(U_i) \leq \dim\left(V_{(i)f}\right)\right\}. \end{aligned}$$

The two sets $P^*_{\mathscr{U},\mathscr{V}}(I,J)$ and $T^*_{\mathscr{U},\mathscr{V}}(I,J)$ were first introduced in [12]. For any indexed families of vector spaces $\mathscr{H}=\{U_i : i\in I\}$, $\mathscr{K}=\{V_j : j\in J\}$ and $\mathscr{Q}=\{W_l : l\in E\}$, it is clear that $g\circ f\in P^*_{\mathscr{H},\mathscr{Q}}(I,E)$ for all $g\in P^*_{\mathscr{H},\mathscr{K}}(I,J)$ and $f\in P^*_{\mathscr{K},\mathscr{Q}}(J,E)$.

Throughout the remainder of this section, let $\mathscr{U}=\{U_i : i\in I\}$ and $\mathscr{V}=\{V_j : j\in J\}$ be indexed families of vector spaces. Also, let $\mathcal{S}$ be either $P^*_{\mathscr{H},\mathscr{K}}(I,J)$ or $\widehat{P}^*_{\mathscr{H},\mathscr{K}}(I,J)$ or $T^*_{\mathscr{H},\mathscr{K}}(I,J)$, and suppose that $\mathcal{S}\neq\emptyset$. Furthermore, let γ be an element of $T(J,I)$ such that $f\circ\gamma\circ g\in\mathcal{S}$ for all $f,g\in\mathcal{S}$. The following results generalize Lemma 1, Theorem 2 and Theorem 3 from [12], respectively, to include the case where $\mathcal{S}=\widehat{P}^*_{\mathscr{H},\mathscr{K}}(I,J)$. The corresponding proofs follow similarly to those in [12], with only minor adjustments required.

Lemma 2.1. *For any $f\in\mathcal{S}_\gamma$ and $A,B\subseteq J$, the function $(\gamma\circ f)\big|_A$ is an $\mathcal{S}$-isomorphism from A onto B if and only if $\gamma|_B$ is injective, $(\gamma\circ f)\big|_A$ is a bijective function from A onto B and*

$$\dim(U_i)\le\dim\left(V_{\left((i)(\gamma|_B)^{-1}\right)((\gamma\circ f)|_A)^{-1}}\right)$$

for all $i\in(B)\gamma$.

Proof. ($\Rightarrow$) Suppose that $(\gamma\circ f)\big|_A$ is an $\mathcal{S}$-isomorphism from A onto B. Then $(\gamma\circ f)\big|_A$ is a bijective function from A onto B and $\gamma\big|_B$ is injective. We also have that there exists $g\in\mathcal{S}$ such that $(\gamma\circ g)\big|_B=\left((\gamma\circ f)\big|_A\right)^{-1}$. Hence, for any $i\in(B)\gamma$, we get

$$\left((i)\left(\gamma\big|_B\right)^{-1}\right)\left((\gamma\circ f)\big|_A\right)^{-1}=\left((i)\left(\gamma\big|_B\right)^{-1}\right)\gamma\circ g=(i)g,$$

which implies

$$\dim(U_i)\le\dim\left(V_{(i)g}\right)=\dim\left(V_{((i)\gamma^{-1})\left((\gamma\circ f)\big|_A\right)^{-1}}\right)$$

as required.

($\Leftarrow$) Suppose that $(\gamma\circ f)\big|_A$ is a bijective function from A onto B, $\gamma\big|_B$ is injective and

$$\dim(U_i)\le\dim\left(V_{\left((i)\left(\gamma\big|_B\right)^{-1}\right)\left((\gamma\circ f)\big|_A\right)^{-1}}\right)$$

for all $i\in(B)\gamma$. We will show that $(\gamma\circ f)\big|_A$ is an $\mathcal{S}$-isomorphism from A onto B. To prove this, let

$$g=\left(\gamma\big|_B\right)^{-1}\circ\left((\gamma\circ f)\big|_A\right)^{-1}$$

if $\mathcal{S}$ is either $P^*_{\mathscr{H},\mathscr{K}}(I,J)$ or $\widehat{P}^*_{\mathscr{H},\mathscr{K}}(I,J)$, and let $g: I\to J$ be defined by

$$(i)g=\begin{cases}\left((i)\left(\gamma\big|_B\right)^{-1}\right)\left((\gamma\circ f)\big|_A\right)^{-1} & \text{if } i\in(B)\gamma,\\ (i)f & \text{otherwise,}\end{cases}$$

if $\mathcal{S}$ is $T^*_{\mathscr{H},\mathscr{K}}(I,J)$. Then $g\in\mathcal{S}$ and $(j)\gamma\circ g=(j)\left((\gamma\circ f)\big|_A\right)^{-1}\in A$ for all $j\in B$, which implies that $(B)\gamma\circ g\subseteq A$. Additionally, $(i)\gamma\circ f\circ\gamma\circ g=i$ for all $i\in A$ and $(j)\gamma\circ g\circ\gamma\circ f=j$ for all $j\in B$. These observations show that $(\gamma\circ f\circ\gamma\circ g)\big|_A=\mathrm{id}_A$ and $(\gamma\circ g\circ\gamma\circ f)\big|_B=\mathrm{id}_B$, respectively. Therefore, $(\gamma\circ f)\big|_A$ is an $\mathcal{S}$-isomorphism from A onto B. $\square$

By Theorem 1.3, along with Lemma 2.1, the following theorem is obtained.

THEOREM 2.2. *Let $f \in \mathcal{S}_\gamma$. Then f is regular if and only if $\gamma|_{\mathrm{Ran}(f)}$ is injective and for all $j \in \mathrm{Ran}(f)$, there exists $k_j \in J$ such that $(k_j)\,\gamma \circ f = j$ and $\dim(U_{(j)\gamma}) \leq \dim\left(V_{k_j}\right)$.*

Proof. ($\Rightarrow$) Suppose that f is regular. Then, by Theorem 1.3, there exists $A \subseteq J$ such that $(\gamma \circ f)\big|_A$ is an $\mathcal{S}$-isomorphism from A onto $\mathrm{Ran}(f)$. Moreover, by Lemma 2.1, it follows that $\gamma|_{\mathrm{Ran}(f)}$ is injective, $(\gamma \circ f)\big|_A$ is a bijective function from A onto $\mathrm{Ran}(f)$ and

$$\dim(U_{(j)\gamma}) \leq \dim\left(V_{(j)((\gamma\circ f)|_A)^{-1}}\right)$$

for all $j \in \mathrm{Ran}(f)$. For each $j \in \mathrm{Ran}(f)$, by the bijectivity of $(\gamma \circ f)\big|_A : A \to \mathrm{Ran}(f)$, we choose $k_j = (j)\left((\gamma \circ f)\big|_A\right)^{-1}$. Then $(k_j)\,\gamma \circ f = j$ and $\dim(U_{(j)\gamma}) \leq \dim\left(V_{k_j}\right)$ for all $j \in \mathrm{Ran}(f)$.

($\Leftarrow$) Suppose that $\gamma|_{\mathrm{Ran}(f)}$ is injective, and that for all $j \in \mathrm{Ran}(f)$, there exists $k_j \in J$ such that $(k_j)\,\gamma\circ f = j$ and $\dim(U_{(j)\gamma}) \leq \dim\left(V_{k_j}\right)$. Let $A = \{k_j : j \in \mathrm{Ran}(f)\}$. Then $(\gamma \circ f)\big|_A$ is a bijective function from A onto $\mathrm{Ran}(f)$ with $(j)\left((\gamma \circ f)\big|_A\right)^{-1} = k_j$ for all $j \in \mathrm{Ran}(f)$. Thus, by Lemma 2.1, the function $(\gamma \circ f)\big|_A$ is an $\mathcal{S}$-isomorphism from A onto $\mathrm{Ran}(f)$. Therefore, by Theorem 1.3, we conclude that f is regular. $\square$

For all $j \in J$, let $\widehat{j} = \{i \in I : \dim(U_i) \leq \dim(V_j)\}$.

THEOREM 2.3. *Suppose that $\left|\widehat{j}\right| > 1$ for all $j \in J$. In the case where $\mathcal{S} = T^*_{\mathscr{H},\mathscr{K}}(I, J)$, suppose further that there exist $k \in J$ and $g \in T^*_{\mathscr{H},\mathscr{K}}(I, J)$ such that $\mathrm{Ran}(\gamma) \subseteq \widehat{k}$ and $k \notin (I \setminus \mathrm{Ran}(\gamma))g$. Then $\mathcal{S}_\gamma$ is regular if and only if γ is a bijective function from J onto I, and for all $f \in \mathcal{S}$ and $j \in \mathrm{Ran}(f)$, there exists $i \in (j)f^{-1}$ such that $\dim\left(U_{(j)\gamma}\right) \leq \dim\left(V_{(i)\gamma^{-1}}\right)$.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{S}_\gamma$ is regular. We first show that γ is a bijective function from J onto I. Suppose to the contrary that γ is not bijective. Then we have two cases to consider as follows.

<u>Case 1</u> γ is not injective: Let $\nu, \mu \in J$ be such that $\nu \neq \mu$ but $(\nu)\gamma = (\mu)\gamma$. Fix two elements p of $\widehat{\nu}$ and q of $\widehat{\mu}$ such that $p \neq q$. If $\mathcal{S}$ is either $P^*_{\mathscr{H},\mathscr{K}}(I, J)$ or $\widehat{P}^*_{\mathscr{H},\mathscr{K}}(I, J)$, and let $f : \{p, q\} \to J$ be defined by $(p)f = \nu$ and $(q)f = \mu$. In the case where $\mathcal{S} = T^*_{\mathscr{H},\mathscr{K}}(I, J)$, let $f : I \to J$ be defined by

$$(i)f = \begin{cases} \nu & \text{if } i = p, \\ \mu & \text{if } i = q, \\ (i)h & \text{otherwise,} \end{cases}$$

where h is a fixed element of $\mathcal{S}$. Then $f \in \mathcal{S}$, and $\gamma|_{\mathrm{Ran}(f)}$ is not injective. Hence, by Theorem 1.5, the function f is not regular, a contradiction.

<u>Case 2</u> γ is not surjective: If $\mathcal{S}$ is either $P^*_{\mathscr{H},\mathscr{K}}(I, J)$ or $\widehat{P}^*_{\mathscr{H},\mathscr{K}}(I, J)$, let $f = h|_{I\setminus \mathrm{Ran}\,\gamma}$, where h is a fixed element of $\mathcal{S}$. In this situation, since $\mathrm{Dom}(f) = I \setminus \mathrm{Ran}\,\gamma$, by Theorem 2.2, it follows that f is not regular. In the case where $\mathcal{S} = T^*_{\mathscr{H},\mathscr{K}}(I, J)$, let

$f : I \to J$ be defined by

$$(i)f = \begin{cases} k & \text{if } i \in \operatorname{Ran}(\gamma) \subseteq \widehat{k}, \\ (i)g & \text{otherwise.} \end{cases}$$

Then $f \in \mathcal{S}$. Since $k \notin (I \setminus \operatorname{Ran}(\gamma))g$, we get $((p)g)f^{-1} \cap \operatorname{Ran}(\gamma) = \emptyset$, where p is a fixed element of $I \setminus \operatorname{Ran}(\gamma)$. Hence, by Theorem 1.5, the function f is not regular. This also contradicts the regularity of $\mathcal{S}_\gamma$.

We have now shown that γ is bijective. To prove the second statement, suppose to the contrary that there exist $f \in \mathcal{S}$ and $j \in \operatorname{Ran}(f)$ such that $\dim\left(U_{(j)\gamma}\right) > \dim\left(V_{(i)\gamma^{-1}}\right)$ for all $i \in (j)f^{-1}$. Then, by Theorem 2.2, we immediately get that f is not regular, which contradicts the regularity of $\mathcal{S}_\gamma$.

($\Leftarrow$) Suppose that γ is a bijective function from J onto I, and that for all $f \in \mathcal{S}$ and $j \in \operatorname{Ran}(f)$, there exists $i \in (j)f^{-1}$ such that $\dim\left(U_{(j)\gamma}\right) \leq \dim\left(V_{(i)\gamma^{-1}}\right)$. To prove that $\mathcal{S}_\gamma$ is regular, let $f \in \mathcal{S}_\gamma$. Then, by the first assumption that γ is injective, we get that $\gamma|_{\operatorname{Ran}(f)}$ is injective. Next, let $j \in \operatorname{Ran}(f)$. Then, by the second assumption, there exists $i \in (j)f^{-1}$ such that $\dim\left(U_{(j)\gamma}\right) \leq \dim\left(V_{(i)\gamma^{-1}}\right)$ Thus, by Theorem 2.2, we obtain that f is regular. □

Remark 2.4. In the case where all vector spaces U_i and V_j have the same dimension, we have that $P^*_{\mathscr{H},\mathscr{K}}(I,J) = P(I,J)$ and $T^*_{\mathscr{H},\mathscr{K}}(I,J) = T(I,J)$, and that $\widehat{j} = I$ for all $j \in J$. Under this condition, it is clear that Theorem 2.2 agrees with Theorem 1.5. Moreover, if γ is bijective, then the condition that for all $f \in \mathcal{S}$ and $j \in \operatorname{Ran}(f)$, there exists $i \in (j)f^{-1}$ such that $\dim\left(U_{(j)\gamma}\right) \leq \dim\left(V_{(i)\gamma^{-1}}\right)$, is automatically satisfied. Additionally, if $|I| > 1$ and $|J| > 1$, then the condition that there exist $k \in J$ and $g \in T^*_{\mathscr{H},\mathscr{K}}(I,J)$ such that $\operatorname{Ran}(\gamma) \subseteq \widehat{k}$ and $k \notin (I \setminus \operatorname{Ran}(\gamma))g$, also holds. Therefore, under these assumptions, Theorem 2.3 agrees with Theorem 1.6.

3. The character of a linear transformation

In this paper, for any vector space U and V over the same field, a linear transformation from a subspace of U into V is called a *partial linear-transformation from U into V*. Let $PL(U,V)$ and $L(U,V)$ denote the sets of all partial linear-transformations and linear transformations from U into V, respectively. The set of all zero linear transformations in $PL(U,V)$ is denoted by $Z(U,V)$. The element of $Z(U,V)$ whose domain is $\{0\}$ is denoted by $e_{(U,V)}$.

In [12], Tang-On and Rakbud introduced the notion of the character of a partial linear-transformation. We now recall this concept, along with related definitions, as presented in their work.

For any vector space U, a family $\mathscr{F} = \{H_i : i \in I\}$ consisting of nonzero subspaces of U is called a *direct-sum partition of U* if $U = \bigoplus_{i\in I} H_i$. Analogous to the definition introduced by Sanyatit et al. [6], for any subspace W of a vector space U, equipped with a direct-sum partition $\mathscr{H} = \{H_i : i \in I\}$, let

$$\operatorname{ind}(W) = \{i \in I : H_i \cap W \neq \{0\}\},$$

which is called the *index of W*.

Now, let U and V be arbitrary vector spaces over the same field $\mathbb{F}$, and let $\mathscr{H} = \{H_i : i \in I\}$ and $\mathscr{K} = \{K_j : j \in J\}$ be arbitrary direct-sum partitions of U and V, respectively. Analogous to the sets of transformations defined by Sanyatit et al. [6], let

$$PL_{\mathscr{H},\mathscr{K}}(U,V)=\{T\in PL(U,V):\forall i\in I,\,\exists j\in J,(H_i\cap\mathrm{Dom}(T))\,T\subseteq K_j\}\,,$$

$$PL^*_{\mathscr{H},\mathscr{K}}(U,V)=\left\{T\in PL_{\mathscr{H},\mathscr{K}}(U,V):\forall i\in I,T\big|_{H_i\cap\mathrm{Dom}(T)}\text{ is injective}\right\},$$

$$QL^*_{\mathscr{H},\mathscr{K}}(U,V)=\left\{T\in PL^*_{\mathscr{H},\mathscr{K}}(U,V):\forall i\in\mathrm{ind}(\mathrm{Dom}(T)),H_i\subseteq\mathrm{Dom}(T)\right\}$$

and

$$L^*_{\mathscr{H},\mathscr{K}}(U,V)=\left\{T\in QL^*_{\mathscr{H},\mathscr{K}}(U,V):\mathrm{Dom}(T)=U\right\}.$$

In the case where $U=V$, $I=J$ and $H_i=K_i$ for all $i\in I$, we simply denote the sets $PL_{\mathscr{H},\mathscr{K}}(U,V)$, $PL^*_{\mathscr{H},\mathscr{K}}(U,V)$, $QL^*_{\mathscr{H},\mathscr{K}}(U,V)$ and $L^*_{\mathscr{H},\mathscr{K}}(U,V)$ by $PL_{\mathscr{H}}(U)$, $PL^*_{\mathscr{H}}(U)$, $QL^*_{\mathscr{H}}(U)$ and $L^*_{\mathscr{H}}(U)$, respectively. From the definitions of $PL_{\mathscr{H},\mathscr{K}}(U,V)$, $PL^*_{\mathscr{H},\mathscr{K}}(U,V)$ and $QL^*_{\mathscr{H},\mathscr{K}}(U,V)$, the following statements clearly hold.

(i) $Z(U,V)\subseteq PL_{\mathscr{H},\mathscr{K}}(U,V)$.

(ii) For all $T\in PL(U,V)$, the element T belongs to $PL_{\mathscr{H},\mathscr{K}}(U,V)$ if and only if for all $i\in\mathrm{ind}(\mathrm{Dom}(T))$ with $(H_i\cap\mathrm{Dom}(T))\,T\neq\{0\}$, there exists $j\in J$ such that $(H_i\cap\mathrm{Dom}(T))\,T\subseteq K_j$. If $T\in PL_{\mathscr{H},\mathscr{K}}(U,V)$, then $T\in PL^*_{\mathscr{H},\mathscr{K}}(U,V)$ if and only if $T\big|_{H_i\cap\mathrm{Dom}(T)}$ is injective for all $i\in\mathrm{ind}(\mathrm{Dom}(T))$.

(iii) An element T of $Z(U,V)$ belongs to $PL^*_{\mathscr{H},\mathscr{K}}(U,V)$ if and only if $\mathrm{Dom}(T)=\{0\}$, that is, $T=e_{(U,V)}$. In fact, $e_{(U,V)}\in QL^*_{\mathscr{H},\mathscr{K}}(U,V)$.

(iv) $Z(U,V)\cap L^*_{\mathscr{H},\mathscr{K}}(U,V)=\varnothing$.

For each $T\in PL_{\mathscr{H},\mathscr{K}}(U,V)$, the *character of* T, denoted by $\chi^{(T)}$, is a function from a subset of I into J defined by

$$\forall(i,j)\in I\times J,(i)\chi^{(T)}=j\Leftrightarrow((H_i\cap\mathrm{Dom}(T))\,T\neq\{0\}\wedge(H_i\cap\mathrm{Dom}(T))\,T\subseteq K_j)$$

if $T\notin Z(U,V)$, and $\chi^{(T)}=\varnothing$ otherwise. It is evident that $\chi^{(T)}$ is a well-defined function for all $T\in PL_{\mathscr{H},\mathscr{K}}(U,V)\setminus Z(U,V)$.

We now define, for each $T\in PL(U,V)$, the set

$$\mathrm{ind}^*(\mathrm{Dom}(T))=\{i\in I:(H_i\cap\mathrm{Dom}(T))\,T\neq\{0\}\}\,.$$

Then, for any $T\in PL(U,V)$, the following statements clearly hold:

(i) $\mathrm{ind}^*(\mathrm{Dom}(T))\subseteq\mathrm{ind}(\mathrm{Dom}(T))$;

(ii) If $T\big|_{H_i\cap\mathrm{Dom}(T)}$ is injective for all $i\in\mathrm{ind}(\mathrm{Dom}(T))$, then $\mathrm{ind}^*(\mathrm{Dom}(T))=\mathrm{ind}(\mathrm{Dom}(T))$;

(iii) $\chi^{(T)}\in P(I,J)$ with $\mathrm{Dom}\left(\chi^{(T)}\right)=\mathrm{ind}^*(\mathrm{Dom}(T))$ and $\mathrm{Ran}\left(\chi^{(T)}\right)\subseteq\mathrm{ind}(\mathrm{Ran}(T))$;

(iv) If $T\in PL^*_{\mathscr{H},\mathscr{K}}(U,V)$, then $\mathrm{Dom}\left(\chi^{(T)}\right)=\mathrm{ind}(\mathrm{Dom}(T))$;

(v) If $T\in QL^*_{\mathscr{H},\mathscr{K}}(U,V)$, then $\chi^{(T)}\in P^*_{\mathscr{H},\mathscr{K}}(I,J)$ with $\dim\left(\mathrm{Ran}(T)\cap K_{(i)\chi^{(T)}}\right)\geq\dim(H_i)$ for all $i\in\mathrm{Dom}(\chi^{(T)})$.

Next, we study an important property of the character concerning composition. The following lemma will serve as a useful tool in establishing this property.

Lemma 3.1. *Let U, V and W be vector spaces over the same field, with direct-sum partitions $\mathscr{H}=\{H_i:i\in I\}$, $\mathscr{K}=\{K_j:j\in J\}$ and $\mathscr{Q}=\{Q_l:l\in E\}$, respectively. Also, let $S\in PL(U,V)$ and $T\in PL(V,W)$. Then, for all $j\in J$, if there exists $i\in\mathrm{ind}(\mathrm{Dom}(S\circ T))$ such that $S\big|_{H_i\cap\mathrm{Dom}(S)}$ is injective and $(H_i\cap\mathrm{Dom}(S))\,S\subseteq K_j$, then $j\in\mathrm{ind}(\mathrm{Dom}(T))$.*

Proof. Let $j \in J$, and suppose that there exists $i \in \operatorname{ind}(\operatorname{Dom}(S \circ T))$ with $S\big|_{H_i \cap \operatorname{Dom}(S)}$ injective and $(H_i \cap \operatorname{Dom}(S))\, S \subseteq K_j$. Since $i \in \operatorname{ind}(\operatorname{Dom}(S \circ T))$, it follows that $H_i \cap \operatorname{Dom}(S \circ T) \neq \{0\}$. Thus, by the injectivity of $S\big|_{H_i \cap \operatorname{Dom}(S)}$, we get $(H_i \cap \operatorname{Dom}(S \circ T))\, S \neq \{0\}$. Since $(H_i \cap \operatorname{Dom}(S \circ T))\, S \subseteq \operatorname{Dom}(T)$ and $\operatorname{Dom}(S \circ T) \subseteq \operatorname{Dom}(S)$, we obtain $(H_i \cap \operatorname{Dom}(S \circ T))\, S \subseteq K_j \cap \operatorname{Dom}(T)$. Hence, $K_j \cap \operatorname{Dom}(T) \neq \{0\}$. □

Analogous to Proposition 1.7, the following proposition concerning the behavior of the character under composition was also stated in [12] without proof.

Proposition 3.2. [12, Proposition 4] *Let U, V and W be vector spaces over the same field, with direct-sum partitions $\mathscr{H} = \{H_i : i \in I\}$, $\mathscr{K} = \{K_j : j \in J\}$ and $\mathscr{Q} = \{Q_l : l \in E\}$, respectively. Also, let $S \in PL_{\mathscr{H},\mathscr{K}}(U, V)$ and $T \in PL_{\mathscr{K},\mathscr{Q}}(V, W)$. Then the following statements hold.*

(1) $S \circ T \in PL_{\mathscr{H},\mathscr{Q}}(U, W)$. *In particular,*

$$S \circ T \in \begin{cases} PL^*_{\mathscr{H},\mathscr{Q}}(U, W) & \text{if } S \in PL^*_{\mathscr{H},\mathscr{K}}(U, V) \text{ and } T \in PL^*_{\mathscr{K},\mathscr{Q}}(V, W), \\ QL^*_{\mathscr{H},\mathscr{Q}}(U, W) & \text{if } S \in QL^*_{\mathscr{H},\mathscr{K}}(U, V) \text{ and } T \in QL^*_{\mathscr{K},\mathscr{Q}}(V, W), \\ L^*_{\mathscr{H},\mathscr{Q}}(U, W) & \text{if } S \in L^*_{\mathscr{H},\mathscr{K}}(U, V) \text{ and } T \in L^*_{\mathscr{K},\mathscr{Q}}(V, W). \end{cases}$$

(2) $\chi^{(S \circ T)} \subseteq \chi^{(S)} \circ \chi^{(T)}$.

(3) $\chi^{(S)} \circ \chi^{(T)} = \chi^{(S \circ T)}$ *if and only if* $\operatorname{Dom}\left(\chi^{(S)} \circ \chi^{(T)}\right) \subseteq \operatorname{Dom}(\chi^{(S \circ T)})$, *which is also equivalent to the condition that* $((H_i \cap \operatorname{Dom}(S))S \cap \operatorname{Dom}(T))T \neq \{0\}$ *for all* $i \in \operatorname{Dom}\left(\chi^{(S)} \circ \chi^{(T)}\right)$.

(4) *If* $T \in QL^*_{\mathscr{K},\mathscr{Q}}(V, W)$, *then* $\chi^{(S \circ T)} = \chi^{(S)} \circ \chi^{(T)}$.

Proof. (1) It is straightforward to show that $S \circ T \in PL_{\mathscr{H},\mathscr{Q}}(U, W)$, and that $S \circ T \in PL^*_{\mathscr{H},\mathscr{Q}}(U, W)$ if $S \in PL^*_{\mathscr{H},\mathscr{K}}(U, V)$ and $T \in PL^*_{\mathscr{K},\mathscr{Q}}(V, W)$. Next, we show that if $S \in QL^*_{\mathscr{H},\mathscr{K}}(U, V)$ and $T \in QL^*_{\mathscr{K},\mathscr{Q}}(V, W)$, then $S \circ T \in QL^*_{\mathscr{H},\mathscr{Q}}(U, W)$. Suppose that $S \in QL^*_{\mathscr{H},\mathscr{K}}(U, V)$ and $T \in QL^*_{\mathscr{K},\mathscr{Q}}(V, W)$. Then $S \circ T \in PL^*_{\mathscr{H},\mathscr{Q}}(U, W)$. It remains to show that $H_i \subseteq \operatorname{Dom}(S \circ T)$ for all $i \in \operatorname{ind}(\operatorname{Dom}(S \circ T))$. Let $i \in \operatorname{ind}(\operatorname{Dom}(S \circ T))$. Then $i \in \operatorname{ind}(\operatorname{Dom}(S))$. Since $S \in QL^*_{\mathscr{H},\mathscr{K}}(U, V)$, we get $H_i \subseteq \operatorname{Dom}(S)$. Moreover, there exists $j \in J$ such that $(H_i)S \subseteq K_j$. Accordingly, by Lemma 3.1, we have $j \in \operatorname{ind}(\operatorname{Dom}(T))$. Since $T \in QL^*_{\mathscr{K},\mathscr{Q}}(V, W)$, it follows that $K_j \subseteq \operatorname{Dom}(T)$, which implies $(H_i)S \subseteq \operatorname{Dom}(T)$. Therefore, $H_i \subseteq \operatorname{Dom}(S \circ T)$, which shows that $S \circ T \in QL^*_{\mathscr{H},\mathscr{Q}}(U, W)$ as desired. Finally, it is clear that if $S \in L^*_{\mathscr{H},\mathscr{K}}(U, V)$ and $T \in L^*_{\mathscr{K},\mathscr{Q}}(V, W)$, then $S \circ T \in L^*_{\mathscr{H},\mathscr{Q}}(U, W)$.

(2) To see that $\chi^{(S \circ T)} \subseteq \chi^{(S)} \circ \chi^{(T)}$, let $(i, l) \in \chi^{(S \circ T)}$. Then $\{0\} \neq (H_i \cap \operatorname{Dom}(S \circ T))\, S \circ T \subseteq Q_l$. Let w be a fixed nonzero element of $(H_i \cap \operatorname{Dom}(S \circ T))\, S \circ T$. Then, by the linearity of $S \circ T$, there exists $u \in H_i \setminus \{0\}$ such that $(u, w) \in S \circ T$. Hence, there exists $v \in V$ such that $(u, v) \in S$ and $(v, w) \in T$. Since T is linear and $w \neq 0$, we have $v \neq 0$. Thus, $(H_i \cap \operatorname{Dom}(S))\, S \neq \{0\}$. Moreover, there exists $j \in J$ such that $(H_i \cap \operatorname{Dom}(S))\, S \subseteq K_j$, which implies that $(i, j) \in \chi^{(S)}$. Since $(v, w) \in T$, it follows that $(K_j \cap \operatorname{Dom}(T))T \neq \{0\}$ and $\{0\} \neq (K_j \cap \operatorname{Dom}(T))T \subseteq Q_l$. This shows that $(j, l) \in \chi^{(T)}$; hence, we conclude that $(i, l) \in \chi^{(S)} \circ \chi^{(T)}$. Accordingly, $\chi^{(S \circ T)} \subseteq \chi^{(S)} \circ \chi^{(T)}$.

(3) It is evident that the necessity holds. Now, we prove the sufficiency. Suppose that $\operatorname{Dom}\left(\chi^{(S)} \circ \chi^{(T)}\right) \subseteq \operatorname{Dom}(\chi^{(S \circ T)})$. To see that $\chi^{(S)} \circ \chi^{(T)} \subseteq \chi^{(S \circ T)}$, let $(i, l) \in \chi^{(S)} \circ \chi^{(T)}$. Then $i \in \operatorname{Dom}\left(\chi^{(S)} \circ \chi^{(T)}\right) \subseteq \operatorname{Dom}(\chi^{(S \circ T)}) = \operatorname{ind}^*(\operatorname{Dom}(S \circ T))$.

Furthermore, there exists $j \in J$ such that $(i,j) \in \chi^{(S)}$ and $(j,l) \in \chi^{(T)}$. Hence, $(H_i \cap \mathrm{Dom}(S))\, S \subseteq K_j$ and $(K_j \cap \mathrm{Dom}(T))T \subseteq Q_l$. Thus, $(H_i \cap \mathrm{Dom}(S\circ T))\, S \subseteq (H_i \cap \mathrm{Dom}(S))\, S \subseteq K_j$. Since $(H_i \cap \mathrm{Dom}(S\circ T))\, S \subseteq \mathrm{Dom}(T)$, we get $(H_i \cap \mathrm{Dom}(S\circ T))S \subseteq K_j \cap \mathrm{Dom}(T)$. Accordingly, $(H_i \cap \mathrm{Dom}(S\circ T))\, S\circ T \subseteq (K_j \cap \mathrm{Dom}(T))T \subseteq Q_l$. Finally, since $i \in \mathrm{ind}^*(\mathrm{Dom}(S\circ T))$, we obtain $(i,l) \in \chi^{(S\circ T)}$.

(4) Suppose that $T \in QL^*_{\mathscr{K},\mathscr{Q}}(V,W)$. To prove that $\chi^{(S\circ T)} \subseteq \chi^{(S)} \circ \chi^{(T)}$, it suffices to show that $\mathrm{Dom}\left(\chi^{(S)} \circ \chi^{(T)}\right) \subseteq \mathrm{Dom}\left(\chi^{(S\circ T)}\right)$. Let $i \in \mathrm{Dom}\left(\chi^{(S)} \circ \chi^{(T)}\right)$. Then there exists $j \in \mathrm{Dom}(\chi^{(T)}) = \mathrm{ind}(\mathrm{Dom}(T))$ such that $(i,j) \in \chi^{(S)}$. This implies that $\{0\} \neq (H_i \cap \mathrm{Dom}(S))\, S \subseteq K_j \subseteq \mathrm{Dom}(T)$. Since $(H_i \cap \mathrm{Dom}(S))\, S \neq \{0\}$, we get $H_i \cap \mathrm{Dom}(S) \neq \{0\}$. Therefore, by the injectivity of $T\big|_{K_j}$, we obtain $((H_i \cap \mathrm{Dom}(S))\, S \cap \mathrm{Dom}(T))\, T = ((H_i \cap \mathrm{Dom}(S))\, S)\, T \neq \{0\}$, which yields that $i \in \mathrm{Dom}\left(\chi^{(S\circ T)}\right)$. Accordingly, $\chi^{(S\circ T)} = \chi^{(S)} \circ \chi^{(T)}$. □

Additional properties of characters are presented below.

Proposition 3.3. *Let U and V be vector spaces over the same field, with direct-sum partitions $\mathscr{H} = \{H_i : i \in I\}$ and $\mathscr{K} = \{K_j : j \in J\}$, respectively. Also, let $T \in PL_{\mathscr{H},\mathscr{K}}(U,V)$, and let A be a subspace of $\mathrm{Dom}(T)$. Then the following statements hold.*

(1) $\chi^{(T|_A)} \subseteq \chi^{(T)}\big|_{\mathrm{ind}(A)}$, *which becomes equality if $T \in PL^*_{\mathscr{H},\mathscr{K}}(U,V)$.*

(2) *For any subspace B of V, if $T \in PL^*_{\mathscr{H},\mathscr{K}}(U,V)$ and $(A)T \subseteq B$, then $(\mathrm{ind}(A))\chi^{(T)} \subseteq \mathrm{ind}(B)$.*

Proof. (1) Let A be a subspace of $\mathrm{Dom}(T)$. We first show that $\chi^{(T|_A)} \subseteq \chi^{(T)}\big|_{\mathrm{ind}(A)}$. Let $(i,j) \in \chi^{(T|_A)}$. Then $(H_i \cap A)\, T \neq \{0\}$ and $(H_i \cap A)\, T \subseteq K_j$. These imply that $i \in \mathrm{ind}(A)$ and $(H_i \cap \mathrm{Dom}(T))\, T \subseteq K_j$, respectively. Hence, $(i,j) \in \chi^{(T)}|_{\mathrm{ind}(A)}$. Therefore, we get $\chi^{(T|_A)} \subseteq \chi^{(T)}\big|_{\mathrm{ind}(A)}$ as required. Now, suppose that $T \in PL^*_{\mathscr{H},\mathscr{K}}(U,V)$. To see that $\chi^{(T|_A)} = \chi^{(T)}\big|_{\mathrm{ind}(A)}$, let $(i,j) \in \chi^{(T)}\big|_{\mathrm{ind}(A)}$. Then $H_i \cap A \neq \{0\}$. Since $T\big|_{H_i \cap \mathrm{Dom}(T)}$ is injective and $\{0\} \neq H_i \cap A \subseteq H_i \cap \mathrm{Dom}(T)$, it follows that $(H_i \cap A)\, T \neq \{0\}$. Moreover, because $(i,j) \in \chi^{(T)}\big|_{\mathrm{ind}(A)} \subseteq \chi^{(T)}$, we get $(H_i \cap A)\, T \subseteq (H_i \cap \mathrm{Dom}(T))\, T \subseteq K_j$. Therefore, $(i,j) \in \chi^{(T|_A)}$. Consequently, $\chi^{(T)}\big|_{\mathrm{ind}(A)} = \chi^{(T|_A)}$.

(2) Let B be a subspace of V. Suppose that $T \in PL^*_{\mathscr{H},\mathscr{K}}(U,V)$ and $(A)T \subseteq B$. To see that $(\mathrm{ind}(A))\chi^{(T)} \subseteq \mathrm{ind}(B)$, let $i \in \mathrm{ind}(A)$. Then $H_i \cap A \neq \{0\}$ and $(H_i \cap A)\, T \subseteq B \cap K_{(i)\chi^{(T)}}$. Since $T\big|_{H_i \cap \mathrm{Dom}(T)}$ is injective and $H_i \cap A \neq \{0\}$, we get $(H_i \cap A)\, T \neq \{0\}$. Therefore, $B \cap K_{(i)\chi^{(T)}} \neq \{0\}$, which implies that $(i)\chi^{(T)} \in \mathrm{ind}(B)$. □

Proposition 3.4. *Let U and V be vector spaces over the same field, with direct-sum partitions $\mathscr{H} = \{H_i : i \in I\}$ and $\mathscr{K} = \{K_j : j \in J\}$, respectively. Also, let $T \in PL^*_{\mathscr{H},\mathscr{K}}(U,V)$ be such that $\mathrm{Dom}(T) = \bigoplus_{i \in \mathrm{ind}(\mathrm{Dom}(T))} \mathrm{Dom}(T) \cap H_i$. Then the following statements hold.*

(1) *If $\chi^{(T)}$ is injective, then T is injective.*

(2) $\mathrm{Ran}\left(\chi^{(T)}\right) = \mathrm{ind}(\mathrm{Ran}(T))$.

Proof. (1) Suppose that $\chi^{(T)}$ is injective. To see that T is injective, let $u \in \mathrm{Dom}(T) \setminus \{0\}$. Then there exist distinct elements $i_1, i_2, ..., i_n$ of $\mathrm{ind}(\mathrm{Dom}(T)) = \mathrm{Dom}\left(\chi^{(T)}\right)$, together with an element $(u_1, u_2, ..., u_n)$ of $\prod_{k=1}^{n} H_{i_k} \cap \mathrm{Dom}(T) \setminus \{0\}$, such that $u =$

$\sum_{k=1}^{n} u_k$. For each $k = 1, 2, \dots, n$, since $T\big|_{H_{i_k} \cap \mathrm{Dom}(T)}$ is injective, we have $(u_k)\,T \neq 0$. Additionally, since $\chi^{(T)}$ is injective, the elements $(i_1)\,\chi^{(T)}, (i_2)\,\chi^{(T)}, \dots, (i_n)\,\chi^{(T)}$ are all distinct. Moreover, for every $k = 1, 2, \dots, n$, since $i_k \in \mathrm{Dom}(\chi^{(T)})$, it follows that $(H_{i_k} \cap \mathrm{Dom}(T))\,T \subseteq K_{(i_k)\chi^{(T)}}$. Since $(i_1)\,\chi^{(T)}, (i_2)\,\chi^{(T)}, \dots, (i_n)\,\chi^{(T)}$ are all distinct and $(u_k)\,T \in K_{(i_k)\chi^{(T)}}$ for all $k = 1, 2, \dots, n$, we have by the directness of the sum $\sum_{j \in J} K_j$ that $(u)T = \sum_{k=1}^{n} (u_k)\,T \neq 0$. Therefore, T is injective.

(2) Let $j \in \mathrm{ind}(\mathrm{Ran}(T))$. Then $\mathrm{Ran}(T) \cap K_j \neq \{0\}$. Fix $v \in \mathrm{Ran}(T) \cap K_j \setminus \{0\}$. Then there exists $u \in \mathrm{Dom}(T)$ such that $u \neq 0$ and $(u)T = v$. Since $\mathrm{Dom}(T) = \bigoplus_{i \in \mathrm{ind}(\mathrm{Dom}(T))} \mathrm{Dom}(T) \cap H_i$, there exist distinct elements $i_1, i_2, ..., i_n$ of $\mathrm{ind}(\mathrm{Dom}(T)) = \mathrm{Dom}\left(\chi^{(T)}\right)$, together with an element $(u_1, u_2, ..., u_n)$ of $\prod_{k=1}^{n} \mathrm{Dom}(T) \cap H_{i_k} \setminus \{0\}$, such that $u = \sum_{k=1}^{n} u_k$. Thus, $v = (u)T = \sum_{k=1}^{n} (u_k)\,T$. Since $(u_k)\,T \in K_{(i_k)\chi^{(T)}} \setminus \{0\}$ for all $k = 1, 2, ..., n$ and $v \in K_j$, by the directness of the sum $\sum_{l \in J} K_l$, it follows that $(i_1)\,\chi^{(T)} = (i_2)\,\chi^{(T)} = \cdots = (i_n)\,\chi^{(T)} = j$. Hence, $j \in \mathrm{Ran}\left(\chi^{(T)}\right)$. □

Proposition 3.5. *Let U, V and W be vector spaces over the same field, with direct-sum partitions $\mathscr{H} = \{H_i : i \in I\}$, $\mathscr{K} = \{K_j : j \in J\}$ and $\mathscr{Q} = \{Q_l : l \in E\}$, respectively. Also, let $S \in PL^*_{\mathscr{H},\mathscr{K}}(U, V)$ and $T \in PL^*_{\mathscr{K},\mathscr{Q}}(V, W)$ be such that $\mathrm{Dom}(S) = \bigoplus_{i \in \mathrm{ind}(\mathrm{Dom}(S))} \mathrm{Dom}(S) \cap H_i$ and $\mathrm{Dom}(T) = \bigoplus_{j \in \mathrm{ind}(\mathrm{Dom}(T))} \mathrm{Dom}(T) \cap K_j$. If $S \circ T \neq e_{(U,V)}$ and $\chi^{(S)}$ is injective, then $\mathrm{Dom}(S \circ T) = \bigoplus_{i \in \mathrm{ind}(\mathrm{Dom}(S \circ T))} \mathrm{Dom}(S \circ T) \cap H_i$.*

Proof. Suppose that $S \circ T \neq e_{(U,V)}$, and that $\chi^{(S)}$ is injective. We want to show that $\mathrm{Dom}(S \circ T) = \bigoplus_{i \in \mathrm{Dom}\left(\chi^{(S \circ T)}\right)} \mathrm{Dom}(S \circ T) \cap H_i$. To do this, it suffices to show that $\mathrm{Dom}(S \circ T) \subseteq \sum_{i \in \mathrm{Dom}\left(\chi^{(S \circ T)}\right)} \mathrm{Dom}(S \circ T) \cap H_i$. Let $u \in \mathrm{Dom}(S \circ T) \setminus \{0\}$. Then $u \in \mathrm{Dom}(S)$ and $(u)S \in \mathrm{Dom}(T)$. Since $\chi^{(S)}$ is injective, statement (1) of Proposition 3.4 implies that S is injective. Hence, $(u)S \neq 0$. Since $(u)S \in \mathrm{Dom}(T)$, there exist distinct elements $j_1, j_2, ..., j_m$ of $\mathrm{Dom}\left(\chi^{(T)}\right)$, together with an element $(v_1, v_2, ..., v_m)$ of $\prod_{l=1}^{m} \mathrm{Dom}(T) \cap K_{j_l} \setminus \{0\}$, such that $(u)S = \sum_{l=1}^{m} v_l$. Since $u \in \mathrm{Dom}(S)$, there exist distinct elements $i_1, i_2, ..., i_n$ of $\mathrm{Dom}\left(\chi^{(S)}\right)$, together with an element $(u_1, u_2, ..., u_n)$ of $\prod_{k=1}^{n} \mathrm{Dom}(S) \cap H_{i_k} \setminus \{0\}$, such that $u = \sum_{k=1}^{n} u_k$. Thus, $(u)S = \sum_{k=1}^{n} (u_k)\,S$. Note that $(u_k)\,S \in K_{(i_k)\chi^{(S)}} \setminus \{0\}$ for all $k = 1, 2, ..., n$. Moreover, since $\chi^{(S)}$ is injective, the elements $(i_1)\,\chi^{(S)}, (i_2)\,\chi^{(S)}, \dots, (i_n)\,\chi^{(S)}$ are all distinct. Hence, by the directness of the sum $\sum_{j \in J} K_j$, we get $n = m$. After reordering the indices if necessary, we may assume that $j_k = (i_k)\,\chi^{(S)}$ and $v_k = (u_k)S$ for all $k = 1, 2, ..., n$. For each $k = 1, 2, ..., n$, since $(i_k)\,\chi^{(S)} = j_k \in \mathrm{Dom}\left(\chi^{(T)}\right)$, it follows that $i_k \in \mathrm{Dom}\left(\chi^{(S \circ T)}\right)$. Therefore, $u = \sum_{k=1}^{n} u_k \in \sum_{i \in \mathrm{Dom}\left(\chi^{(S \circ T)}\right)} \mathrm{Dom}(S \circ T) \cap H_i$. □

We define another interesting subset of $PL_{\mathscr{H},\mathscr{K}}(U, V)$ as follows: Let

$$QL^*_{\mathscr{H},\mathscr{K}}(U, V)_{\oplus} = \left\{ T \in QL^*_{\mathscr{H},\mathscr{K}}(U, V) : \mathrm{Dom}(T) = \bigoplus_{i \in \mathrm{ind}(\mathrm{Dom}(T))} H_i \right\} \cup \left\{ e_{(U,V)} \right\}.$$

It is clear that $L^*_{\mathscr{H},\mathscr{K}}(U, V) \subseteq QL^*_{\mathscr{H},\mathscr{K}}(U, V)_{\oplus}$. The following corollary follows immediately from Proposition 3.5.

Corollary 3.6. *Let U, V and W be vector spaces over the same field, with direct-sum partitions $\mathscr{H} = \{H_i : i \in I\}$, $\mathscr{K} = \{K_j : j \in J\}$ and $\mathscr{Q} = \{Q_l : l \in E\}$,* *respectively. Also, let* $S \in QL^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus$ *and* $T \in QL^*_{\mathscr{K},\mathscr{Q}}(V,W)_\oplus$. *If* $\chi^{(S)}$ *is injective, then* $S \circ T \in QL_{\mathscr{H},\mathscr{Q}}(U,W)^*_\oplus$.

4. Some sandwich semigroups of linear transformations defined by the notion of the character

In this section and throughout the remainder of the paper, let U and V be vector spaces over the same field, with direct-sum partitions $\mathscr{H} = \{H_i : i \in I\}$ and $\mathscr{K} = \{K_j : j \in J\}$, respectively. For any nonempty subset $\mathcal{S}$ of $P^*_{\mathscr{H},\mathscr{K}}(I,J)$, let

$$QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^* = \left\{T \in QL^*_{\mathscr{H},\mathscr{K}}(U,V) : \chi^{(T)} \in \mathcal{S}\right\},$$

$$QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*_\oplus = \left\{T \in QL^*_{\mathscr{H},\mathscr{K}}(U,V))_\oplus : \chi^{(T)} \in \mathcal{S}\right\},$$

and

$$L^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^* = \left\{T \in L^*_{\mathscr{H},\mathscr{K}}(U,V) : \chi^{(T)} \in \mathcal{S}\right\}.$$

In particular, we denote $QL^{\left(\widehat{P}^*_{\mathscr{H},\mathscr{K}}(I,J)\right)}_{\mathscr{H},\mathscr{K}}(U,V)^*_\oplus$ by $\widehat{QL}^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus$. Note that the first and third sets were originally introduced by Tang-On and Rakbud [12]. Based on the proof of Proposition 5 in that paper, we now restate the proposition in a modified form as follows.

Proposition 4.1. [12, Proposition 5] *Let* $f \in P^*_{\mathscr{H},\mathscr{K}}(I,J)$. *Then there exists an element* T *of* $QL^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus$ *such that* $\chi^{(T)} = f$. *Moreover, if* $f \in T^*_{\mathscr{H},\mathscr{K}}(I,J)$, *then we can choose such a linear transformation* T *from* $L^*_{\mathscr{H},\mathscr{K}}(U,V)$.

From Proposition 4.1, we have for any nonempty subset $\mathcal{S}$ of $P^*_{\mathscr{H},\mathscr{K}}(I,J)$ that the sets $QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*$ and $QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*_\oplus$ are not empty. Moreover, $L^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^* \neq \varnothing$ when $\mathcal{S} \cap T^*_{\mathscr{H},\mathscr{K}}(I,J) \neq \varnothing$. Note that

$$QL^*_{\mathscr{H},\mathscr{K}}(U,V) = QL^{\left(P^*_{\mathscr{H},\mathscr{K}}(I,J)\right)}_{\mathscr{H},\mathscr{K}}(U,V)^*,$$

and that

$$QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^* = QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*_\oplus = L^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*$$

if $\mathcal{S}$ is a nonempty subset of $T^*_{\mathscr{H},\mathscr{K}}(I,J)$; hence, in this situation, the set $L^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*$ may be regarded as a special case of both $QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*$ and $QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*_\oplus$. In particular,

$$L^*_{\mathscr{H},\mathscr{K}}(U,V) = L^{\left(T^*_{\mathscr{H},\mathscr{K}}(I,J)\right)}_{\mathscr{H},\mathscr{K}}(U,V)^* = QL^{\left(T^*_{\mathscr{H},\mathscr{K}}(I,J)\right)}_{\mathscr{H},\mathscr{K}}(U,V)^* = QL^{\left(T^*_{\mathscr{H},\mathscr{K}}(I,J)\right)}_{\mathscr{H},\mathscr{K}}(U,V)^*_\oplus.$$

Proposition 4.2. *Let* $R \in QL^*_{\mathscr{K},\mathscr{H}}(V,U)$, *and let* $\mathcal{S}_{\chi^{(R)}}$ *be a sandwich semigroup of transformations from* I *into* J *with* $\mathcal{S} \subseteq P^*_{\mathscr{H},\mathscr{K}}(I,J)$. *Then* $S \circ R \circ T \in QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*$ *for all* $S, T \in QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*$. *Moreover, if every element of* $\mathcal{S}$ *is injective and* $R \in QL^*_{\mathscr{K},\mathscr{H}}(V,U)_\oplus$ *with* $\chi^{(R)}$ *injective, then* $S \circ R \circ T \in QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*_\oplus$ *for all* $S, T \in QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*_\oplus$.

Proof. Let $S, T \in QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*$. Then, by statements (1) and (4) of Proposition 3.2, we have $S \circ R \circ T \in QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*$. If every element of $\mathcal{S}$ is injective and $R \in QL^*_{\mathscr{K},\mathscr{H}}(V,U)_{\oplus}$ with $\chi^{(R)}$ injective, then, by Corollary 3.6, we get $S \circ R \circ T \in QL_{\mathscr{H},\mathscr{K}}(U,V)^*_{\oplus}$ for all $S, T \in QL_{\mathscr{H},\mathscr{K}}(U,V)^*_{\oplus}$. □

From now on, we fix an element R of $QL^*_{\mathscr{K},\mathscr{H}}(V,U) \setminus \{e_{(U,V)}\}$, and also fix a sandwich semigroup of transformations $\mathcal{S}_{\chi^{(R)}}$ from I into J with $\mathcal{S} \subseteq P^*_{\mathscr{H},\mathscr{K}}(I,J)$. Then, by Proposition 4.2, a sandwich semigroup of transformations from U into V, namely $\left(QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*\right)_R$, is obtained. Additionally, if every element of $\mathcal{S}$ is injective and $R \in QL^*_{\mathscr{K},\mathscr{H}}(V,U)_{\oplus}$ with $\chi^{(R)}$ injective, we obtain another sandwich semigroup of transformations from U into V, namely $\left(QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*_{\oplus}\right)_R$. In particular, when $\mathcal{S} = \widehat{P}^*_{\mathscr{H},\mathscr{K}}(I,J)$, we have the sandwich semigroup $\left(\widehat{QL}^*_{\mathscr{H},\mathscr{K}}(U,V)_{\oplus}\right)_R$.

5. Regularity of elements of $\left(L^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*\right)_R$ and $\left(QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*_{\oplus}\right)_R$

Throughout this section, the vector spaces H_i and K_j are assumed to be finite dimensional for all $i \in I$ and $j \in J$. We aim to investigate the regularity of $\left(L^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*\right)_R$ when $\mathcal{S} \subseteq T^*_{\mathscr{H},\mathscr{K}}(I,J)$, and of $\left(QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U,V)^*_{\oplus}\right)_R$ when every element of $\mathcal{S}$ is injective and $R \in QL^*_{\mathscr{K},\mathscr{H}}(V,U)_{\oplus}$ with $\chi^{(R)}$ injective. To proceed, we use the following lemma from Magill and Subbiah [2], restated in a form adapted to the present setting.

Lemma 5.1. [2, Lemma 3.1] *Let Z and W be nonempty sets. For any $f \in P(Z,W)$ and $\theta \in P(W,Z)$, the transformation f is θ-idempotent if and only if $(y)\theta \circ f = y$ for all $y \in \mathrm{Ran}(f)$.*

Additionally, Definition 1.1 and Definition 1.2 are also modified here to align with our current framework.

Definition 5.2. Let X and Y be nonempty sets, and let $\mathcal{H} \subseteq P(X,Y)$. Also, let $\theta \in P(Y,X)$. A subset A of Y is called an *$(\mathcal{H},\theta)$-retract of Y* if there exists a θ-idempotent transformation f in $\mathscr{H}$ from X into Y such that $A = \mathrm{Ran}(f)$.

Definition 5.3. Let X and Y be nonempty sets, and let $\mathcal{H} \subseteq P(X,Y)$. Also, let $\theta \in P(Y,X)$. For any subsets A and B of Y, we say that *A is $(\mathcal{H},\theta)$-isomorphic to B* if there exist $f, g \in \mathcal{H}$ such that $(A)\theta \circ f \subseteq B$, $(B)\theta \circ g \subseteq A$, $(\theta \circ f \circ \theta \circ g)\big|_A = \mathrm{id}_A$ and $(\theta \circ g \circ \theta \circ f)\big|_B = \mathrm{id}_B$. In this situation, the map $(\theta \circ f)\big|_A$ is called an *$(\mathcal{H},\theta)$-isomorphism from A onto B.*

The following lemmas will serve as our primary tools for the investigation.

Lemma 5.4. *Let $\gamma \in P^*_{\mathscr{H},\mathscr{K}}(I,J)$, and let A be a subspace of V such that* $\dim(H_i) \leq \dim(A \cap K_{(i)\gamma})$ *for all* $i \in \mathrm{Dom}(\gamma)$ *and* $A = \bigoplus_{j \in \mathrm{ind}(A)} A \cap K_j$. *If γ is a $\chi^{(R)}$-idempotent transformation and* $\mathrm{ind}(A) = \mathrm{Ran}(\gamma)$, *then there exists an R-idempotent transformation T in $QL^*_{\mathscr{H},\mathscr{K}}(U,V)_{\oplus}$ such that $A = \mathrm{Ran}(T)$ and $\chi^{(T)} = \gamma$. Moreover, if $\gamma \in T^*_{\mathscr{H},\mathscr{K}}(I,J)$, then such an R-idempotent transformation T can be chosen from $L^*_{\mathscr{H},\mathscr{K}}(U,V)$.*

Proof. Suppose that γ is a $\chi^{(R)}$-idempotent transformation and $\operatorname{ind}(A)=\operatorname{Ran}(\gamma)$. Then, by Lemma 5.1, we have $(j)\chi^{(R)}\circ\gamma=j$ for all $j\in\operatorname{Ran}(\gamma)$. Let $W=\sum_{i\in\operatorname{Dom}(\gamma)}H_i$. Since the sum $\sum_{i\in I}H_i$ is direct, the sums $\sum_{i\in\operatorname{Dom}(\gamma)}H_i$ is also direct; hence, $W=\bigoplus_{i\in\operatorname{Dom}(\gamma)}H_i$. Since $(j)\chi^{(R)}\gamma=j$ for all $j\in\operatorname{Ran}(\gamma)$, we obtain for every $j\in\operatorname{Ran}(\gamma)$ that $(j)\chi^{(R)}\in(j)\gamma^{-1}$. First, let $j\in\operatorname{Ran}(\gamma)$ be arbitrarily given, and let $\mathscr{C}_j$ be a basis for $A\cap K_j$. Since $R\big|_{A\cap K_j}$ is injective, the set $(\mathscr{C}_j)\,R$ is a linearly independent set of $H_{(j)\chi^{(R)}}$ with $|(\mathscr{C}_j)\,R|=|\mathscr{C}_j|$. For each $i\in(j)\gamma^{-1}$, let $\mathscr{B}_i^{(j)}$ be a fixed basis for H_i if $i\neq(j)\chi^{(R)}$, and let $\mathscr{B}_{(j)\chi^{(R)}}^{(j)}$ be a fixed basis for $H_{(j)\chi^{(R)}}$ that contains $(\mathscr{C}_j)\,R$. Since $\mathscr{B}_{(j)\chi^{(R)}}^{(j)}$ contains $(\mathscr{C}_j)\,R$ with $|\mathscr{C}_j|=|(\mathscr{C}_j)\,R|$ and $\dim\left(H_{(j)\chi^{(R)}}\right)\le\dim\left(A\cap K_j\right)$, we get $\mathscr{B}_{(j)\chi^{(R)}}^{(j)}=(\mathscr{C}_j)\,R$. For each $i\in(j)\gamma^{-1}\setminus\left\{(j)\chi^{(R)}\right\}$, by the inequality $\dim(H_i)\le\dim\left(A\cap K_j\right)$, we fix an injective function $f_i^{(j)}$ from $\mathscr{B}_i^{(j)}$ into $\mathscr{C}_j$. Note that $\left(R\big|_{\mathscr{C}_j}\right)^{-1}$ is a bijective function from $(\mathscr{C}_j)\,R=\mathscr{B}_{(j)\chi^{(R)}}^{(j)}$ onto $\mathscr{C}_j$. Let $\mathscr{B}_j=\bigcup_{i\in(j)\gamma^{-1}}\mathscr{B}_i^{(j)}$, and let $T_0^{(j)}:\mathscr{B}_j\to V$ be defined by

$$T_0^{(j)}\big|_{\mathscr{B}_i^{(j)}}=\begin{cases}f_i^{(j)} & \text{if } i\in(j)\gamma^{-1}\setminus\left\{(j)\chi^{(R)}\right\},\\ \left(R\big|_{\mathscr{C}_j}\right)^{-1} & \text{if } i=(j)\chi^{(R)}.\end{cases}$$

Then $T_0^{(j)}\big|_{\mathscr{B}_i^{(j)}}$ is injective for all $i\in(j)\gamma^{-1}$. Additionally, $\operatorname{Ran}\left(T_0^{(j)}\big|_{\mathscr{B}_{(j)\chi^{(R)}}^{(j)}}\right)=\mathscr{C}_j$ and $\operatorname{Ran}\left(T_0^{(j)}\big|_{\mathscr{B}_i^{(j)}}\right)\subseteq\mathscr{C}_j$ if $i\neq(j)\chi^{(R)}$. Let $\mathscr{B}=\bigcup_{j\in\operatorname{Ran}(\gamma)}\mathscr{B}_j$, and let $T_0:\mathscr{B}\to V$ be defined by $T_0\big|_{\mathscr{B}_j}=T_0^{(j)}$ for all $j\in\operatorname{Ran}(\gamma)$. Moreover, let T be the linear extension of T_0 to W. We claim that $T\in QL^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus$, and T is R-idempotent with $A=\operatorname{Ran}(T)$ and $\chi^{(T)}=\gamma$. Obviously, $\operatorname{ind}(\operatorname{Dom}(T))=\operatorname{ind}(W)=\operatorname{Dom}(\gamma)$. Since $T\big|_{\mathscr{B}_i^{(j)}}=T_0^{(j)}\big|_{\mathscr{B}_i^{(j)}}$ is an injective function from $\mathscr{B}_i^{(j)}$ into $K_{(i)\gamma}$ for all $j\in\operatorname{Ran}(\gamma)$ and $i\in(j)\gamma^{-1}$, we get $T\big|_{H_i}$ is an injective linear transformation form H_i into $K_{(i)\gamma}$ for all $i\in\operatorname{Dom}(\gamma)$. This implies that $T\in PL^*_{\mathscr{H},\mathscr{K}}(U,V)=QL^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus$ with $\chi^{(T)}=\gamma$. It is clear that $T\in L^*_{\mathscr{H},\mathscr{K}}(U,V)$ if $\gamma\in T^*_{\mathscr{H},\mathscr{K}}(I,J)$. For each $j\in\operatorname{Ran}(\gamma)$, since $\operatorname{Ran}\left(T_0^{(j)}\big|_{\mathscr{B}_i^{(j)}}\right)\subseteq\mathscr{C}_j$ for all $i\in(j)\gamma^{-1}\setminus\left\{(j)\chi^{(R)}\right\}$ and $\operatorname{Ran}\left(T_0^{(j)}\big|_{\mathscr{B}_{(j)\chi^{(R)}}^{(j)}}\right)=\mathscr{C}_j$, we get $\operatorname{Ran}\left(T\big|_{\mathscr{B}_j}\right)=\mathscr{C}_j$ for all $j\in\operatorname{Ran}(\gamma)$. Let $\mathscr{C}=\bigcup_{j\in\operatorname{Ran}(\gamma)}\mathscr{C}_j$. Since $A=\bigoplus_{j\in\operatorname{Ran}(\gamma)}A\cap K_{(j)\gamma}$, the set $\mathscr{C}$ is a basis for A. From the definition of T, we have $(\mathscr{B})T=\left(\bigcup_{j\in\operatorname{Ran}(\gamma)}\mathscr{B}_j\right)T=\bigcup_{j\in\operatorname{Ran}(\gamma)}(\mathscr{B}_j)\,T=\bigcup_{j\in\operatorname{Ran}(\gamma)}\mathscr{C}_j=\mathscr{C}$. This implies that $\operatorname{Ran}(T)=A$. To complete the proof, it remains to show that T is R-idempotent. By Lemma 5.1, we will prove that $(v)R\circ T=v$ for all $v\in\operatorname{Ran}(T)=A$. To this end, let $v\in\mathscr{C}$. Then there exists $j\in\operatorname{Ran}(\gamma)$ such that $v\in\mathscr{C}_j$. Thus,

$$(v)R\circ T=\left((v)R\big|_{\mathscr{C}_j}\right)T\big|_{(\mathscr{C}_j)R}=\left((v)R\big|_{\mathscr{C}_j}\right)T_0^{(j)}\big|_{(\mathscr{C}_j)R}=\left((v)R\big|_{\mathscr{C}_j}\right)\left(R\big|_{\mathscr{C}_j}\right)^{-1}=v.$$

Therefore, T is R-idempotent as required. □

LEMMA 5.5. *Let $T\in QL^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus\setminus\{e_{(U,V)}\}$. Then the following statements hold.*

(1) $\operatorname{Ran}(T) = \bigoplus_{j \in \operatorname{Ran}(\chi^{(T)})} \operatorname{Ran}(T) \cap K_j$.

(2) *For all* $j \in \operatorname{Ran}(\chi^{(T)})$, $\operatorname{Ran}(T) \cap K_j = \sum_{i \in (j)\left(\chi^{(T)}\right)^{-1}} (H_i)\, T$.

Proof. (1) Since the sum $\sum_{j \in J} K_j$ is direct, the sum $\sum_{j \in \operatorname{Ran}(\chi^{(T)})} \operatorname{Ran}(T) \cap K_j$ is also direct. To prove that $\operatorname{Ran}(T) = \bigoplus_{j \in \operatorname{Ran}(\chi^{(T)})} \operatorname{Ran}(T) \cap K_j$, it remains to show that $\operatorname{Ran}(T) \subseteq \sum_{j \in \operatorname{Ran}(\chi^{(T)})} \operatorname{Ran}(T) \cap K_j$. Let $u \in \operatorname{Dom}(T) \setminus \{0\}$. Since $\operatorname{Dom}(T) = \bigoplus_{i \in \operatorname{ind}(\operatorname{Dom}(T))} H_i$, there exist distinct elements $i_1, i_2, ..., i_n$ of $\operatorname{ind}(\operatorname{Dom}(T)) = \operatorname{Dom}(\chi^{(T)})$, together with an element $(u_1, u_2, ..., u_n)$ of $\prod_{k=1}^{n} H_{i_k} \setminus \{0\}$, such that $u = \sum_{k=1}^{n} u_k$. For each $k = 1, 2, ..., n$, since $u_k \neq 0$ and $T\big|_{H_{i_k}}$ is injective, we get $0 \neq (u_k)\, T \in K_{(i_k)\chi^{(T)}}$. Therefore, $(u)T = \sum_{k=1}^{n} (u_k)\, T \in \sum_{j \in \operatorname{Ran}(\chi^{(T)})} \operatorname{Ran}(T) \cap K_j$.

(2) Let $j \in \operatorname{Ran}\left(\chi^{(T)}\right)$. It is clear that $\sum_{i \in (j)\left(\chi^{(T)}\right)^{-1}} (H_i)\, T \subseteq \operatorname{Ran}(T) \cap K_j$. To see that the reverse inclusion holds, let $v \in \operatorname{Ran}(T) \cap K_j$. Then there exist distinct elements $i_1, i_2, ..., i_n$ of $\operatorname{ind}(\operatorname{Dom}(T)) = \operatorname{Dom}\left(\chi^{(T)}\right)$, together with an element $(u_1, u_2, ..., u_n)$ of $\prod_{k=1}^{n} H_{i_k} \setminus \{0\}$, such that $\sum_{k=1}^{n} (u_k) T = v$. Note that $(u_k)T \in \operatorname{Ran}(T) \cap K_{(i_k)\chi^{(T)}} \setminus \{0\}$ for all $k = 1, 2, \ldots, n$. Since $\operatorname{Ran}(T) = \bigoplus_{l \in \operatorname{Ran}(\chi^{(T)})} \operatorname{Ran}(T) \cap K_l$ and $v \in K_j$, it follows that $i_k \in (j)\left(\chi^{(T)}\right)^{-1}$ for all $k = 1, 2, \ldots, n$. Hence, $v \in \sum_{i \in (j)\left(\chi^{(T)}\right)^{-1}} (H_i)\, T$. □

Proposition 5.6. *For any subspace A of V and $\mathcal{H} \subseteq P^{*}_{\mathscr{H},\mathscr{K}}(I, J)$, the set A is a $\left(QL^{(\mathcal{H})}_{\mathscr{H},\mathscr{K}}(U, V)^{*}_{\oplus}, R\right)$-retract of V if and only if there exists a $\chi^{(R)}$-idempotent transformation γ in $\mathcal{H}$ such that* $\operatorname{ind}(A) = \operatorname{Ran}(\gamma)$, $A = \bigoplus_{j \in \operatorname{Ran}(\gamma)} A \cap K_j$ *and* $\dim(H_i) \leq \dim\left(A \cap K_{(i)\gamma}\right)$ *for all* $i \in \operatorname{Dom}(\gamma)$.

Proof. Let A be a subspace of V, and let $\mathcal{H} \subseteq P^{*}_{\mathscr{H},\mathscr{K}}(I, J)$.

($\Rightarrow$) Suppose that A is a $\left(QL^{(\mathcal{H})}_{\mathscr{H},\mathscr{K}}(U, V)^{*}_{\oplus}, R\right)$-retract of V. Then there exists an R-idempotent transformation T in $QL^{(\mathcal{H})}_{\mathscr{H},\mathscr{K}}(U, V)^{*}_{\oplus}$ such that $A = \operatorname{Ran}(T)$. By statement (2) of Proposition 3.4, we get $\operatorname{ind}(A) = \operatorname{ind}(\operatorname{Ran}(T)) = \operatorname{Ran}(\chi^{(T)})$. Furthermore, by statement (4) of Proposition 3.2, the function $\chi^{(T)}$ is a $\chi^{(R)}$-idempotent transformation in $\mathcal{H}$. Moreover, by statement (1) of Lemma 5.5, we get $A = \operatorname{Ran}(T) = \bigoplus_{j \in \operatorname{Ran}\left(\chi^{(T)}\right)} \operatorname{Ran}(T) \cap K_j$. Finally, since $T \in QL^{*}_{\mathscr{H},\mathscr{K}}(U, V)$, we have $\dim(H_i) \leq \dim\left(\operatorname{Ran}(T) \cap K_{(i)\chi^{(T)}}\right)$ for all $i \in \operatorname{Dom}\left(\chi^{(T)}\right)$.

($\Leftarrow$) Suppose that there exists a $\chi^{(R)}$-idempotent transformation γ in $\mathcal{H}$ such that $\operatorname{ind}(A) = \operatorname{Ran}(\gamma)$, $A = \bigoplus_{j \in \operatorname{Ran}(\gamma)} A \cap K_j$ and $\dim(H_i) \leq \dim\left(A \cap K_{(i)\gamma}\right)$ for all $i \in \operatorname{Dom}(\gamma)$. Then, by Lemma 5.4, there exists an R-idempotent transformation T in $QL^{(\mathcal{H})}_{\mathscr{H},\mathscr{K}}(U, V)^{*}_{\oplus}$ such that $A = \operatorname{Ran}(T)$. Therefore, A is a $\left(QL^{(\mathcal{H})}_{\mathscr{H},\mathscr{K}}(U, V)^{*}_{\oplus}, R\right)$-retract of V. □

Proposition 5.7. *Let A and B be subspaces of V such that $A = \bigoplus_{j \in \operatorname{ind}(A)} A \cap K_j$ and $B = \bigoplus_{j \in \operatorname{ind}(B)} B \cap K_j$. Also, let $\mathcal{H} \subseteq P^{*}_{\mathscr{H},\mathscr{K}}(I, J)$ and $T \in Q^{(\mathcal{H})}_{\mathscr{H},\mathscr{K}}(U, V)^{*}_{\oplus}$. Then $(R \circ T)\big|_A$ is a $\left(QL^{(\mathcal{H})}_{\mathscr{H},\mathscr{K}}(U, V)^{*}_{\oplus}, R\right)$-isomorphism from A onto B if and only if the following conditions hold:*

(i) $(\chi^{(R)} \circ \chi^{(T)})\big|_{\operatorname{ind}(A)}$ *is an* $\left(\mathcal{H}, \chi^{(R)}\right)$*-isomorphism from* $\operatorname{ind}(A)$ *onto* $\operatorname{ind}(B)$*;*

(ii) $(R\circ T)\big|_{A\cap K_j}$ *is an isomorphism from* $A\cap K_j$ *onto* $B\cap K_{(j)\chi^{(R\circ T)}}$ *for all* $j\in \mathrm{ind}(A)$.

Proof. ($\Rightarrow$) Suppose that $(R\circ T)\big|_A$ is a $\left(QL^{(\mathcal{H})}_{\mathscr{H},\mathscr{K}}(U,V)^*_{\oplus}, R\right)$-isomorphism from A onto B. Then $(A)(R\circ T)\subseteq B$. Moreover, there exists $S\in QL^{(\mathcal{H})}_{\mathscr{H},\mathscr{K}}(U,V)^*_{\oplus}$ such that $(B)(R\circ S)\subseteq A$, $(R\circ T\circ R\circ S)\big|_A=\mathrm{id}_A$ and $(R\circ S\circ R\circ T)\big|_B=\mathrm{id}_B$. To get that $(\chi^{(R)}\circ\chi^{(T)})\big|_{\mathrm{ind}(A)}$ is an $\big(\mathcal{H},\chi^{(R)}\big)$-isomorphism from $\mathrm{ind}(A)$ onto $\mathrm{ind}(B)$, we prove the following statements:

(a) $(\mathrm{ind}(A))\chi^{(R)}\circ\chi^{(T)}\subseteq \mathrm{ind}(B)$;
(b) $(\mathrm{ind}(B))\chi^{(R)}\circ\chi^{(S)}\subseteq \mathrm{ind}(A)$;
(c) $\left(\chi^{(R)}\circ\chi^{(T)}\circ\chi^{(R)}\circ\chi^{(S)}\right)\big|_{\mathrm{ind}(A)}=\mathrm{id}_{\mathrm{ind}(A)}$;
(d) $\left(\chi^{(R)}\circ\chi^{(S)}\circ\chi^{(R)}\circ\chi^{(T)}\right)\big|_{\mathrm{ind}(B)}=\mathrm{id}_{\mathrm{ind}(B)}$.

Since $(A)(R\circ T)\subseteq B$ and $(B)(R\circ S)\subseteq A$, by statement (4) of Proposition 3.2 and statement (2) of Proposition 3.3, we get $(\mathrm{ind}(A))\chi^{(R)}\circ\chi^{(T)}=(\mathrm{ind}(A))\chi^{(R\circ T)}\subseteq\mathrm{ind}(B)$ and $(\mathrm{ind}(B))\chi^{(R)}\circ\chi^{(S)}=(\mathrm{ind}(B))\chi^{(R\circ S)}\subseteq\mathrm{ind}(A)$. Since $(R\circ T\circ R\circ S)\big|_A=\mathrm{id}_A$, by statement (4) of Proposition 3.2 and statement (1) of Proposition 3.3, it follows that $\left(\chi^{(R)}\circ\chi^{(T)}\circ\chi^{(R)}\circ\chi^{(S)}\right)\big|_{\mathrm{ind}(A)}=\chi^{((R\circ T\circ R\circ S)|_A)}=\chi^{(\mathrm{id}_A)}=\mathrm{id}_{\mathrm{ind}(A)}$. Similarly, since $(R\circ S\circ R\circ T)\big|_B=\mathrm{id}_B$, we obtain $\left(\chi^{(R)}\circ\chi^{(S)}\circ\chi^{(R)}\circ\chi^{(T)}\right)\big|_{\mathrm{ind}(B)}=\mathrm{id}_{\mathrm{ind}(B)}$. Accordingly, $(\chi^{(R)}\circ\chi^{(T)})\big|_{\mathrm{ind}(A)}$ is an $\big(\mathcal{H},\chi^{(R)}\big)$-isomorphism from $\mathrm{ind}(A)$ onto $\mathrm{ind}(B)$ as required. This implies that $\chi^{(R\circ T)}\big|_A=(\chi^{(R)}\circ\chi^{(T)})\big|_{\mathrm{ind}(A)}:\mathrm{ind}(A)\to\mathrm{ind}(B)$ is bijective. Additionally, since $R\circ T\in QL^*_{\mathscr{K}}(V)$ and $A\subseteq\mathrm{Dom}(R\circ T)$, we have $\mathrm{ind}(A)\subseteq\mathrm{ind}(\mathrm{Dom}(R\circ T))=\mathrm{ind}^*(\mathrm{Dom}(R\circ T))$. Thus, by the inclusion $(A)(R\circ T)\subseteq B$, we get $(A\cap K_j)(R\circ T)\subseteq B\cap K_{(j)\chi^{(R\circ T)}}$ for all $j\in\mathrm{ind}(A)$. Furthermore, since $(R\circ T)\big|_A$ is a bijective function A onto B and $\chi^{(R\circ T)}\big|_A=(\chi^{(R)}\circ\chi^{(T)})\big|_{\mathrm{ind}(A)}$ is injective, we have for each $j\in\mathrm{ind}(A)$ that $(R\circ T)\big|_{K_j\cap A}$ is a bijective function from $A\cap K_j$ onto $B\cap K_{(j)\chi^{(R\circ T)}}$.

($\Leftarrow$) Suppose that conditions (i) and (ii) hold. Then, by condition (i), the function $\chi^{(R\circ T)}\big|_{\mathrm{ind}(A)}=\left(\chi^{(R)}\circ\chi^{(T)}\right)\big|_{\mathrm{ind}(A)}:\mathrm{ind}(A)\to\mathrm{ind}(B)$ is bijective. It also yields that there exists $\gamma\in\mathcal{H}$ such that $\left(\chi^{(R)}\circ\gamma\right)\big|_{\mathrm{ind}(B)}=\left(\chi^{(R\circ T)}\big|_{\mathrm{ind}(A)}\right)^{-1}$. This implies that $\mathrm{ind}(A)\subseteq\mathrm{Ran}(\gamma)$ and $\left((j)\chi^{(R\circ T)}\right)\chi^{(R)}\in(j)\gamma^{-1}$ for all $j\in\mathrm{ind}(A)$. Let $j\in\mathrm{Ran}(\gamma)$ be arbitrarily fixed. For each $i\in(j)\gamma^{-1}$, let $\mathscr{B}^{(j)}_i$ be a fixed basis for H_i, and let $\mathscr{B}_j=\bigcup_{i\in(j)\gamma^{-1}}\mathscr{B}^{(j)}_i$. Then, by the inequality $\dim(H_i)\le\dim(K_j)$ for all $i\in(j)\gamma^{-1}$, we fix, for each $i\in(j)\gamma^{-1}$, an injective function $f^{(j)}_i$ from $\mathscr{B}^{(j)}_i$ into K_j. If $j\notin\mathrm{ind}(A)$, let $S^{(j)}_0:\mathscr{B}_j\to V$ be simply defined by $S^{(j)}_0\big|_{\mathscr{B}^{(j)}_i}=f^{(j)}_i$ for all $i\in(j)\gamma^{-1}$. Assume that $j\in\mathrm{ind}(A)$. In this case, we have $\left((j)\chi^{(R\circ T)}\right)\chi^{(R)}\in(j)\gamma^{-1}$. Let $\mathscr{E}_j$ be a basis for $A\cap K_j$, and let $\mathscr{C}_j=(\mathscr{E}_j)\,R\circ T$. Then, by condition (ii), the set $\mathscr{C}_j$ is basis for $B\cap K_{(j)\chi^{(R\circ T)}}$. In this case, we replace the basis $\mathscr{B}^{(j)}_{\left((j)\chi^{(R\circ T)}\right)\chi^{(R)}}$ for $H_{\left((j)\chi^{(R\circ T)}\right)\chi^{(R)}}$ by another fixed one containing $(\mathscr{C}_j)\,R$. Let $\mathscr{E}'_j$ be a basis for K_j containing $\mathscr{E}_j$. Since $\dim\left(H_{\left((j)\chi^{(R\circ T)}\right)\chi^{(R)}}\right)\le\dim(K_j)$ and $|(\mathscr{C}_j)\,R|=|\mathscr{C}_j|=|\mathscr{E}_j|$, it follows that

$$\left|\mathscr{B}^{(j)}_{\left((j)\chi^{(R\circ T)}\right)\chi^{(R)}}\setminus(\mathscr{C}_j)\,R\right|\le\left|\mathscr{E}'_j\setminus\mathscr{E}_j\right|.$$

Therefore, we are able to fix an injective function g_j from $\mathscr{B}^{(j)}_{\left((j)\chi^{(R\circ T)}\right)\chi^{(R)}} \setminus (\mathscr{C}_j)\, R$ into $\mathscr{E}'_j \setminus \mathscr{E}_j$. In this case, we replace the function $f^{(j)}_{\left((j)\chi^{(R\circ T)}\right)\chi^{(R)}}$ by the one defined by

$$f^{(j)}_{\left((j)\chi^{(R\circ T)}\right)\chi^{(R)}}\Big|_{\mathscr{B}^{(j)}_{\left((j)\chi^{(R\circ T)}\right)\chi^{(R)}}\setminus(\mathscr{C}_j)R} = g_j$$

and

$$f^{(j)}_{\left((j)\chi^{(R\circ T)}\right)\chi^{(R)}}\Big|_{(\mathscr{C}_j)R} = \left(R\big|_{\mathscr{C}_j}\right)^{-1} \circ \left((R\circ T)\big|_{\mathscr{E}_j}\right)^{-1}.$$

Note that by condition (ii) and the injectivity of $R\big|_{K_j}$, the function $f^{(j)}_{\left((j)\chi^{(R\circ T)}\right)\chi^{(R)}}$ is well-defined. Similar to the case when $j \notin \operatorname{ind}(A)$, let $S^{(j)}_0 : \mathscr{B}_j \to V$ be defined by $S^{(j)}_0\big|_{\mathscr{B}^{(j)}_i} = f^{(j)}_i$ for all $i \in (j)\gamma^{-1}$. Now, we have, for every $j \in \operatorname{Ran}(\gamma)$, the function $S^{(j)}_0 : \mathscr{B}_j \to V$, as defined above. Let $\mathscr{B} = \bigcup_{j\in\operatorname{Ran}(\gamma)} \mathscr{B}_j$ and $Z = \sum_{j\in\operatorname{Ran}(\gamma)} H_j$. Then $Z = \bigoplus_{j\in\operatorname{Ran}(\gamma)} H_j$; so, $\mathscr{B}$ is a basis for Z. Additionally, let $S_0 : \mathscr{B} \to V$ be defined by $S_0\big|_{\mathscr{B}_j} = S^{(j)}_0$ for all $j \in \operatorname{Ran}(\gamma)$, and let S be the linear extension of S_0 to Z. It is easy to see that $S \in QL_{\mathscr{H},\mathscr{K}}(U,V)^*_\oplus$ with $\chi^{(S)} = \gamma$; thus, $S \in QL^{(\mathcal{H})}_{\mathscr{H},\mathscr{K}}(U,V)^*_\oplus$. Finally, we will show that S satisfies the condition for $(R\circ T)\big|_A$ to be a $\left(QL^{(\mathcal{H})}_{\mathscr{H},\mathscr{K}}(U,V)^*_\oplus, R\right)$-isomorphism from A onto B. By condition (ii) and the assumption that $A = \bigoplus_{j\in\operatorname{ind}(A)} A\cap K_j$, we immediately have $A \subseteq \operatorname{Dom}(R\circ T)$ and $(A)R\circ T \subseteq B$. For each $j \in \operatorname{ind}(B)$, let $k_j = (j)\left(\chi^{(R\circ T)}\big|_{\operatorname{ind}(A)}\right)^{-1}$. Since $\mathscr{C}_{k_j}$ is a basis for $B\cap K_j$ and $\left(\mathscr{C}_{k_j}\right) R \subseteq \operatorname{Dom}(S)$ for all $j \in \operatorname{ind}(B)$, we get for each $j \in \operatorname{ind}(B)$ that $\mathscr{C}_{k_j} \subseteq \operatorname{Dom}(R\circ S)$. Thus, by the assumption that $B = \bigoplus_{j\in\operatorname{ind}(B)} B\cap K_j$, we obtain $B \subseteq \operatorname{Dom}(R\circ S)$. Moreover, since

$$\begin{aligned}\left(\mathscr{C}_{k_j}\right) R\circ S &= \left(\left(\mathscr{C}_{k_j}\right) R\right)\left(R\big|_{\mathscr{C}_{k_j}}\right)^{-1} \circ \left((R\circ T)\big|_{\mathscr{E}_{k_j}}\right)^{-1} \\ &= \left(\left(\mathscr{E}_{k_j}\right) R\circ T\right)\left((R\circ T)\big|_{\mathscr{E}_{k_j}}\right)^{-1} = \mathscr{E}_{k_j} \subseteq A\end{aligned}$$

for all $j \in \operatorname{ind}(B)$ and $B = \bigoplus_{j\in\operatorname{ind}(B)} B\cap K_j$, we get $(B)R\circ S \subseteq A$. Note that for each $j \in \operatorname{ind}(A)$, we have

$$\begin{aligned}(v)(R\circ T\circ R\circ S)\big|_{\mathscr{E}_j} &= \left((v)(R\circ T)\big|_{\mathscr{E}_j}\right)(R\circ S)\big|_{\mathscr{C}_j} \\ &= \left(\left((v)(R\circ T)\big|_{\mathscr{E}_j}\right) R\right)\left(R\big|_{\mathscr{C}_j}\right)^{-1} \circ \left((R\circ T)\big|_{\mathscr{E}_j}\right)^{-1} \\ &= \left((v)(R\circ T)\big|_{\mathscr{E}_j}\right)\left((R\circ T)\big|_{\mathscr{E}_j}\right)^{-1} = v\end{aligned}$$

for all $v \in \mathscr{E}_j$. Since $A = \bigoplus_{j\in\mathrm{ind}(A)} A \cap K_j$, it follows that $(R \circ T \circ R \circ S)\big|_A = \mathrm{id}_A$. For each $j \in \mathrm{ind}(B)$, we see that if $k \in \mathrm{ind}(A)$ such that $(k)\chi^{(R\circ T)} = j$, then

$$\begin{aligned}(v)(R \circ S \circ R \circ T)\big|_{\mathscr{C}_k} &= \left((v)(R \circ S)\big|_{\mathscr{C}_k}\right)(R \circ T)\big|_{\mathscr{E}_k}\\ &= \left(((v)R)\left(R\big|_{\mathscr{C}_k}\right)^{-1} \circ \left((R \circ T)\big|_{\mathscr{E}_k}\right)^{-1}\right)(R \circ T)\big|_{\mathscr{E}_k}\\ &= \left((v)\left((R \circ T)\big|_{\mathscr{E}_k}\right)^{-1}\right)(R \circ T)\big|_{\mathscr{E}_k} = v\end{aligned}$$

for all $v \in \mathscr{C}_k$. Since $B = \bigoplus_{j\in\mathrm{ind}(B)} B \cap K_j$, we obtain $(R \circ S \circ R \circ T)\big|_B = \mathrm{id}_B$. Therefore, $(R \circ T)\big|_A$ is a $\left(QL^{(\mathcal{H})}_{\mathscr{H},\mathscr{K}}(U, V)^*_\oplus, R\right)$-isomorphism from A onto B. □

If $\mathcal{S} \subseteq T^*_{\mathscr{H},\mathscr{K}}(I, J)$, then $QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U, V)^*_\oplus = L^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U, V)^*$. In this situation, we deal with the semigroup $\left(L^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U, V)^*\right)_R$. If every element of $\mathcal{S}$ is injective and $R \in QL^*_{\mathscr{K},\mathscr{H}}(V, U)_\oplus$ with $\chi^{(R)}$ is injective, then the semigroup $\left(QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U, V)^*_\oplus\right)_R$ arises in this case. From Theorem 1.3, Proposition 5.7 and Proposition 5.6, the regularity of $\left(L^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U, V)^*\right)_R$ and $\left(QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U, V)^*_\oplus\right)_R$ can immediately be characterized as follows.

Theorem 5.8. *Let Λ be either $L^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U, V)^*$ or $QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U, V)^*_\oplus$, where $L^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U, V)^*$ is considered when $\mathcal{S} \subseteq T^*_{\mathscr{H},\mathscr{K}}(I, J)$ and $QL^{(\mathcal{S})}_{\mathscr{H},\mathscr{K}}(U, V)^*_\oplus$ is considered when every element of $\mathcal{S}$ is injective and $R \in QL^*_{\mathscr{K},\mathscr{H}}(V, U)_\oplus$ with $\chi^{(R)}$ injective. Then, for any $T \in \Lambda_R$, the following statements are equivalent.*

1. *T is regular.*
2. *There exist a subspace A of V and an idempotent element γ of $\mathcal{S}_{\chi^{(R)}}$ such that $\mathrm{ind}(A) = \mathrm{Ran}(\gamma)$, $A = \bigoplus_{j\in\mathrm{Ran}(\gamma)} A \cap K_j$, $\dim(H_i) \le \dim(A \cap K_{(i)\gamma})$ for all $i \in \mathrm{Dom}(\gamma)$, $\left(\chi^{(R)} \circ \chi^{(T)}\right)\big|_{\mathrm{ind}(A)}$ is an $\mathcal{S}$-isomorphism from $\mathrm{Ran}(\gamma)$ onto $\mathrm{Ran}(\chi^{(T)})$ and $(R \circ T)\big|_{A\cap K_i}$ is an isomorphism from $A \cap K_j$ onto $\mathrm{Ran}(T) \cap K_{(j)\chi^{(R\circ T)}}$ for all $j \in \mathrm{Ran}(\gamma)$.*
3. *There exists a subspace A of V such that $A = \bigoplus_{j\in\mathrm{ind}(A)} A\cap K_j$, $\left(\chi^{(R)} \circ \chi^{(T)}\right)\big|_{\mathrm{ind}(A)}$ is an $\mathcal{S}$-isomorphism from $\mathrm{ind}(A)$ onto $\mathrm{Ran}\left(\chi^{(T)}\right)$ and $(R \circ T)\big|_{A\cap K_j}$ is an isomorphism from $A \cap K_j$ onto $\mathrm{Ran}(T) \cap K_{(j)\chi^{(R\circ T)}}$ for all $j \in \mathrm{ind}(A)$.*

6. Regularity of elements of specific sandwich semigroups of linear transformations

In this section, we assume that H_i and K_j are finite dimensional for all $i \in I$ and $j \in J$. We will apply Theorem 5.8 to investigate the regularity of the semigroups $\left(L^*_{\mathscr{H},\mathscr{K}}(U, V)\right)_R$ and $\left(\widehat{QL}^*_{\mathscr{H},\mathscr{K}}(U, V)_\oplus\right)_R$, where $R \in L^*_{\mathscr{K},\mathscr{H}}(U, V)$. To this end, we first present an auxiliary lemma, analogous to Lemma 3.10 in [6], which will serve as a preparatory step toward establishing our main result.

Lemma 6.1. *Let Z, Y and W be vector spaces over the same field, and let $S \in PL(Z,Y)$ and $T \in PL(Y,W)$. Then, for any nonzero subspace B of* $\mathrm{Ran}(T)$*, there exists a nonzero subspace A of* $\mathrm{Dom}(S)$ *such that $(S \circ T)\big|_A$ is an isomorphism from A onto B if and only if $(v)T^{-1} \cap \mathrm{Ran}(S) \neq \{0\}$ for all $v \in B \setminus \{0\}$.*

Proof. Let B be a nonzero subspace of $\mathrm{Ran}(T)$.

($\Rightarrow$) Suppose that there exists a nonzero subspace A of $\mathrm{Dom}(S)$ such that $(S\circ T)\big|_A$ is an isomorphism from A onto B, and let $v \in B \setminus \{0\}$. Then there exists $u \in A \setminus \{0\}$ such that $(u)S \circ T = v$, which yields that $(u)S \in (v)T^{-1} \cap \mathrm{Ran}(S)$. Since $S\big|_A \circ T = (S \circ T)\big|_A$ is injective, we get $S\big|_A$ is injective. This implies that $(u)S \neq 0$. Therefore, $(v)T^{-1} \cap \mathrm{Ran}(S) \neq \{0\}$.

($\Leftarrow$) Suppose that $(v)T^{-1} \cap \mathrm{Ran}(S) \neq \{0\}$ for all $v \in B \setminus \{0\}$. Let $\mathscr{B}$ be a basis for B. Since $(v)T^{-1} \cap \mathrm{Ran}(S) \neq \{0\}$ for all $v \in \mathscr{B}$, we have for each $v \in \mathscr{B}$ that there exists $z_v \in (v)T^{-1}$ such that $z_v \neq 0$ and $z_v \in \mathrm{Ran}(S)$. Thus, for each $v \in \mathscr{B}$, there exists $a_v \in \mathrm{Dom}(S)$ such that $a_v \neq 0$ and $(a_v)\,S = z_v$. Let $A = \langle\{a_v : v \in \mathscr{B}\}\rangle$. Then A is a subspace of $\mathrm{Dom}(S)$. Since $(\{a_v : v \in \mathscr{B}\})\,S\circ T = \{((a_v)\,S)\,T : v \in \mathscr{B}\} = \mathscr{B}$, it follows that $(A)S\circ T = B$. Hence, $(S\circ T)\big|_A$ is a surjective function from A onto B. To prove injective, let $a \in A \setminus \{0\}$. Then there exist distinct elements $v_1, v_2, \ldots, v_n$ of B and $\alpha_1, \alpha_1, \ldots, \alpha_n \in \mathbb{F} \setminus \{0\}$ such that $a = \sum_{i=1}^n \alpha_i a_{v_i}$. Thus, the linear independence of $\mathscr{B}$, we get

$$(a)S \circ T = \left(\sum_{i=1}^{n} \alpha_i a_{v_i}\right) S \circ T = \sum_{i=1}^{n} \alpha_i \left((a_{v_i})\, S\right) T = \sum_{i=1}^{n} \alpha_i v_i \neq 0.$$

Therefore, $(S\circ T)\big|_A$ is a injective. Consequently, $(S\circ T)\big|_A$ is an isomorphism from A onto B. □

For the remainder of this section, we assume that $R \in L^*_{\mathscr{K},\mathscr{H}}(V,U)$. In the case of the semigroup $\left(\widehat{QL}^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus\right)_R$, we further suppose that $\chi^{(R)}$ is injective. The following theorem states the main result of this section.

Theorem 6.2. *Let Λ be either $L^*_{\mathscr{H},\mathscr{K}}(U,V)$ or $\widehat{QL}^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus$. Then, for any $T \in \Lambda_R$, the element T is regular if and only if $\chi^{(R)}\big|_{\mathrm{Ran}\left(\chi^{(T)}\right)}$ is injective and for each $j \in \mathrm{Ran}\left(\chi^{(T)}\right)$, there exists $k_j \in J$ satisfying the following conditions:*

(i) $(k_j)\,\chi^{(R)} \circ \chi^{(T)} = j$;
(ii) $\dim(H_{(j)\chi^{(R)}}) \leq \dim\left(K_{k_j}\right)$;
(iii) $(v)T^{-1} \cap \left(K_{k_j}\right) R \neq \{0\}$ *for all* $v \in \mathrm{Ran}(T) \cap K_j \setminus \{0\}$.

Proof. Let $\mathcal{S}$ be $T^*_{\mathscr{H},\mathscr{K}}(I,J)$ or $\widehat{P}^*_{\mathscr{H},\mathscr{K}}(I,J)$ if Λ is $L^*_{\mathscr{H},\mathscr{K}}(U,V)$ or $\widehat{QL}^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus$, respectively. Also, let $T \in \Lambda_R$.

($\Rightarrow$) Suppose that T is regular. Then, by Theorem 5.8, there exists a subspace A of V satisfying the following conditions:

(a) $\left(\chi^{(R)} \circ \chi^{(T)}\right)\big|_{\mathrm{ind}(A)}$ is an $\mathcal{S}$-isomorphism from $\mathrm{ind}(A)$ onto $\mathrm{Ran}\left(\chi^{(T)}\right)$;
(b) $(R \circ T)\big|_{A\cap K_j}$ is an isomorphism from $A \cap K_j$ onto $\mathrm{Ran}(T) \cap K_{(j)\chi^{(R\circ T)}}$ for all $j \in \mathrm{ind}(A)$.

Since $\left(\chi^{(R)}\circ\chi^{(T)}\right)\big|_{\mathrm{ind}(A)}$ is an $\mathcal{S}$-isomorphism from $\mathrm{ind}(A)$ onto $\mathrm{Ran}\left(\chi^{(T)}\right)$, we have by Lemma 2.1 that $\chi^{(R)}\big|_{\mathrm{Ran}(\chi^{(T)})}$ is injective and $\left(\chi^{(R)}\circ\chi^{(T)}\right)\big|_{\mathrm{ind}(A)} : \mathrm{ind}(A)\to\mathrm{Ran}(\chi^{(T)})$ is bijective. We also get

$$\dim\left(H_i\right)\le\dim\left(K_{\left((i)\left(\chi^{(R)}\big|_{\mathrm{Ran}(\chi^{(T)})}\right)^{-1}\right)\left(\left(\chi^{(R)}\circ\chi^{(T)}\right)\big|_{\mathrm{ind}(A)}\right)^{-1}}\right)$$

for all $i\in(\mathrm{Ran}(\chi^{(T)}\circ\chi^{(R)})$, which is equivalent to

$$\dim\left(H_{(j)\chi^{(R)}}\right)\le\dim\left(K_{(j)\left(\left(\chi^{(R)}\circ\chi^{(T)}\right)\big|_{A}\right)^{-1}}\right)$$

for all $j\in\mathrm{Ran}\left(\chi^{(T)}\right)$. Let $j\in\mathrm{Ran}\left(\chi^{(T)}\right)$. Then, by the surjectivity of the function $\left(\chi^{(R)}\circ\chi^{(T)}\right)\big|_{\mathrm{ind}(A)}$, there exists $k_j\in\mathrm{ind}(A)$ such that $(k_j)\chi^{(R)}\circ\chi^{(T)}=j$. Since $(k_j)\chi^{(R)}\circ\chi^{(T)}=j$ and $k_j\in\mathrm{ind}(A)$, we have $(j)\left(\left(\chi^{(R)}\circ\chi^{(T)}\right)\big|_{A}\right)^{-1}=k_j$. Hence, $\dim\left(H_{(j)\chi^{(R)}}\right)\le\dim\left(K_{k_j}\right)$. Additionally, since $(R\circ T)\big|_{A\cap K_{k_j}} : A\cap K_{k_j}\to\mathrm{Ran}(T)\cap K_j$ is bijective, we obtain by Lemma 6.1 that $(v)T^{-1}\cap\left(K_{k_j}\right)R\neq\{0\}$ for all $v\in\mathrm{Ran}(T)\cap K_j\setminus\{0\}$.

($\Leftarrow$) Suppose that $\chi^{(R)}\big|_{\mathrm{Ran}\left(\chi^{(T)}\right)}$ is injective and for each $j\in\mathrm{Ran}\left(\chi^{(T)}\right)$, there exists $k_j\in J$ satisfying the following conditions:

(i) $(k_j)\,\chi^{(R)}\circ\chi^{(T)}=j$;
(ii) $\dim(H_{(j)\chi^{(R)}})\le\dim\left(K_{k_j}\right)$;
(iii) $(v)T^{-1}\cap\left(K_{k_j}\right)R\neq\{0\}$ for all $v\in\mathrm{Ran}(T)\cap K_j\setminus\{0\}$.

Note that by statement (2) of Proposition 3.4, we have $\mathrm{Ran}\left(\chi^{(T)}\right)=\mathrm{ind}(\mathrm{Ran}(T))$, which implies that $\mathrm{Ran}(T)\cap K_j\neq\{0\}$ for all $j\in\mathrm{Ran}\left(\chi^{(T)}\right)$. For each $j\in\mathrm{Ran}\left(\chi^{(T)}\right)$, since $(v)T^{-1}\cap\left(K_{k_j}\right)R\neq\{0\}$ for all $v\in\mathrm{Ran}(T)\cap K_j\setminus\{0\}$, by Lemma 6.1, there exists a nonzero subspace A_j of K_{k_j} such that $(R\circ T)\big|_{A_j}$ is an isomorphism from A_j onto $\mathrm{Ran}(T)\cap K_j$. Let $A=\bigoplus_{j\in\mathrm{Ran}\left(\chi^{(T)}\right)}A_j$. Now, we show that $A\cap K_{k_j}=A_j$ for all $j\in\mathrm{Ran}\left(\chi^{(T)}\right)$. To prove this, let $j\in\mathrm{Ran}\left(\chi^{(T)}\right)$. It is clear that $A_j\subseteq A\cap K_{k_j}$. Let $x\in A\cap K_{k_j}$. Since $x\in A=\bigoplus_{r\in\mathrm{Ran}\left(\chi^{(T)}\right)}A_r$, there exist distinct elements $j_1,j_2,\ldots,j_n$ of $\mathrm{Ran}\left(\chi^{(T)}\right)$, together with an element $(u_1,u_2,\ldots,u_n)$ of $\prod_{l=1}^{n}A_{j_l}\setminus\{0\}$, such that $x=\sum_{l=1}^{n}u_l$. Since $u_l\in K_{k_{j_l}}$ for all $l=1,2,\ldots,n$ and $x\in K_{k_j}$, by the directness of the sum $\sum_{j\in J}K_j$, we have $n=1$ and $k_{j_1}=k_j$. Thus, $x=u_1\in A_j$, and so, we get $A\cap K_{k_j}=A_j$ as desired. Since $A\cap K_{k_j}=A_j$ for all $j\in\mathrm{Ran}\left(\chi^{(T)}\right)$, it follows that $\mathrm{ind}(A)=\left\{k_j : j\in\mathrm{Ran}\left(\chi^{(T)}\right)\right\}$. Since $(k_j)\,\chi^{(R)}\circ\chi^{(T)}=j$ for all $j\in\mathrm{Ran}\left(\chi^{(T)}\right)$, the function $\left(\chi^{(R)}\circ\chi^{(T)}\right)\big|_{\mathrm{ind}(A)} : \mathrm{ind}(A)\to\mathrm{Ran}\left(\chi^{(T)}\right)$ is bijective. Since $\dim(H_{(j)\chi^{(R)}})\le\dim\left(K_{k_j}\right)$ and $k_j=(j)\left(\left(\chi^{(R)}\circ\chi^{(T)}\right)\big|_{\mathrm{ind}(A)}\right)^{-1}$ for all $j\in\mathrm{Ran}\left(\chi^{(T)}\right)$, we have for each $j\in\mathrm{Ran}\left(\chi^{(T)}\right)$ that

$$\dim(H_{(j)\chi^{(R)}})\le\dim\left(K_{(j)\left(\left(\chi^{(R)}\circ\chi^{(T)}\right)\big|_{\mathrm{ind}(A)}\right)^{-1}}\right).$$

Recall that $\chi^{(R)}|_{\mathrm{Ran}\left(\chi^{(T)}\right)}$ is injective. Moreover, we have shown that $\left(\chi^{(\theta)}\chi^{(T)}\right)|_{\mathrm{ind}(A)}$ is a bijective function from $\mathrm{ind}(A)$ onto $\mathrm{Ran}\left(\chi^{(T)}\right)$ and

$$\dim(H_{(j)\chi^{(R)}}) \leq \dim\left(K_{(j)\left(\left(\chi^{(R)}\circ\chi^{(T)}\right)|_{\mathrm{ind}(A)}\right)^{-1}}\right)$$

for all $j \in \mathrm{Ran}\left(\chi^{(T)}\right)$. Therefore, by Lemma 2.1, the function $\left(\chi^{(R)}\circ\chi^{(T)}\right)|_{\mathrm{ind}(A)}$ is an $\mathcal{S}$-isomorphism from $\mathrm{ind}(A)$ onto $\mathrm{Ran}\left(\chi^{(T)}\right)$. Since the followings hold:

(a) $\left\{k_j : j \in \mathrm{Ran}\left(\chi^{(T)}\right)\right\} = \mathrm{ind}(A)$ and $A = \bigoplus_{j\in\mathrm{Ran}\left(\chi^{(T)}\right)} A_j = \bigoplus_{j\in\mathrm{Ran}\left(\chi^{(T)}\right)} A \cap K_{k_j}$;

(b) $(R\circ T)|_{A_j}$ is an isomorphism from $A_j = A\cap K_{k_j}$ onto $\mathrm{Ran}(T)\cap K_j = \mathrm{Ran}(T)\cap K_{(k_j)\chi^{(R\circ T)}}$ for all $j \in \mathrm{Ran}\left(\chi^{(T)}\right)$,

by Theorem 5.8, the linear transformation T is regular. □

The following proposition provides a simple necessary condition for both semigroups $\left(L^*_{\mathscr{H},\mathscr{K}}(U,V)\right)_R$ and $\left(\widehat{QL}^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus\right)_R$ to be regular, under certain circumstances.

Proposition 6.3. *Let Λ be either $L^*_{\mathscr{H},\mathscr{K}}(U,V)$ or $\widehat{QL}^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus$. Suppose that $\left|\widehat{j}\right| > 1$ for all $j\in J$. In the case where $\Lambda = L^*_{\mathscr{H},\mathscr{K}}(U,V)$, suppose further that there exist $k\in J$ and $g\in T^*_{\mathscr{H},\mathscr{K}}(I,J)$ such that $\mathrm{Ran}(\gamma)\subseteq\widehat{k}$ and $k\notin(I\setminus\mathrm{Ran}(\gamma))g$. If Λ is regular, then $\chi^{(R)}$ is bijective.*

Proof. Suppose that Λ_R is regular. Then, by Proposition 4.1 and statement (4) of Proposition 3.2, it follows that $\mathcal{S}_{\chi^{(R)}}$ is regular, where $\mathcal{S} = T^*_{\mathscr{H},\mathscr{K}}(I,J)$ if $\Lambda = L^*_{\mathscr{H},\mathscr{K}}(U,V)$ and $\mathcal{S} = \widehat{P}^*_{\mathscr{H},\mathscr{K}}(I,J)$ if $\Lambda = \widehat{QL}^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus$. Thus, by Theorem 2.3, the function $\chi^{(R)}$ is surjective. □

In the case where $|I|>1$ and all vector spaces H_i and K_j have the same dimension, the bijectivity of $\chi^{(R)}$ is also a sufficient condition for the semigroup $\widehat{QL}^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus$ to be regular. This is stated in the following proposition.

Proposition 6.4. *Suppose that $|I|>1$, and that all vector spaces H_i and K_j have the same dimension. Then $\left(\widehat{QL}^*_{\mathscr{H},\mathscr{K}}(U,V)_\oplus\right)_R$ is regular if and only if $\chi^{(R)}$ is bijective.*

Proof. By Proposition 6.3, it remains to show that the sufficiency holds. Suppose that $\chi^{(R)}$ is bijective. To see that Λ_R is regular, let $T\in\Lambda_R$. Since $\chi^{(R)}$ is injective, the injectivity of $\chi^{(R)}|_{\mathrm{Ran}(\chi^{(T)})}$ follows immediately. Next, let $j\in\mathrm{Ran}(\chi^{(T)})$ be arbitrarily given. Then, by the injectivity of $\chi^{(T)}$, there exists a unique $i\in I$ such that $(i)\chi^{(T)} = j$. Moreover, by the surjectivity of $\chi^{(R)}$, there exists $k\in J$ such that $(k)\chi^{(R)} = i$. Since all elements of $\mathscr{H}$ and $\mathscr{K}$ have the same dimension, we have $\dim H_{(j)\chi^{(R)}} = \dim K_j = \dim K_k$ and $(K_k)\,R = H_i$. By statement (2) of Lemma 5.5, we have $(H_i)T = \mathrm{Ran}(T)\cap K_j$. This implies that $(v)T^{-1}\cap(K_k)\,R\neq\{0\}$ for all $v\in\mathrm{Ran}(T)\cap K_j\setminus\{0\}$. Therefore, by Theorem 6.2, the linear transformation T is regular. □

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