

A FINITENESS THEOREM FOR UNIVERSAL m -GONAL FORMS WITH COEFFICIENTS 1 OR 2

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ABSTRACT. In 2022, Kim [16] proved a finiteness theorem for a restricted class of universal generalized m -gonal forms; namely, a generalized m -gonal form f with coefficients 1 or 2 is universal if $m \geq 10$ and f represents 1, $m - 4$ and $m - 2$. In this paper, we prove a similar finiteness theorem for universal m -gonal forms. If m is even, $m \geq 10$ and an m -gonal form f with coefficients 1 or 2 represents $2m - 1$ and $4m - 2$, then f is universal, and if m is odd, $m \geq 7$ and f represents either $2m - 1$ and $2m - 2$ or $2m - 2$ and $5m - 4$, then f is universal.

1. introduction

An m -gonal number is $P_m(x) = (m - 2)\frac{x(x-1)}{2} + x = \frac{(m-2)x^2 - (m-4)x}{2}$ for some positive integer x . An m -gonal form is $f(x_1, x_2, \dots, x_n) = a_1P_m(x_1) + a_2P_m(x_2) + \dots + a_nP_m(x_n) = \sum_{i=1}^n a_iP_m(x_i)$ for some positive integers a_1, a_2, \dots, a_n . An m -gonal form $\sum_{i=1}^n a_iP_m(x_i)$ is also denoted by $\langle a_1, a_2, \dots, a_n \rangle_m$. Also, $\langle a_1, a_2, \dots, a_n \rangle_m$ is written by $\langle a^n \rangle_m$ if $a_1 = a_2 = \dots = a_n = a$. An m -gonal form f represents a positive integer k if $f(x_1, x_2, \dots, x_n) = k$ for some non-negative integers x_1, x_2, \dots, x_n , and an m -gonal form f is called universal if it represents all positive integers. It originates from the problem representing a positive integer as a sum of m -gonal numbers, especially Fermat's conjecture that every positive integer is represented by a sum of m m -gonal numbers for all $m \geq 3$, or equivalently a sum $\sum_{i=1}^m P_m(x_i) = \langle 1^m \rangle_m$ of m m -gonal numbers is universal. This conjecture was proved by Lagrange for $m = 4$, Gauss for $m = 3$ and finally by Cauchy for $m \geq 5$.

Liouville and Ramanujan found all universal 3-gonal and universal 4-gonal forms, or equivalently, all ternary universal triangular forms and all quaternary universal diagonal quadratic forms respectively.

Griffiths [7–9] studied the universal m -gonal forms in another direction. Griffiths defined the weight $w = w(f)$ of an m -gonal form $\langle a_1, a_2, \dots, a_n \rangle_m$ by $w = a_1 + a_2 + \dots + a_n$. She thought that an m -gonal form f is a sum $\langle 1^w \rangle_m$ of w m -gonal numbers

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with additional conditions $x_1 = x_2 = \cdots = x_{w_1} = y_1$, $x_{w_1+1} = x_{w_1+2} = \cdots = x_{w_2} = y_2$, \dots , $x_{w_{n-1}+1} = x_{w_{n-1}+2} = \cdots = x_{w_n} = y_n$ where $w_i = a_1 + a_2 + \cdots + a_i$. She classified [7] the universal m -gonal forms with weight m as a generalization of Cauchy's Theorem. She also gave some contributions [8, 9] to the universal m -gonal forms with weight $m + 1$ and $m + 2$.

A generalized m -gonal number is $P_m(x) = \frac{(m-2)x^2 - (m-4)x}{2}$ for some $x \in \mathbb{Z}$. Note that x is not necessarily non-negative. It is easy to see that every generalized m -gonal number is an m -gonal number if and only if m is 3 or 4. A generalized m -gonal form is a sum $f(x_1, x_2, \dots, x_n) = \sum_{i=1}^n a_i P_m(x_i)$ with $a_i \in \mathbb{Z}^+$, and f represents a positive integer k if $f(x_1, x_2, \dots, x_n) = k$ for some integers x_1, x_2, \dots, x_n . A generalized m -gonal form is also called universal if it represents all positive integers. Notations $\langle a_1, a_2, \dots, a_n \rangle_m$ and $\langle a^n \rangle_m$ are also used for generalized m -gonal forms. Dickson [6] studied universal generalized m -gonal forms by showing that for each $m \geq 10$, the equation

$$\sum_{i=1}^{m-4} P_m(x_i) = k$$

has integer solutions $x_1, x_2, \dots, x_{m-4} \geq -2$ for all positive integers k . As a consequence, generalized m -gonal form $\langle 1^{m-4} \rangle_m = \sum_{i=1}^{m-4} P_m(x_i)$ is universal. Also he proved that the generalized 9-gonal form $\langle 1^5 \rangle_9$ is universal. From Gauss' and Lagrange's Theorem, $\langle 1^3 \rangle_3$ and $\langle 1^4 \rangle_4$ are universal. Since every generalized 6-gonal number is a triangular number, the generalized form $\langle 1^3 \rangle_6$ is universal. Guy noted [10] that the generalized form $\langle 1^3 \rangle_5$ is universal and Sun showed [18] that the generalized forms $\langle 1^4 \rangle_7$ and $\langle 1^4 \rangle_8$ are both universal.

The study of m -gonal and generalized m -gonal forms is influenced by the finiteness theorems for quadratic forms. A positive definite integral quadratic form is a polynomial $q(x_1, x_2, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{i,j} x_i x_j$ such that $a_{i,i} \in \mathbb{Z}^+$, $2a_{i,j} \in \mathbb{Z}$ for $i \neq j$, $a_{i,j} = a_{j,i}$ for all i, j and $q(x_1, x_2, \dots, x_n) > 0$ for all $(x_1, x_2, \dots, x_n) \neq (0, 0, \dots, 0)$. It is called classic if $a_{i,j} \in \mathbb{Z}$ for all distinct i, j . Conway and Schneeberger [2, 5] proved that a classic positive definite integral quadratic form is universal if it represents all elements of $\{1, 2, 3, 5, 6, 7, 10, 14, 15\}$, and conjectured that a general positive definite integral quadratic form, not necessarily classic, is universal if it represents all elements of

$$\{1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, \\ 34, 35, 37, 42, 58, 93, 110, 145, 203, 290\}.$$

This conjecture was proved by Bhargava and Hanke [3]. Similarly, for each $m \geq 3$ there are universality criterion sets for universal m -gonal forms and universal generalized m -gonal forms, which are sets $S_m, S_m^* \subset \mathbb{Z}^+$ such that any m -gonal form is universal and any generalized m -gonal form is universal if they represent all elements of S_m and S_m^* respectively.

It is a consequence of the Conway and Schneeberger's Theorem that $S_4 = S_4^* = \{1, 2, 3, 5, 6, 7, 10, 14, 15\}$. Bosma and Kane proved [4] that $S_3 = S_3^* = S_6^* = \{1, 2, 4, 5, 8\}$. Ju [11] showed that

$$S_5^* = \{1, 3, 8, 9, 11, 18, 19, 25, 27, 43, 98, 109\},$$

and Ju and Oh [13] proved that

$$S_8^* = \{1, 2, 3, 4, 6, 7, 9, 12, 13, 14, 18, 60\}.$$

The only known universality criterion set for universal m -gonal forms is, aside from $m = 3, 4$,

$$S_5 = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 13, 14, 17, 18, 19, 23, 28, 31, 33, 34, 39, 42, 63\}$$

proved by Ju and Kim [12]. In 2019, Kane and Liu proved [14] that for all $\epsilon > 0$, there is $C = C_{m,\epsilon} > 0$ such that every element of S_m^* is at most $Cm^{7+\epsilon}$. The second author of this paper and Park improved [15] this bound to Cm for some $C = C_m > 0$.

In 2021, Kim [16] proved that if $m \geq 10$ and a generalized m -gonal form $f = \sum_{i=1}^{\alpha} P_m(x_i) + \sum_{i=1}^{\beta} 2P_m(x_{\alpha+i})$ with coefficients 1 or 2 represents $1, m - 4$ and $m - 2$, then f is universal. It is a finiteness theorem for the restricted class of universal generalized m -gonal forms. He proved similar finiteness theorems for $m = 7$ and $m = 9$. In 2022, Banerjee et. al. found [1] the least β such that $\sum_{i=1}^{r-1} P_m(x_i) + \sum_{i=1}^{\beta} rP_m(x_{r+i-1})$ is universal for each $r \geq 2$. In this paper, we prove a finiteness theorem for universal m -gonal forms with coefficients 1 or 2 except $m = 5$ as follows.

THEOREM 1.1. *Let f be an m -gonal form with coefficients 1 or 2. If m is even, $m \geq 10$ and f represents $2m - 1$ and $4m - 2$, then f is universal. If m is odd, $m \geq 7$ and f represents either $2m - 1$ and $2m - 2$ or $2m - 2$ and $5m - 4$, then f is universal. If $m = 8$ and f represents either 15 and 30 or 15 and 78 , then f is universal. If $m = 6$ and f represents 11 and at least one of $22, 82$ and 100 , then f is universal.*

This theorem is an m -gonal form version of this theorem.

2. preliminaries

We use a brief notation by adopting the exponents in coefficients to reduce the repeated coefficients of $f = \langle a_1, a_2, \dots, a_n \rangle_m$. Concretely, if $\alpha_1, \alpha_2, \dots, \alpha_k \geq 0, n = \sum_{i=1}^k \alpha_i$ and

$$a_{s_{i-1}+1} = a_{s_{i-1}+2} = \dots = a_{s_i} = b_i$$

for all $i = 1, 2, \dots, k$ where $s_0 = 0$ and $s_i = \sum_{t=1}^i \alpha_t$, then $f = \langle b_1^{\alpha_1}, b_2^{\alpha_2}, \dots, b_k^{\alpha_k} \rangle_m$. If f is of coefficients 1 or 2, then $f = \langle 1^\alpha, 2^\beta \rangle_m$ for some $\alpha, \beta \geq 0$. The partial weight w_i of an m -gonal form $f = \langle a_1, a_2, \dots, a_n \rangle_m$ is the sum $a_1 + a_2 + \dots + a_i$ of the first i coefficients of f . The weight $w = w_n$ of f is the sum of all coefficients of f .

Let \mathfrak{G}_m be the class of all (generalized) m -gonal forms and \mathfrak{S} be a subclass of \mathfrak{G}_m . A universality criterion set of universal (generalized) m -gonal forms for \mathfrak{S} is a set $S \subset \mathbb{Z}^+$ such that every (generalized) m -gonal form $f \in \mathfrak{S}$ representing all the elements of S is universal. A minimal universality criterion set of universal (generalized) m -gonal forms for \mathfrak{S} is a universality criterion set of universal (generalized) m -gonal forms for \mathfrak{S} each of whose proper subset is no more critical. In 2022, Kim [16] found the universality criterion sets of universal generalized m -gonal forms for $\mathfrak{S} = \{ \langle 1^\alpha, 2^\beta \rangle_m \mid \alpha, \beta \geq 0 \}$.

THEOREM 2.1 (Kim, [16]). *If $m \geq 10$, then $\{1, m - 4, m - 2\}$ is a universality criterion set of universal generalized m -gonal forms $\langle 1^\alpha, 2^\beta \rangle_m$ with $\alpha, \beta \geq 0$. Also $\{1, 3, 5, 10, 19, 23\}$ is a universality criterion set of universal generalized 7-gonal forms $\langle 1^\alpha, 2^\beta \rangle_7$ with $\alpha, \beta \geq 0$, and $\{1, 5, 7, 34\}$ is a universality criterion set of universal generalized 9-gonal forms $\langle 1^\alpha, 2^\beta \rangle_9$ with $\alpha, \beta \geq 0$.*

From this result, another interesting problem, the uniqueness of minimal universality criterion sets of universal (generalized) m -gonal forms for a subclass \mathfrak{S} of (generalized) m -gonal forms, is naturally raised. Concretely, the minimal universality criterion sets of universal generalized m -gonal forms for \mathfrak{S}_m consist of all positive integers N such that there is a generalized m -gonal form that represents all positive integers except N . It follows that these minimal universality criterion sets are unique. From Kim's Theorem [16] and by some considerations, we can obtain all minimal universality criterion sets of universal generalized m -gonal forms for a subclass $\mathfrak{S} = \{\langle 1^\alpha, 2^\beta \rangle_m \mid \alpha, \beta \geq 0\}$. When m is odd, $\{m - 4, m - 2\}$ is the unique minimal universality criterion set, and when m is even, $\{a, m - 4, m - 2\}$ is a minimal universality criterion set for all odd a . If m is even, then there are infinitely many minimal universality criterion sets, and thus the uniqueness fails for even m . As a consequence, the uniqueness of universality criterion sets of universal (generalized) m -gonal forms for various families \mathfrak{S} is an interesting problem.

THEOREM 2.2 (Griffiths [7–9]). *Let $m \geq 5$ and $f = \langle a_1, a_2, \dots, a_n \rangle_m$ be an m -gonal form with weight m . Then, f is universal if and only if f is one of the following.*

- (1) $f = \langle 1, 1, 1, 1, a_5, a_6, \dots, a_n \rangle_m, a_5 \neq 3, a_k \leq w_{k-1} - 1$ for all $k = 5, 6, \dots, n$,
- (2) $f = \langle 1, 1, 1, 2, a_5, a_6, \dots, a_n \rangle_m, a_k \leq w_{k-1} - 1$ for all $k = 5, 6, \dots, n$,
- (3) $f = \langle 1, 1, 1, 2 \rangle_m$ for $m = 5$.

Moreover, if f is of weight $m + 1$ or $m + 2$ and satisfies (1) or (2) given above, then f is universal.

COROLLARY 2.3. *If $f = \langle 1^\alpha, 2^\beta \rangle_m, \alpha \geq 3$ and $\alpha + 2\beta \geq 3$, then f is representd by a sum of four m -gonal numbers.*

THEOREM 2.4 (Cauchy). *If $m \geq 3$, then $\langle 1^m \rangle_m$ is universal.*

LEMMA 2.5 (Cauchy). *The system*

$$(1) \quad \begin{cases} x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2q + r \\ x_1 + x_2 + x_3 + x_4 = r. \end{cases}$$

has a non-negative integer solution (x_1, x_2, x_3, x_4) if $\Delta(2q + r, r) = 4(2q + r) - r^2 = 8q + 4r - r^2 \geq 0, \Delta^*(2q + r - 1, r + 1) = 3(2q + r - 1) - (r + 1)^2 = 6q + r - r^2 - 4 < 0$ and $\Delta(2q + r, r) \neq 4^u(8l + 7)$ for all $u, l \geq 0$.

From the identity

$$\begin{aligned} P_m(x_1) + P_m(x_2) + P_m(x_3) + P_m(x_4) &= \frac{(m - 2)(\sum_{i=1}^4 x_i^2) - (m - 4)(\sum_{i=1}^4 x_i)}{2} \\ &= \frac{(m - 2)(2q + r) - (m - 4)r}{2} = q(m - 2) + r, \end{aligned}$$

we have the following corollary.

COROLLARY 2.6. *If $\Delta(2q + r, r) = 8q + 4r - r^2 \geq 0, \Delta^*(2q + r, r) = 6q + r - r^2 - 4 < 0$ and $\Delta(2q + r, r) \neq 4^u(8l + 7)$ for all $u, l \geq 0$, then $q(m - 2) + r$ is represented by the sum $\langle 1^4 \rangle_m$ of four m -gonal numbers.*

If r is odd, then since $\Delta(2q + r, r) \equiv 8q + 4r - r^2 \equiv 3 \pmod{8}$, we have the following corollary.

COROLLARY 2.7. *If r is odd and*

$$\frac{1 + \sqrt{1 + 24q}}{2} < r \leq 2 + \sqrt{4 + 8q},$$

then $q(m - 2) + r$ is represented by a sum $\langle 1^4 \rangle_m$ of four m -gonal numbers.

LEMMA 2.8. *If $m \geq \alpha \geq 5$, then for each $q \geq 2$, a sum $\langle 1^\alpha \rangle_m$ of α m -gonal numbers represents all positive integers N such that $q(m - 2) + \frac{5 + \sqrt{1 + 24q}}{2} < N \leq q(m - 2) + \alpha - 4 + \sqrt{4 + 8q}$.*

Proof. Let r_q be the smallest number r such that $\langle 1^4 \rangle_m$ represents $q(m - 2) + r$ and $\frac{1 + \sqrt{1 + 24q}}{2} < r \leq 2 + \sqrt{4 + 8q}$. Since $q \geq 2$,

$$\begin{aligned} 2 + \sqrt{4 + 8q} - \frac{1 + \sqrt{1 + 24q}}{2} &\geq \frac{3}{2} + \left(\sqrt{\frac{1}{2} + q} \right) (\sqrt{8} - \sqrt{6}) \\ &\geq \frac{3}{2} + \left(\sqrt{\frac{5}{2}} \right) (\sqrt{8} - \sqrt{6}) = \frac{3}{2} + \sqrt{20} - \sqrt{15} > 2. \end{aligned}$$

Thus, there is an odd number r such that $\frac{1 + \sqrt{1 + 24q}}{2} < r \leq \frac{5 + \sqrt{1 + 24q}}{2} \leq 2 + \sqrt{4 + 8q}$. By Corollary 2.7, $\langle 1^4 \rangle_m$ represents $q(m - 2) + r$. Thus, we have $r_q \leq \frac{5 + \sqrt{1 + 24q}}{2}$. Similarly, we can define r_q^* by the largest number r such that $\langle 1^4 \rangle_m$ represents $q(m - 2) + r$ and $\frac{1 + \sqrt{1 + 24q}}{2} < r \leq 2 + \sqrt{4 + 8q}$, for each $q \geq 2$, and we can see that $r_q^* \geq \sqrt{4 + 8q}$. Since $\langle 1^4 \rangle_m$ represents $q(m - 2) + r$ for all odd integers r such that $\frac{1 + \sqrt{1 + 24q}}{2} < r \leq 2 + \sqrt{4 + 8q}$, $\langle 1^\alpha \rangle_m$ represents all integers N such that $q(m - 2) + r_q \leq N \leq q(m - 2) + r_q^*$. Since $\langle 1^{\alpha - 4} \rangle_m$ represents $0, 1, \dots, \alpha - 4$, $\langle 1^\alpha \rangle_m$ represents all integers N such that $q(m - 2) + r_q^* \leq N \leq q(m - 2) + r_q^* + \alpha - 4$. Thus, $\langle 1^\alpha \rangle_m$ represents all integers N such that $q(m - 2) + \frac{5 + \sqrt{1 + 24q}}{2} < N \leq q(m - 2) + \alpha - 4 + \sqrt{4 + 8q}$. \square

COROLLARY 2.9. *Suppose that $m \geq 8$.*

1) *A sum $\langle 1^{m-1} \rangle_m$ of $m - 1$ m -gonal numbers represents all positive integers larger than $214(m - 2) + 37$.*

2) *A sum $\langle 1^{m-2} \rangle_m$ of $m - 2$ m -gonal numbers represents all positive integers larger than $298(m - 2) + 43$.*

Proof. 1) If $q \geq 214$, then by Lemma 2, $\sum_{i=1}^{m-1} P_m(x_i)$ represents all positive integers N such that $q(m-2) + \frac{5+\sqrt{1+24q}}{2} < N \leq q(m-2) + m - 5 + \sqrt{4+8q}$. Since

$$\begin{aligned}
q(m-2) + m - 5 &+ \sqrt{4+8q} - \left((q+1)(m-2) + \frac{5 + \sqrt{1+24(q+1)}}{2} \right) \\
&= -\frac{11}{2} + \sqrt{8\left(q + \frac{1}{2}\right)} - \sqrt{6\left(q + \frac{25}{24}\right)} \\
&= -\frac{11}{2} + \frac{2q - \frac{9}{4}}{\sqrt{8\left(q + \frac{1}{2}\right)} + \sqrt{6\left(q + \frac{25}{24}\right)}} \\
&\geq -\frac{11}{2} + \frac{2\left(q + \frac{25}{24}\right) - \frac{13}{3}}{\sqrt{8\left(q + \frac{25}{24}\right)} + \sqrt{6\left(q + \frac{25}{24}\right)}} \\
&= -\frac{11}{2} + \frac{2\sqrt{q + \frac{25}{24}}}{\sqrt{8} + \sqrt{6}} - \frac{13}{3(\sqrt{8} + \sqrt{6})\sqrt{\left(q + \frac{25}{24}\right)}} \\
&\geq -\frac{11}{2} + \frac{2\sqrt{214 + \frac{25}{24}}}{\sqrt{8} + \sqrt{6}} - \frac{13}{3(\sqrt{8} + \sqrt{6})\sqrt{\left(214 + \frac{25}{24}\right)}} \\
&\geq 0.0008 > 0
\end{aligned}$$

for all $q \geq 214$, $\langle 1^{m-1} \rangle_m$ represents all positive integers larger than $214(m-2) + r_{214} = 214(m-2) + 37$.

2) If $q \geq 298$, then $\langle 1^{m-2} \rangle_m$ represents all positive integers N such that $q(m-2) + \frac{5+\sqrt{1+24q}}{2} < N \leq q(m-2) + m - 6 + \sqrt{4+8q}$. Since

$$\begin{aligned}
q(m-2) + m - 6 &+ \sqrt{4+8q} - \left((q+1)(m-2) + \frac{5 + \sqrt{1+24(q+1)}}{2} \right) \\
&\geq -\frac{13}{2} + \frac{2\sqrt{q + \frac{25}{24}}}{\sqrt{8} + \sqrt{6}} - \frac{13}{3(\sqrt{8} + \sqrt{6})\sqrt{\left(q + \frac{25}{24}\right)}} \\
&\geq -\frac{13}{2} + \frac{2\sqrt{298 + \frac{25}{24}}}{\sqrt{8} + \sqrt{6}} - \frac{13}{3(\sqrt{8} + \sqrt{6})\sqrt{\left(298 + \frac{25}{24}\right)}} \\
&\geq 0.005 > 0,
\end{aligned}$$

for all $q \geq 298$, $\langle 1^{m-2} \rangle_m$ represents all positive integers larger than $298(m-2) + r_{298} = 298(m-2) + 43$. \square

PROPOSITION 2.10. *Suppose that $m \geq 9$.*

1) A sum $\langle 1^{m-1} \rangle_m$ of $m-1$ m -gonal numbers represents all positive integers except $2m-1$ and $5m-4$.

2) A sum $\langle 1^{m-2} \rangle_m$ of $m-2$ m -gonal numbers represents all positive integers except $m-1, 2m-2, 2m-1, 4m-4, 5m-5$ and $5m-4$.

q	s_q	t_q	$t_q - s_q$	$t_q - s_{q+1}$	q	s_q	t_q	$t_q - s_q$	$t_q - s_{q+1}$	q	s_q	t_q	$t_q - s_q$	$t_q - s_{q+1}$
0	0	5	5	3	22	9	17	8	6	44	16	23	7	13
1	2	6	4	2	23	11	17	6	7	45	10	23	13	11
2	4	7	3	4	24	10	18	8	7	46	12	24	12	10
3	3	8	5	3	25	11	18	7	5	47	14	24	10	11
4	5	9	4	2	26	13	18	5	7	48	13	24	11	9
5	7	10	3	6	27	11	19	8	11	49	15	24	9	7
6	4	10	6	4	28	8	19	11	9	50	17	25	8	11
7	6	11	5	3	29	10	19	9	7	51	14	25	11	9
8	8	11	3	4	30	12	20	8	9	52	16	25	9	7
9	7	12	5	7	31	11	20	9	7	53	18	25	7	8
10	5	12	7	5	32	13	20	7	5	54	17	25	8	10
11	7	13	6	5	33	15	20	5	8	55	15	26	11	10
12	8	13	5	5	34	12	21	9	7	56	16	26	10	10
13	8	14	6	4	35	14	21	7	12	57	16	26	10	8
14	10	14	4	8	36	9	21	12	10	58	18	26	8	7
15	6	15	9	7	37	11	21	10	8	59	19	26	7	10
16	8	15	7	5	38	13	22	9	10	60	16	27	11	9
17	10	15	5	6	39	12	22	10	8	61	18	27	9	7
18	9	16	7	5	40	14	22	8	6	62	20	27	7	8
19	11	16	5	6	41	16	22	6	9	63	19	27	8	10
20	10	16	6	9	42	13	23	10	9	64	17	27	10	8
21	7	17	10	8	43	14	23	9	7	65	19	28	9	

Table 1. $t_q - s_q$ and $t_q - s_{q+1}$ for $0 \leq q \leq 64$

Proof. Let w be $m - 1$ or $m - 2$. From a computation performed by a computer, for each $q = 1, 2, \dots, 65$, we find consecutive integers $q(m - 2) + s_q, q(m - 2) + s_q + 1, \dots, q(m - 2) + t_q$ represented by $\langle 1^5 \rangle_m$. The numbers s_q and t_q are given in Table 1. Thus, $\langle 1^{w-1} \rangle_m$ represents all positive integers N such that $q(m - 2) + s_q \leq N \leq q(m - 2) + t_q + w - 6$. From Table 1, we have that if $33 \leq q \leq 64$, then $t_q - s_{q+1} \geq 6$. Since

$$\begin{aligned} q(m - 2) + t_q + w - 6 &- ((q + 1)(m - 2) + s_{q+1}) = w - m - 4 + (t_q - s_{q+1}) \\ &\geq (m - 3) - m - 4 + 6 = -1, \end{aligned}$$

$\langle 1^{w-1} \rangle_m$ represents all positive integers N such that

$$33(m - 2) + 15 = 33(m - 2) + s_{33} \leq N \leq 65(m - 2) + t_{65} + w - 6.$$

Since $w \geq m - 2$, $65(m - 2) + t_{65} + w - 6 \geq 65(m - 2) + 28 + m - 8 = 66(m - 2) + 22$. Thus, $\langle 1^{w-1} \rangle_m$ represents $33(m - 2) + 15, 33(m - 2) + 16, \dots, 66(m - 2) + 22$. It follows that for each $x \geq 0$, $\langle 1^w \rangle_m$ represents all positive integers N such that $P_m(x) + 33(m - 2) + 15 \leq N \leq P_m(x) + 66(m - 2) + 22$. If $0 \leq x \leq 33$, then since

$$\begin{aligned} P_m(x) + 66(m - 2) + 22 &- (P_m(x + 1) + 33(m - 2) + 15) \\ &= 33(m - 2) + 7 + P_m(x) - P_m(x + 1) \\ &= 33(m - 2) + 7 - x(m - 2) - 1 \\ &= (33 - x)(m - 2) + 6 \geq 0, \end{aligned}$$

$\langle 1^w \rangle_m$ represents all positive integers N such that

$$P_m(0)+33(m-2)+15 = 33(m-2)+15 \leq N \leq P_m(33)+66(m-2)+22 = 594(m-2)+55.$$

By Corollary 2.9, $\langle 1^w \rangle_m$ represents all positive integers at least $33(m-2) + 15$.

If $w = m - 1$, then from Table 1, $t_q - s_{q+1} \geq 3$ for all q with $0 \leq q \leq 32$ except $q = 1, 4$. Thus, if $0 \leq q \leq 32$ and $q \neq 1, 4$, then $q(m-2) + t_q + m - 6 - ((q+1)(m-2) + s_{q+1}) = -4 + (t_q - s_{q+1}) \geq -1$. It follows that $\langle 1^{m-1} \rangle_m$ represents all positive integers N such that $1 = s_0 \leq N \leq (m-2) + t_1 + m - 6 = 2m - 2$, all integers N such that $2m = 2(m-2) + s_2 \leq N \leq 4(m-2) + t_4 + m - 6 = 5m - 5$, and all integers N such that $5m - 3 = 5(m-2) + s_5 \leq N \leq 32(m-2) + t_{32} + m - 6 = 33(m-2) + 16$. Hence, $\langle 1^{m-1} \rangle_m$ represents all positive integers except $2m - 1$ and $5m - 4$.

Suppose that $\langle 1^{m-1} \rangle_m$ represents $2m - 1$. Then, $\sum_{i=1}^{m-1} P_m(x_i) = 2m - 1$ for some $x_1, x_2, \dots, x_{m-1} \geq 0$. Since $P_m(3) = 3m - 3 > 2m - 1$, $0 \leq x_i \leq 2$ for all $i = 1, 2, \dots, m-1$. Also the number of i such that $x_i = 2$ is at most one. If $x_i < 2$ for all i , then $2m - 1 = \sum_{i=1}^{m-1} P_m(x_i) \leq m - 1$. This is a contradiction. If there is one i such that $x_i = 2$, then $2m - 1 = \sum_{i=1}^{m-1} P_m(x_i) \leq m + (m-2) = 2m - 2$. This is a contradiction. Thus, $\langle 1^{m-1} \rangle_m$ cannot represent $2m - 1$. Suppose that $\langle 1^{m-1} \rangle_m$ represents $5m - 4$. Then, $\sum_{i=1}^{m-1} P_m(x_i) = 5m - 4$ for some $x_1, x_2, \dots, x_{m-1} \geq 0$. Since $P_m(4) = 6m - 8 > 5m - 4$, $0 \leq x_i \leq 3$ for all $i = 1, 2, \dots, m-1$. Let n_1 and n_2 be the number of i such that $x_i = 3$ and $x_i = 2$ respectively. We can see $n_1 \leq 1$. Also we have $n_2 \leq 4$ when $n_1 = 0$, and $n_2 \leq 2$ when $n_1 = 1$. If $n_1 = 0$, then $5m - 4 = \sum_{i=1}^{m-1} P_m(x_i) \leq n_2 m + (m-1 - n_2) = (n_2 + 1)(m-1) \leq 5m - 5$. This is a contradiction. If $n_1 = 1$, then $5m - 4 = \sum_{i=1}^{m-1} P_m(x_i) \leq (3m-3) + n_2 m + (m-1 - n_2 - 1) = (n_2 + 4)(m-1) - 1 \leq 5m - 6$. This is a contradiction. Thus, $\langle 1^{m-1} \rangle_m$ cannot represent $5m - 4$.

If $w = m - 2$, then from Table 1 and by the same method as above, $t_q - s_{q+1} \geq 4$ for all q with $0 \leq q \leq 32$ except $q = 0, 1, 3, 4, 7$. Since $q(m-2) + t_q + m - 7 - ((q+1)(m-2) + s_{q+1}) = -5 + (t_q - s_{q+1}) \geq -1$ for all q such that $0 \leq q \leq 32$ and $q \neq 0, 1, 3, 4, 7$, a sum $\langle 1^{m-2} \rangle_m$ represents

$$\begin{aligned} &1, 2, \dots, m-2, & m, m+1, \dots, 2m-3, \\ &2m, 2m+1, \dots, 4m-5, & 4m-3, 4m-2, \dots, 5m-6, \\ &5m-3, 5m-2, \dots, 8m-10, & \text{and } 8m-8, 8m-7, \dots, 33m-50. \end{aligned}$$

Hence, $\langle 1^{m-2} \rangle_m$ represents all positive integers except $m-1, 2m-2, 2m-1, 4m-4, 5m-5, 5m-4$ and $8m-9$. But if $x_1 = x_2 = \dots = x_7 = 2$ and $x_8 = x_9 = \dots = x_{m-2} = 1$, then $\sum_{i=1}^{m-2} P_m(x_i) = 7m + (m-2) = 8m-9$. Thus, $\langle 1^{m-2} \rangle_m$ represents all positive integers except $m-1, 2m-2, 2m-1, 4m-4, 5m-5$ and $5m-4$. The proofs that $\langle 1^{m-2} \rangle_m$ cannot represent these numbers are similar to above proofs. \square

PROPOSITION 2.11. Suppose that $m \geq 14$.

1) If m is even, then m -gonal form $\langle 1, 2^{m-2} \rangle_m$ represents all positive integers except $2m - 1$.

2) If m is odd, then m -gonal form $\langle 1, 2^{m-2} \rangle_m$ represents all positive integers except $2m - 2$.

3) If m is even, then m -gonal form $\langle 1^2, 2^{\frac{m-2}{2}} \rangle_m$ represents all positive integers except $4m - 2$.

4) If m is odd, then m -gonal form $\langle 1^2, 2^{\frac{m-1}{2}} \rangle_m$ is universal.

Proof. 1) Since $\langle 1^{m-2} \rangle_m$ represents all positive integers except $m-1, 2m-2, 2m-1, 4m-4, 5m-5$ and $5m-4$, every even number is represented by $\langle 2^{m-2} \rangle_m$ except $2m-2, 4m-4, 4m-2, 8m-8, 10m-10$ and $10m-8$. It follows that $\langle 1, 2^{m-2} \rangle_m$ represents all positive integers except $2m-2, 2m-1, 4m-4, 4m-3, 4m-2, 4m-1, 8m-8, 8m-7, 10m-10, 10m-9, 10m-8$ and $10m-7$. Since $m-10, m-4, m-2, m, m+2$ are all even and less than $2m-14$, $\langle 2^{m-7} \rangle_m$ represents all of them. By

$$\begin{aligned} 2m-2 &= P_m(2) + (m-2), & 4m-4 &= P_m(2) + 2P_m(2) + (m-4), \\ 4m-3 &= P_m(3) + m, & 4m-2 &= P_m(2) + 2P_m(2) + (m-2), \\ 4m-1 &= P_m(3) + (m+2), & 8m-8 &= P_m(4) + 2P_m(2), \\ 8m-7 &= P_m(3) + 2(2P_m(2)) + (m-4), & 10m-10 &= P_m(2) + 4(2P_m(2)) + (m-10), \\ 10m-9 &= P_m(3) + 2P_m(3) + m, & 10m-8 &= P_m(4) + 2(2P_m(2)), \\ 10m-7 &= P_m(3) + 2(2P_m(2)) + (m-4), \end{aligned}$$

we have that $\langle 1, 2^{m-2} \rangle_m$ represents all positive integers except $2m-1$.

2) By the same method as above, we can see that $\langle 1, 2^{m-2} \rangle_m$ represents all positive integers except $2m-2, 2m-1, 4m-4, 4m-3, 4m-2, 4m-1, 8m-8, 8m-7, 10m-10, 10m-9, 10m-8$ and $10m-7$. Since $m-7, m-3, m-1, m+1, 6, 8$ are all even and less than $2m-14$, $\langle 2^{m-7} \rangle_m$ represents all of them. By

$$\begin{aligned} 2m-1 &= P_m(2) + (m-1), & 4m-4 &= P_m(3) + (m-1), \\ 4m-3 &= P_m(2) + 2P_m(2) + (m-3), & 4m-2 &= P_m(3) + (m+1), \\ 4m-1 &= P_m(2) + 2P_m(2) + (m-1), & 8m-8 &= P_m(4) + 2P_m(2), \\ 8m-7 &= P_m(2) + 3(2P_m(2)) + (m-7), & 10m-10 &= P_m(3) + 3(2P_m(2)) + (m-7), \\ 10m-9 &= P_m(5) + 6, & 10m-8 &= P_m(3) + 2P_m(3) + (m+1), \\ 10m-7 &= P_m(5) + 8, \end{aligned}$$

we have that $\langle 1, 2^{m-2} \rangle_m$ represents all positive integers except $2m-2$.

3) If $q \geq 446$, then by Lemma 2, $\langle 1^{\frac{m-2}{2}} \rangle_m$ represents all positive integers N such that $q(m-2) + \frac{5+\sqrt{1+24q}}{2} < N \leq q(m-2) + \frac{m-10}{2} + \sqrt{4+8q}$. Thus, $\langle 2^{\frac{m-2}{2}} \rangle_m$ represents all positive even integers N such that $2q(m-2) + 5 + \sqrt{1+24q} < N \leq 2q(m-2) + m - 10 + 2\sqrt{4+8q}$, and hence $\langle 1, 2^{\frac{m-2}{2}} \rangle_m$ represents all positive integers N such that $2q(m-2) + 6 + \sqrt{1+24q} < N \leq 2q(m-2) + m - 10 + 2\sqrt{4+8q}$. It follows that $\langle 1^2, 2^{\frac{m-2}{2}} \rangle_m$ represents all positive integers N such that either

$$2q(m-2) + 6 + \sqrt{1+24q} < N \leq 2q(m-2) + m - 10 + 2\sqrt{4+8q}$$

or

$$2q(m-2) + m + 6 + \sqrt{1+24q} < N \leq 2q(m-2) + 2m - 10 + 2\sqrt{4+8q}.$$

Since

$$\begin{aligned}
2q(m-2) + m - 10 + 2\sqrt{4+8q} - \left(2q(m-2) + m + 6 + \sqrt{1+24q}\right) \\
&= -16 + \sqrt{32q+16} - \sqrt{24q+1} \geq -16 + \sqrt{32q+16} - \sqrt{24q+12} \\
&\geq -16 + (\sqrt{32} - \sqrt{24})\sqrt{q + \frac{1}{2}} \geq -16 + (\sqrt{32} - \sqrt{24})\sqrt{\frac{893}{2}} \geq 0.14,
\end{aligned}$$

$\left\langle 1^2, 2^{\frac{m-2}{2}} \right\rangle_m$ represents all positive integers N such that $2q(m-2) + 6 + \sqrt{1+24q} < N \leq 2q(m-2) + 2m - 10 + 2\sqrt{4+8q}$ for all $q \geq 446$. Since $\sqrt{24q+25} \leq \sqrt{24q+12} + 1$, we have

$$\begin{aligned}
2q(m-2) + 2m - 10 + 2\sqrt{4+8q} - \left(2(q+1)(m-2) + 6 + \sqrt{1+24(q+1)}\right) \\
&= -12 + \sqrt{32q+16} - \sqrt{24q+25} \geq -13 + (\sqrt{32} - \sqrt{24})\sqrt{q + \frac{1}{2}} \\
&\geq -13 + (\sqrt{32} - \sqrt{24})\sqrt{\frac{893}{2}} \geq 3.14
\end{aligned}$$

for all $q \geq 446$. Thus, $\left\langle 1^2, 2^{\frac{m-2}{2}} \right\rangle_m$ represents all positive integers larger than $2(446(m-2) + r_{446}) = 992(m-2) + 110$.

In the proof of Proposition 1, for each $q = 1, 2, \dots, 65$, s_q and t_q are defined such that $q(m-2) + s_q, q(m-2) + s_q + 1, \dots, q(m-2) + t_q$ are consecutive integers represented by $\langle 1^5 \rangle_m$. The numbers s_q and t_q are computed by a computer and given in Table 1. Thus, $\left\langle 1^{\frac{m-4}{2}} \right\rangle_m$ represents all positive integers N such that $q(m-2) + s_q \leq N \leq q(m-2) + t_q + \frac{m-14}{2}$. Thus, $\left\langle 2^{\frac{m-4}{2}} \right\rangle_m$ represents all positive even integers N such that $2q(m-2) + 2s_q \leq N \leq 2q(m-2) + 2t_q + m - 14$. It follows that $\left\langle 1^2, 2^{\frac{m-4}{2}} \right\rangle_m$ represents all positive integers N such that either $2q(m-2) + 2s_q \leq N \leq 2q(m-2) + 2t_q + m - 13$ or $2q(m-2) + m + 2s_q \leq N \leq 2q(m-2) + 2t_q + 2m - 13$. From Table 1, we have that if $34 \leq q \leq 64$, then $t_q - s_q \geq 6$. Since

$$2q(m-2) + 2t_q + m - 13 - (2q(m-2) + m + 2s_q) = 2(t_q - s_q) - 13 \geq 2 \cdot 6 - 13 = -1,$$

$\left\langle 1^2, 2^{\frac{m-4}{2}} \right\rangle_m$ represents all positive integers N such that $2q(m-2) + 2s_q \leq N \leq 2q(m-2) + 2t_q + 2m - 13$. From Table 1, we have that if $34 \leq q \leq 64$, then $t_q - s_{q+1} \geq 4$. Since

$$\begin{aligned}
2q(m-2) + 2t_q + 2m - 13 - (2(q+1)(m-2) + 2s_{q+1}) &= -9 + 2(t_q - s_{q+1}) \\
&\geq -9 + 2 \cdot 4 = -1,
\end{aligned}$$

for all $q = 34, 35, \dots, 64$, $\left\langle 1^2, 2^{\frac{m-4}{2}} \right\rangle_m$ represents all positive integers N such that $68(m-2) + 2s_{34} = 68(m-2) + 24 \leq N \leq 130(m-2) + 2t_{65} + 2m - 13 = 132(m-2) + 47$. It follows that for each $x \geq 0$, $\left\langle 1^2, 2^{\frac{m-2}{2}} \right\rangle_m$ represents all positive integers N such that

$2P_m(x) + 68(m - 2) + 24 \leq N \leq 2P_m(x) + 132(m - 2) + 47$. If $0 \leq x \leq 32$, then since

$$\begin{aligned} 2P_m(x) + 132(m - 2) + 47 &= (2P_m(x) + 68(m - 2) + 24) \\ &= 64(m - 2) + 23 + 2(P_m(x) - P_m(x + 1)) \\ &= 64(m - 2) + 23 - 2x(m - 2) - 2 \\ &= (64 - 2x)(m - 2) + 21 \geq 0, \end{aligned}$$

$\left\langle 1^2, 2^{\frac{m-4}{2}} \right\rangle_m$ represents all positive integers N such that

$$2P_m(0) + 68(m - 2) + 24 = 68(m - 2) + 24 \leq N \leq 2P_m(32) + 132(m - 2) + 47 = 1124(m - 2) + 111.$$

Since $1124(m - 2) + 111 > 992(m - 2) + 110$, $\left\langle 1^2, 2^{\frac{m-2}{2}} \right\rangle_m$ represents all positive integers at least $68(m - 2) + 24$.

By the same method as above, we can see that for each $q = 0, 1, \dots, 34$, $\left\langle 1^2, 2^{\frac{m-2}{2}} \right\rangle_m$ represents all positive integers N such that either $2q(m - 2) + 2s_q \leq N \leq 2q(m - 2) + 2t_q + m - 11$ or $2q(m - 2) + m + 2s_q \leq N \leq 2q(m - 2) + 2t_q + 2m - 11$. From Table 1, $t_q - s_q \geq 5$ for all q with $0 \leq q \leq 34$ except $q = 1, 2, 4, 5, 8, 14$. Thus, if $0 \leq q \leq 34$ with $q \neq 1, 2, 4, 5, 8, 14$, then

$$2q(m - 2) + 2t_q + m - 11 - (2q(m - 2) + m + 2s_q) = -11 + 2(t_q - s_q) \geq -1.$$

It follows that if $0 \leq q \leq 34$, then $\left\langle 1^2, 2^{\frac{m-2}{2}} \right\rangle_m$ represents all positive integers N such that $2q(m - 2) + 2s_q \leq N \leq 2q(m - 2) + 2t_q + 2m - 11$ except that $q = 1, 2, 4, 5, 8, 14$ and $2q(m - 2) + 2t_q + m - 11 < N < 2q(m - 2) + m + 2s_q$. These numbers N are

$$3m - 2, 3m - 1, 5m - 4, 5m - 3, 5m - 2, 5m - 1, 9m - 8, 9m - 7, 11m - 10, 11m - 9, \\ 11m - 8, 11m - 7, 17m - 20, 17m - 19, 17m - 18, 17m - 17, 29m - 38, 29m - 37.$$

We can check that all of them are represented by $\left\langle 1^2, 2^{\frac{m-2}{2}} \right\rangle_m$. Thus, if $0 \leq q \leq 34$ and $2q(m - 2) + 2s_q \leq N \leq 2q(m - 2) + 2t_q + 2m - 11$, then $\left\langle 1^2, 2^{\frac{m-2}{2}} \right\rangle_m$ represents N . From Table 1, $t_q - s_{q+1} \geq 3$ for all q with $0 \leq q \leq 34$ except $q = 1, 4$. Thus,

$$2q(m - 2) + 2t_q + 2m - 11 - (2(q + 1)(m - 2) + 2s_{q+1}) = -7 + 2(t_q - s_{q+1}) \geq -1$$

for all q such that $0 \leq q \leq 34$ and $q \neq 1, 4$. It follows that $\left\langle 1^2, 2^{\frac{m-2}{2}} \right\rangle_m$ represents all positive integers N such that $0 = 2s_0 \leq N \leq 68(m - 2) + 2t_{34} + 2m - 11 = 70(m - 2) + 35$ except that $q = 1, 4$ and $2q(m - 2) + 2t_q + 2m - 11 < N < 2(q + 1)(m - 2) + 2s_{q+1}$, which are $4m - 2, 4m - 1, 10m - 8$ and $10m - 7$. By some computations, we have that $\left\langle 1^2, 2^{\frac{m-2}{2}} \right\rangle_m$ represents all positive integers except $4m - 2$.

4) If $q \geq 392$, then by Lemma 2, $\left\langle 1^{\frac{m-1}{2}} \right\rangle_m$ represents all positive integers N such that $q(m - 2) + \frac{5 + \sqrt{1 + 24q}}{2} < N \leq q(m - 2) + \frac{m-9}{2} + \sqrt{4 + 8q}$. Thus, $\left\langle 2^{\frac{m-1}{2}} \right\rangle_m$ represents all positive even integers N such that $2q(m - 2) + 5 + \sqrt{1 + 24q} < N \leq 2q(m - 2) + m - 9 + 2\sqrt{4 + 8q}$, and hence $\left\langle 1, 2^{\frac{m-1}{2}} \right\rangle_m$ represents all positive integers N such that $2q(m - 2) + 6 + \sqrt{1 + 24q} < N \leq 2q(m - 2) + m - 9 + 2\sqrt{4 + 8q}$. It

follows that $\left\langle 1^2, 2^{\frac{m-1}{2}} \right\rangle_m$ represents all positive integers N such that either

$$2q(m-2) + 6 + \sqrt{1+24q} < N \leq 2q(m-2) + m - 9 + 2\sqrt{4+8q}$$

or

$$2q(m-2) + m + 6 + \sqrt{1+24q} < N \leq 2q(m-2) + 2m - 9 + 2\sqrt{4+8q}.$$

Since

$$\begin{aligned} & 2q(m-2) + m - 9 + 2\sqrt{4+8q} - \left(2q(m-2) + m + 6 + \sqrt{1+24q}\right) \\ &= -15 + \sqrt{32q+16} - \sqrt{24q+1} \geq -15 + \sqrt{32q+16} - \sqrt{24q+12} \\ &\geq -15 + (\sqrt{32} - \sqrt{24})\sqrt{q + \frac{1}{2}} \geq -15 + (\sqrt{32} - \sqrt{24})\sqrt{\frac{785}{2}} \geq 0.014, \end{aligned}$$

$\left\langle 1^2, 2^{\frac{m-1}{2}} \right\rangle_m$ represents all positive integers N such that $2q(m-2) + 6 + \sqrt{1+24q} < N \leq 2q(m-2) + 2m - 9 + 2\sqrt{4+8q}$ for all $q \geq 392$. Since $\sqrt{24q+25} \leq \sqrt{24q+12} + 1$, we have

$$\begin{aligned} & 2q(m-2) + 2m - 9 + 2\sqrt{4+8q} - \left(2(q+1)(m-2) + 6 + \sqrt{1+24(q+1)}\right) \\ &= -11 + \sqrt{32q+16} - \sqrt{24q+25} \geq -12 + (\sqrt{32} - \sqrt{24})\sqrt{q + \frac{1}{2}} \\ &\geq -12 + (\sqrt{32} - \sqrt{24})\sqrt{\frac{785}{2}} \geq 3.014 \end{aligned}$$

for all $q \geq 392$. Thus, $\left\langle 1^2, 2^{\frac{m-1}{2}} \right\rangle_m$ represents all positive integers larger than $2(392(m-2) + r_{392}) = 784(m-2) + 102$.

For each $q = 1, 2, \dots, 35$, s_q and t_q be the same as in the proof of Proposition 1. As noted earlier, s_q and t_q are computed by a computer and given in Table 1. Thus, $\left\langle 1^{\frac{m-3}{2}} \right\rangle_m$ represents all positive integers N such that $q(m-2) + s_q \leq N \leq q(m-2) + t_q + \frac{m-13}{2}$. As a consequence, $\left\langle 2^{\frac{m-3}{2}} \right\rangle_m$ represents all positive even integers N such that $2q(m-2) + 2s_q \leq N \leq 2q(m-2) + 2t_q + m - 13$. It follows that $\left\langle 1^2, 2^{\frac{m-3}{2}} \right\rangle_m$ represents all positive integers N such that either $2q(m-2) + 2s_q \leq N \leq 2q(m-2) + 2t_q + m - 12$ or $2q(m-2) + m + 2s_q \leq N \leq 2q(m-2) + 2t_q + 2m - 12$. From Table 1, we have that if $34 \leq q \leq 65$, then $t_q - s_q \geq 6$. Since

$$2q(m-2) + 2t_q + m - 12 - (2q(m-2) + m + 2s_q) = 2(t_q - s_q) - 12 \geq 2 \cdot 6 - 12 = 0,$$

$\left\langle 1^2, 2^{\frac{m-3}{2}} \right\rangle_m$ represents all positive integers N such that $2q(m-2) + 2s_q \leq N \leq 2q(m-2) + 2t_q + 2m - 12$ for all q with $34 \leq q \leq 65$. From Table 1, we have that if $34 \leq q \leq 65$, then $t_q - s_{q+1} \geq 4$. Since

$$\begin{aligned} & 2q(m-2) + 2t_q + 2m - 12 - (2(q+1)(m-2) + 2s_{q+1}) = -8 + 2(t_q - s_{q+1}) \\ &\geq -8 + 2 \cdot 4 = 0, \end{aligned}$$

for all $q = 34, 35, \dots, 64$, all positive integers N such that

$$68(m-2) + 2s_{34} = 68(m-2) + 24 \leq N \leq 130(m-2) + 2t_{65} + 2m - 12 = 132(m-2) + 48$$

are represented by $\langle 1^2, 2^{\frac{m-3}{2}} \rangle_m$. It follows that for each $x \geq 0$, $\langle 1^2, 2^{\frac{m-3}{2}} \rangle_m$ represents all positive integers N such that $2P_m(x) + 68(m - 2) + 24 \leq N \leq 2P_m(x) + 132(m - 2) + 48$. If $0 \leq x \leq 32$, then since

$$\begin{aligned} 2P_m(x) + 132(m - 2) + 48 &= (2P_m(x) + 68(m - 2) + 24) \\ &= 64(m - 2) + 24 + 2(P_m(x) - P_m(x + 1)) \\ &= 64(m - 2) + 24 - 2x(m - 2) - 2 \\ &= (64 - 2x)(m - 2) + 22 \geq 0, \end{aligned}$$

$\langle 1^2, 2^{\frac{m-1}{2}} \rangle_m$ represents all positive integers N such that

$$2P_m(0) + 68(m - 2) + 24 = 68(m - 2) + 24 \leq N \leq 2P_m(31) + 132(m - 2) + 48 = 1124(m - 2) + 112.$$

Since $1124(m - 2) + 112 > 992(m - 2) + 110$, $\langle 1^2, 2^{\frac{m-1}{2}} \rangle_m$ represents all positive integers at least $68(m - 2) + 24$.

By the same method as that used in 3), we can see that for each $q = 0, 1, \dots, 34$, $\langle 1^2, 2^{\frac{m-1}{2}} \rangle_m$ represents all positive integers N such that either $2q(m - 2) + 2s_q \leq N \leq 2q(m - 2) + 2t_q + m - 10$ or $2q(m - 2) + m + 2s_q \leq N \leq 2q(m - 2) + 2t_q + 2m - 10$. From Table 1, $t_q - s_q \geq 5$ for all q with $0 \leq q \leq 34$ except $q = 1, 2, 4, 5, 8, 14$. Thus, if $0 \leq q \leq 34$ except $q = 1, 2, 4, 5, 8, 14$, then $2q(m - 2) + 2t_q + m - 10 - (2q(m - 2) + m + 2s_q) = -10 + 2(t_q - s_q) \geq 0$. It follows that if $0 \leq q \leq 34$, then $\langle 1^2, 2^{\frac{m-1}{2}} \rangle_m$ represents all positive integers N such that $2q(m - 2) + 2s_q \leq N \leq 2q(m - 2) + 2t_q + 2m - 9$ except $2q(m - 2) + 2t_q + m - 10 < N < 2q(m - 2) + m + 2s_q$ for $q = 1, 2, 4, 5, 8, 14$. These numbers are

$$\begin{aligned} &3m - 1, 5m - 3, 5m - 2, 5m - 1, 9m - 7, 11m - 9, 11m - 8, 11m - 7, \\ &17m - 19, 17m - 18, 17m - 17, 29m - 37. \end{aligned}$$

We can check that all of them are represented by $\langle 1^2, 2^{\frac{m-1}{2}} \rangle_m$. From Table 1, $t_q - s_{q+1} \geq 3$ for all q with $0 \leq q \leq 34$ except $q = 1, 4$. Since $2q(m - 2) + 2t_q + 2m - 10 - (2(q + 1)(m - 2) + 2s_{q+1}) = -6 + 2(t_q - s_{q+1}) \geq 0$ when $0 \leq q \leq 34$ and $q \neq 1, 4$, all positive integers N such that $0 = 2s_0 \leq N \leq 68(m - 2) + 2t_{34} + 2m - 11 = 70(m - 2) + 35$ except $4m - 2, 4m - 1, 10m - 8$ and $10m - 7$ are represented by $\langle 1^2, 2^{\frac{m-2}{2}} \rangle_m$. By some computations, we have that $\langle 1^2, 2^{\frac{m-1}{2}} \rangle_m$ is universal. \square

When $5 \leq m \leq 13$, we use computer to find all numbers not represented by some m -gonal forms $\langle 1^\alpha, 2^\beta \rangle_m$ treated in Propositions 2.10 and 2.11. These computations are based on the following two theorems.

THEOREM 2.12 (Legendre). *If $m \geq 3$ and m is odd, then for all $N \geq 28(m - 2)^3$ is represented by a sum $\langle 1^4 \rangle_m$ of four m -gonal numbers. If $m \geq 4$ and m is even, then for all $N \geq 28(m - 2)^3$ is represented by $\langle 1^4 \rangle_m + \delta_m(N)$ where $\langle 1^4 \rangle_m$ is a sum of four m -gonal numbers and $\delta_m(N)$ is 0 or 1, and hence, every $N \geq 28(m - 2)^3$ is represented by $\langle 1^5 \rangle_m$.*

THEOREM 2.13 (Meng and Sun). *If $m \geq 4$ and $m \equiv 0 \pmod{4}$, then for all $N \geq 28(m - 2)^3$ is represented by a sum $\langle 1^4 \rangle_m$ of four m -gonal numbers. If $m \geq 3$ and $m \not\equiv 0 \pmod{4}$, then for all $N \geq 1628(m - 2)^3$ is represented by $\langle 1^2, 2^2 \rangle_m$.*

The results of computations by a computer are as follows.

PROPOSITION 2.14. 1) If $6 \leq m \leq 8$, then $\langle 1^{m-1} \rangle_m$ represents all positive integers except $2m - 1$ and $5m - 4$.

2) A sum $\langle 1^4 \rangle_5$ represents all positive integers except 9, 21, 31, 43, 55 and 89, and $\langle 1, 2^4 \rangle_5$ is universal.

3) If $6 \leq m \leq 13$, then $\langle 1, 2^{m-2} \rangle_m$ represents all positive integers except $2m - 1$ for even m , and it represents all positive integers except $2m - 2$ for odd m .

4) If $m = 10, 12$, then $\langle 1^2, 2^{\frac{m-2}{2}} \rangle_m$ represents all positive integers except $4m - 2$.

5) Two m -gonal forms $\langle 1^2, 2^3 \rangle_8$ and $\langle 1^2, 2^2 \rangle_6$ represent all positive integers except 30, 78 and all positive integers except 22, 82, 100 respectively.

6) If m is odd and $5 \leq m \leq 13$, then the m -gonal form $\langle 1^2, 2^{\frac{m-1}{2}} \rangle_m$ is universal.

It is also obtained that $\langle 1, 2^3 \rangle_5$ represents all positive integers less than 10^6 except 8, 43, 67 and 135. It is probable that $\langle 1, 2^3 \rangle_5$ actually represents all positive integers except these four numbers. As a consequence of Propositions 2.10, 2.11 and 2.14, we have the following finiteness Theorem for sums of m -gonal numbers.

COROLLARY 2.15. Let $f = \langle 1^\alpha \rangle_m$ be a sum of m -gonal numbers, or equivalently, an m -gonal form with all coefficients 1. If $m \geq 6$ and f represents $2m - 1$ or $5m - 4$, then f is universal. In other words, both $\{2m - 1\}$ and $\{5m - 4\}$ are minimal universality criterion sets for universal m -gonal forms with all coefficients 1, and no more minimal universality criterion sets for this class exist. If $m = 5$ and f represents at least one of 9, 21, 31, 43, 55 and 89, then f is universal. In other words, all of $\{9\}$, $\{21\}$, $\{31\}$, $\{43\}$, $\{55\}$ and $\{89\}$ are minimal universality criterion sets for universal m -gonal form with all coefficients 1, and no more minimal universality criterion sets for this class exist.

We say that an m -gonal form $g(x_1, x_2, \dots, x_k)$ is induced by another m -gonal form $f(x_1, x_2, \dots, x_n)$ if g is obtained from f by additional conditions

$$\begin{cases} x_{i_1} = x_{i_2} = \dots = x_{i_s} = 0, \\ x_{j_1,1} = x_{j_1,2} = \dots = x_{j_1,t_1}, \\ x_{j_2,1} = x_{j_2,2} = \dots = x_{j_2,t_2}, \\ \vdots \\ x_{j_k,1} = x_{j_k,2} = \dots = x_{j_k,t_k}. \end{cases}$$

If $f = \langle a_1, a_2, \dots, a_n \rangle_m$ and $k < n$, then $g = \langle a_1, a_2, \dots, a_k \rangle_m$ is induced by f since $g(x_1, x_2, \dots, x_k) = f(x_1, x_2, \dots, x_k, 0, 0, \dots, 0)$. Also every m -gonal form is induced by a sum $\langle 1^\alpha \rangle_m$ of m -gonal numbers since $f = \langle a_1, a_2, \dots, a_n \rangle_m$ is induced by $\langle 1^\alpha \rangle_m$ with $\alpha = \sum_{i=1}^n a_i$ and

$$\begin{cases} x_1 = x_2 = \dots = x_{a_1}, \\ x_{a_1+1} = x_{a_1+2} = \dots = x_{a_1+a_2}, \\ \vdots \\ x_{a_1+a_2+\dots+a_{n-1}+1} = x_{a_1+a_2+\dots+a_{n-1}+2} = \dots = x_{a_1+a_2+\dots+a_n}. \end{cases}$$

Obviously, if an m -gonal form g is induced by another form f and g represents a positive integer N , then f represents N . We have the following universality conditions for $\langle 1^\alpha, 2^\beta \rangle_m$.

COROLLARY 2.16. *Let $m \geq 5$. An m -gonal form $f = \langle 1^\alpha, 2^\beta \rangle_m$ with $\beta \geq 1$ is universal if and only if the one of the following conditions is satisfied.*

- 1) $\langle 1, 2^\beta \rangle_m$ satisfying $\beta \geq m - 1$,
- 2) $\langle 1^2, 2^\beta \rangle_m$ satisfying $\beta \geq \frac{m}{2}$ for even m and $\beta \geq \frac{m-1}{2}$ for odd m ,
- 3) $\langle 1^\alpha, 2^\beta \rangle_m$ satisfying $\alpha \geq 3$ and $\alpha + 2\beta \geq m$.

Proof. Let $f = \langle 1^\alpha, 2^\beta \rangle_m$. When $\alpha = 1$ and $m \geq 14$, by Proposition 2.11 1) and 2), f is universal if and only if $\beta \geq m - 1$. When $\alpha = 1$ and $5 \leq m \leq 13$, by Proposition 2.14 2) and 3), f is universal if and only if $\beta \geq m - 1$. When $\alpha = 2$ and $m \geq 14$, by Proposition 2.11 3) and 4), f is universal if and only if $\beta \geq \frac{m}{2}$ for even m and $\beta \geq \frac{m-1}{2}$ for odd m respectively. When $\alpha = 2$ and $5 \leq m \leq 13$, by Proposition 2.14 4), 5) and 6), f is universal if and only if $\beta \geq \frac{m}{2}$ for even m and $\beta \geq \frac{m-1}{2}$ for odd m respectively. Note that if $\beta \leq \frac{m-3}{2}$, then since $\langle 1^2, 2^\beta \rangle_m$ is induced by $\langle 1^{m-1} \rangle_m$, it is not universal. When $\alpha \geq 3$, by Theorem 1, f is universal if $\alpha + 2\beta \geq m$. Conversely, if $\alpha + 2\beta \leq m - 1$, then since f is induced by $\langle 1^{m-1} \rangle_m$, f is not universal. \square

Proof of Theorem 1. Suppose that m is even, $m \geq 10$ and f represents $2m - 1$ and $4m - 2$. If $\alpha = 1$, then since f represents $2m - 1$, by Proposition 2.11 1), we have $\beta \geq m - 1$. Thus, f is universal. If $\alpha = 2$, then since f represents $4m - 2$, by Proposition 2.11 3), we have $\beta \geq \frac{m}{2}$. Thus, f is universal. If $\alpha \geq 3$, then since f represents $2m - 1$, f is not induced by $\langle 1^{m-1} \rangle_m$. Thus, we have $\alpha + 2\beta \geq m$. By Theorem 1, f is universal.

Suppose that m is odd, $m \geq 7$ and f represents $2m - 2$ and $2m - 1$. If $\alpha = 1$, then since f represents $2m - 2$, by Proposition 2.11 2), we have $\beta \geq m - 1$. Thus, f is universal. If $\alpha = 2$, then since f represents $2m - 1$, by Proposition 2.11 4), we have $\beta \geq \frac{m-1}{2}$. Thus, f is universal. If $\alpha \geq 3$, then since f represents $2m - 1$, f is not induced by $\langle 1^{m-1} \rangle_m$. Thus, we have $\alpha + 2\beta \geq m$. By Theorem 1, f is universal.

Similarly, we can see that if m is odd, $m \geq 7$ and f represents $2m - 2$ and $5m - 4$, then f is universal.

The proof for the cases $m = 6$ and $m = 8$ are similar. \square

REMARK 2.17. If m is even and $m \geq 10$, then since $\langle 1, 2^{m-2} \rangle_m$ represents all positive integers except $2m - 1$ and $\langle 1^2, 2^{\frac{m-2}{2}} \rangle_m$ represents all positive integers except $4m - 2$, $\{2m - 1, 4m - 2\}$ is the unique minimal universality criterion set for the class $\mathfrak{S} = \{\langle 1^\alpha, 2^\beta \rangle_m \mid \alpha, \beta \geq 0\}$ of m -gonal forms. If $m = 8$, then since $\langle 1, 2^6 \rangle_8$ represents all positive integers except 15 and $\langle 1^2, 2^3 \rangle_8$ represents all positive integers except 30, 78, we have that $\{15, 30\}$ and $\{15, 78\}$ are only minimal universality criterion sets for the class $\mathfrak{S} = \{\langle 1^\alpha, 2^\beta \rangle_8 \mid \alpha, \beta \geq 0\}$ of 8-gonal forms. If $m = 6$, then since $\langle 1, 2^4 \rangle_6$ represents all positive integers except 11 and $\langle 1^2, 2^2 \rangle_6$ represents all positive integers except 22, 82, 100, we have that $\{11, 22\}$, $\{11, 82\}$ and $\{11, 100\}$ are all of minimal universality criterion sets for the class $\mathfrak{S} = \{\langle 1^\alpha, 2^\beta \rangle_6 \mid \alpha, \beta \geq 0\}$ of 6-gonal forms.

If m is odd and $m \geq 7$, then $\langle 1, 2^{m-2} \rangle_m$ represents all positive integers except $2m - 2$. Since $\langle 1^{m-1} \rangle_m$ represents all positive integers except $2m - 1$ and $5m - 4$, both $\{2m - 2, 2m - 1\}$ and $\{2m - 2, 5m - 4\}$ are minimal universality criterion sets for the class $\mathfrak{S} = \{\langle 1^\alpha, 2^\beta \rangle_m \mid \alpha, \beta \geq 0\}$ of m -gonal forms, and no other minimal universality criterion sets for the same class of m -gonal forms exist.

For $m = 5$, as noted earlier, it seems that $\langle 1, 2^3 \rangle_5$ actually represents all positive integers except 8, 43, 67 and 135. If this prediction is true, then the minimal universality criterion sets for the class $\mathfrak{S} = \{\langle 1^\alpha, 2^\beta \rangle_5 \mid \alpha, \beta \geq 0\}$ are $\{43\}$ or $\{a, b\}$ such that $a \in \{8, 67, 135\}$ and $b \in \{9, 21, 31, 55, 89\}$.

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