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# SKELLAM DISTRIBUTION SERIES RELATED TO CATEGORIES OF BI-UNIVALENT FUNCTIONS

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Abstract. The Skellam distribution is a probability that describes the difference between two independent random variables following a Poisson distribution. It has been widely used to model the difference in the number of events in various fields, such as sports analysis, physics, economics, and medical statistics. More recently, this distribution has received increasing attention in mathematical studies that link probability theory to complex analysis and its applications. In this research, we present a new analytic function based on the Skellam distribution. The study leads to the construction of two new spiralike categories associated with the Gregory number. We also present a new Lemma, a generalization of the lemma adopted by Zaparwa [44], which is important for studying the Fekete-Szegö inequality. Using this lemma, Fekete-Szegö inequalities of the new spiralike analytic categories are derived.

## 1. Introduction and preliminaries

We define $\mathcal{A}$ as the category of analytic functions $f$ in the open disk

$$\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| < 1\},$$

such that $f(0) = f'(0) - 1 = 0$, taking the following form:

$$f(\xi) = \xi + \sum_{j=2}^{\infty} a_j \xi^j. \tag{1}$$

We also define the category of Carathéodory function $\mathbb{P}$ as $\mathfrak{p}(\xi)$ in the domain $\mathbb{D}$, such that $Re(\mathfrak{p}(\xi)) > 0$, and take the following sequential representation:

$$\mathfrak{p}(\xi) = 1 + \sum_{j=1}^{\infty} \mathfrak{p}_j \xi^j. \tag{2}$$

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The category $\mathcal{E}$ refers to Schwarz functions $\varpi(\xi)$ defined on the domain $\mathbb{D}$, which satisfy the conditions $\varpi(0) = 0$ and $|\varpi(\xi)| < 1$, and take the sequential representation:

$$\varpi(\xi) = \sum_{j=1}^{\infty} s_j \xi^j. \tag{3}$$

When a Schwarz function exists, $\varpi(\xi) \in \mathcal{E}$, such that $f(\xi) = q(\varpi(\xi))$ for all $\varpi \in \mathcal{E}$, the analytic function $f(\xi)$ is subordinated to $q(\xi)$. This is denoted by $f(\xi) \prec q(\xi)$. Furthermore, the fact that $q(\xi)$ is a univalent in $\mathcal{E}$ makes the relation $f(\xi) \prec q(\xi)$ equivalent to the congruence conditions $f(0) = q(0)$ and the image containing $f(\mathcal{E}) \subset q(\mathcal{E})$ (see [11, 28]).

The function $f$ is spirallike-shape if (see [40])

$$\Re e \left( e^{-i\tau} \frac{\xi f'(\xi)}{f(\xi)} \right) > 0, \quad (|\tau| < \frac{\pi}{2},\ \xi \in \mathbb{D}).$$

The function $f$ of family $\mathcal{A}$ is an element of the $\tau$-spiralike category, denoted by $\mathcal{S}^{*,\tau}_{\nu}$, if the following holds:

$$\Re e \left( e^{-i\tau} \frac{\xi f'(\xi)}{f(\xi)} \right) > \nu\ cos\tau, \quad (|\tau| < \frac{\pi}{2},\ \nu \in [0, 1); \xi \in \mathbb{D}).$$

This category was initially explored by Libra [27] (also see [25]).

Nowadays, Salehtan and Motamednezhad [36] provided the new category, which is called $\gamma$-$\mu$-spiralike, is given by

$$\Re e \left( e^{-i\tau} \frac{\xi f'(\xi) + \mu \xi^2 f''(\xi)}{(1-\mu) f(\xi) + \mu \xi f'(\xi)} \right) > \nu\ cos\tau, \quad (|\tau| < \frac{\pi}{2},\ \nu \in [0, 1), \mu \in [0, 1]; \xi \in \mathbb{D}).$$

Murugusundaramoorthy [30] introduced a category of the following

$$\Re e \left( e^{-i\tau} \left( 1 + \frac{\xi f''(\xi)}{f'(\xi)} \right) \right) > \nu\ cos\tau, \quad (|\tau| < \frac{\pi}{2},\ \nu \in [0, 1); \xi \in \mathbb{D}).$$

Wahid et al. [25] derived the category of symmetric spiralike as follows:

$$\mathrm{Re} \left\{ e^{i\tau} \frac{2\xi f'(\xi)}{f(\xi) + \overline{f(\overline{\xi})}} \right\} > \delta,\ (0 \leqslant \delta < 1,\ |\tau| < \frac{\pi}{2}; \xi \in \mathbb{D}).$$

Nikitin [34], Bhoosnurmath and Devadas [8, 9] investigated the accuracy of the geometric properties of the categories that they studied as follows:

$$e^{i\tau} \frac{\xi f'(\xi)}{f(\xi)} \prec \frac{1 + X\xi}{1 + Y\xi} \cos\tau + i \sin\tau, \quad (-1 \leqq Y < X \leqq 1)$$

and

$$e^{i\tau} \frac{\xi f''(\xi)}{f'(\xi)} \prec \frac{1 + X\xi}{1 + Y\xi} \cos\tau + i \sin\tau, \quad (-1 \leqq Y < X \leqq 1),$$

where $\frac{1+X\xi}{1+Y\xi}$ is the Janowski function [20, 21].

The familiarize inverse function $f^{-1} : \mathbb{D}_2 \to \mathbb{D}_1$ is defined as:

$$f^{-1}(f(\xi)) = \xi, \quad (\xi \in \mathbb{D}_1)$$

and

$$f(f^{-1}(\chi)) = \chi \quad \left( |\chi| < r_0(f); r_0(f) \geq \frac{1}{4} \right).$$

The expansion form of $f^{-1}$ is given by

$$f^{-1}(\chi) = \chi - a_2\chi^2 + (2a_2^2 - a_3)\chi^3 - (5a_2^3 - 5a_2a_3 + a_4)\chi^4 + \cdots .$$

We say that the function $f$ belongs to the family $\Sigma$ (the family of one-to-one functions in $\mathbb{D}$) if $f$ is an analytic function, and both $f(\xi)$ and its inverse $f^{-1}(\xi)$ are univalent within $\mathbb{D}$. For depth details, see [1, 12, 29, 35, 39, 42, 43].

When studying the family $\Sigma$, Brannan and Clunie postulated [10] that $a_2 < \sqrt{2}$, while Lewin [26] arrived at the inequality that $a_2 < 1.51$. Later, Netanyahu [32] proved that $\max a_2 = 3/4$. However, finding a general estimate for the coefficients of $a_j$ $(j \geqq 3)$, remains an open question.

Using $q$-calculus, Kamble and Shrigan [22, 23] investigated new categories of bi-univalent functions with derivational limits for the coefficients $|a_2|$ and $|a_3|$. Yusuf et al. [42] also studied the Fekete-Szegö problem for specific subcategories, while Srivastava et al. [38] addressed the Hankel second determinant for different categories of these functions. It should be noted that the Fekete-Szegö problem, introduced by Fekete and Szegö [17], is one of the fundamental problems related to the coefficients of functions on $\Sigma$. The Fekete-Szegö inequality is stated as

$$|a_3 - \varrho a_2^2| \leqslant 1 + 2e^{-2\varrho/(1-\varrho)}, \quad \varrho \in \mathbb{R}.$$

Recently, the study of bi-univalent functions associated with orthogonal polynomials has seen considerable interest in modern literature, such as [3, 5, 7, 16, 18, 19].

Gregory coefficients, also known as inverse logarithmic numbers, Bernoulli numbers of the second kind, or Cauchy numbers of the first kind, are a sequence of alternating and decreasing rational numbers, such as $\frac{1}{2}, -\frac{1}{12}, \frac{1}{24}, -\frac{19}{720}, \ldots$. These coefficients naturally appear in the Maclaurin expansion of the function $\frac{\xi}{\log(1+\xi)}$, which can be represented as follows:

$$\mathcal{G}(\xi) = \frac{\xi}{\log(1+\xi)} = 1 + \frac{1}{2}\xi - \frac{1}{12}\xi^2 + \frac{1}{24}\xi^3 - \frac{19}{720}\xi^4 + \cdots, \quad \xi \in \mathbb{D}.$$

Figure 1 shows image of the disk $\mathbb{D}$ under the function $\mathcal{G}(\xi)$, i.e., the set $\mathcal{G}(\mathbb{D})$. The parameters associated with this function are known as Gregory coefficients, named after James Gregory, who first introduced them in 1670 in the context of his studies in numerical integration. These parameters have enjoyed renewed interest throughout the ages, with numerous mathematicians reanalyzing and developing them, thus solidifying their place in the work of many prominent researchers. They continue to be widely recognized in contemporary mathematical literature.

The generating function of the Gregory coefficients $w_j$ (see [24]), are given by

$$\mathcal{G}(\xi) = \xi/\log(\xi+1) = \sum_{j=0}^{\infty} w_j\xi^j = 1 + \frac{1}{2}\xi - \frac{1}{12}\xi^2 + \cdots, \quad \xi \in \mathbb{D}. \tag{4}$$

Clearly, $w_j$ for some values of $k \in \mathbb{N}$ are

$$w_0 = 1,\ w_1 = \frac{1}{2},\ w_2 = \frac{-1}{12},\ w_3 = \frac{1}{24},\ w_4 = \frac{-19}{720}, \cdots .$$

A variable $x$ is said to have Skellam distribution if it takes the values $0, 1, 2, \ldots$ with probabilities

$$e^{-(\epsilon_1+\epsilon_2)}I_0(2\sqrt{\epsilon_1\epsilon_2}), e^{-(\epsilon_1+\epsilon_2)}\left(\frac{\epsilon_1}{\epsilon_2}\right)^{1/2}I_1(2\sqrt{\epsilon_1\epsilon_2}), e^{-(\epsilon_1+\epsilon_2)}\left(\frac{\epsilon_1}{\epsilon_2}\right)I_2(2\sqrt{\epsilon_1\epsilon_2}), \ldots,$$

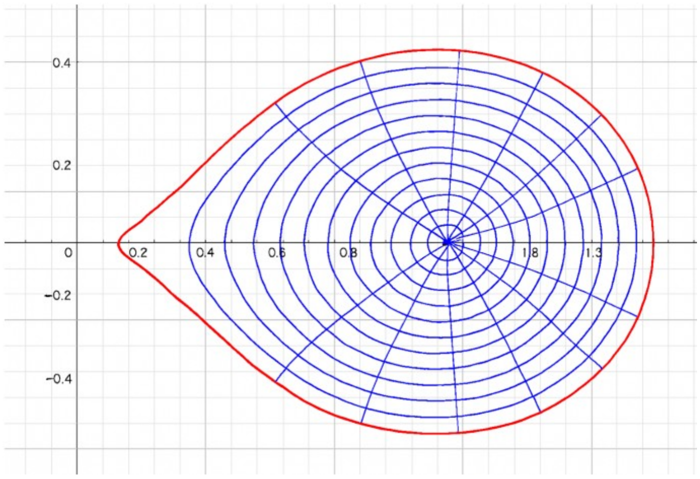



FIGURE 1. The image of $\mathbb{D}$ under $\mathcal{G}(\xi)$.

respectively. Thus,

$$P(x=j)=e^{-(\epsilon_1+\epsilon_2)}\left(\frac{\epsilon_1}{\epsilon_2}\right)^{j/2} I_{|j|}(2\sqrt{\epsilon_1\epsilon_2}),\ j=0,\mp1,\mp2,\dots, \tag{5}$$

where

$$I_{|j|}(x)=\sum_{l=0}^{\infty}\frac{1}{l!\Gamma(l+j+1)}\left(\frac{x}{2}\right)^{2l+j}$$

is the modified Bessel function of the first kind of order $j$ and the mean of Poisson variables $\epsilon_1,\epsilon_2>0$.

For the specific case, if $\epsilon_1=\epsilon_2=\kappa$, the probability (5) leads to

$$P(x=j)=e^{-2\kappa}I_j(2\kappa),\quad j\geqq 0, \tag{6}$$

where

$$I_j(2\kappa)=\sum_{l=0}^{\infty}\frac{\kappa^{2l+j}}{l!\Gamma(l+j+1)}.$$

Now, we introduce Skellam distribution series as follows:

$$\Lambda_\kappa(\xi)=\xi+\sum_{j=2}^{\infty}e^{-2\kappa}I_{j-1}(2\kappa)\xi^j,\ \xi\in\mathbb{D}. \tag{7}$$

We can easily show that series (7) is convergent and the radius of convergence is infinity.

We purpose from (7) that

$$\Lambda_1^\kappa(\xi)=2\xi-\Lambda_\kappa(\xi)=\xi-\sum_{j=2}^{\infty}e^{-2\kappa}I_{j-1}(2\kappa)\xi^j,\ \xi\in\mathbb{D}. \tag{8}$$

By employing convolution, an operator $\mathcal{Q}_\kappa:\mathcal{A}\to\mathcal{A}$ can be prepared as follows:

$$\mathcal{Q}_\kappa f(\xi):=\Lambda_\kappa(\xi)*f(\xi)=\xi+\sum_{j=2}^{\infty}\sigma_{\kappa,j}a_j\xi^j,\ \xi\in\mathbb{D}, \tag{9}$$

where

$$\sigma_{\kappa,j}=e^{-2\kappa}I_{j-1}(2\kappa). \tag{10}$$

Building on a wide range of previous findings that have explored the relationships between multiple categories of univalent functions using distribution series functions (for example [14, 15, 31, 33, 41]), along with a number of recent studies in the bi-univalent area, in the present paper we consider a new operator $\mathscr{Q}_\kappa$ using skellam distribution series. The study establishes categories of spiralike functions concerning to the Gregory coefficients involving this operator. Furthermore, the study delved into the Fekete-Szegö problems and some related inequalities. Our results demonstrate clear extensions and overlap with many previous findings in this field, thus strengthening scientific coherence and highlighting the contribution of this work within the existing theoretical framework.

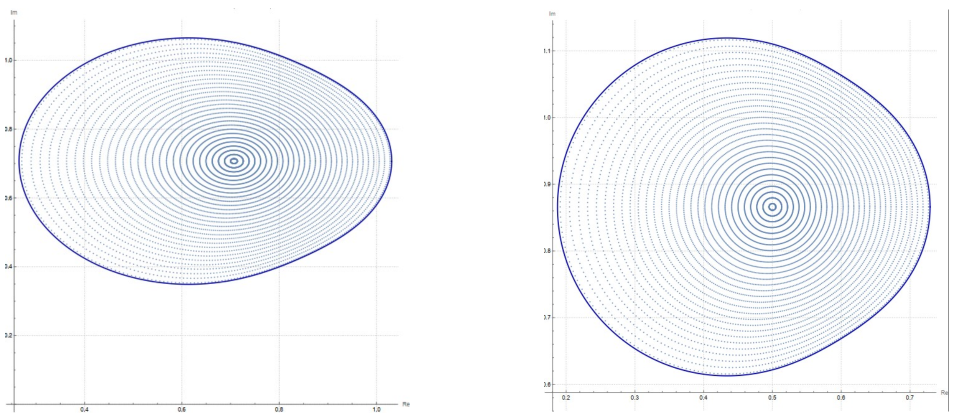
Definition 1.1. The family $\overline{\mathcal{S}}^\kappa_\tau[\mathcal{G}]$ and $\overline{\mathcal{C}}^\kappa_\tau[\mathcal{G}]$ involve the function $f(\xi)$ if

$$\overline{\mathcal{S}}^\kappa_\tau[\mathcal{G}] = \left\{ e^{i\tau} \frac{\xi(\mathscr{Q}_\kappa f(\xi))'}{\mathscr{Q}_\kappa f(\xi)} \prec \mathcal{G}(\xi)\cos\tau + i\sin\tau, \quad |\tau| < \frac{\pi}{2}; \xi \in \mathbb{D} \right\} \tag{11}$$

and

$$\overline{\mathcal{C}}^\kappa_\tau[\mathcal{G}] = \left\{ e^{i\tau} \frac{\xi(\mathscr{Q}_\kappa f'(\xi))'}{\mathscr{Q}_\kappa f'(\xi)} \prec \mathcal{G}(\xi)\cos\tau + i\sin\tau, \ |\tau| < \frac{\pi}{2}; \xi \in \mathbb{D} \right\}, \tag{12}$$

where $\mathcal{G}(\xi)$ is the generating of Gregory function in (4).



(a) When $\tau = \frac{\pi}{4}$ (b) When $\tau = \frac{\pi}{3}$

Figure 2. The image of $\mathcal{G}(\xi)\cos\tau + i\sin\tau$.

Remark 1.2. Some of Special cases for Definitions 1.3 and 1.4 are listed below

1. If $\kappa = 0$, then the families reduce to

$$\overline{\mathcal{S}}^0_\tau[\mathcal{G}] = \left\{ e^{i\tau} \frac{\xi f'(\xi)}{f(\xi)} \prec \mathcal{G}(\xi)\cos\tau + i\sin\tau, \quad |\tau| < \frac{\pi}{2}; \xi \in \mathbb{D} \right\}$$

and

$$\overline{\mathcal{C}}^0_\tau[\mathcal{G}] = \left\{ e^{i\tau} \frac{(\xi f'(\xi))'}{f'(\xi)} \prec \mathcal{G}(\xi)\cos\tau + i\sin\tau, \ |\tau| < \frac{\pi}{2}; \xi \in \mathbb{D} \right\}.$$

2. If $\tau = 0, \kappa = 0$, then

$$\overline{\mathcal{S}}_0^{\kappa}[\mathcal{G}] = \left\{ \frac{\xi f'(\xi)}{f(\xi)} \prec \mathcal{G}(\xi), \quad \xi \in \mathbb{D} \right\}$$

and

$$\overline{\mathcal{C}}_0^{\kappa}[\mathcal{G}] = \left\{ \frac{(\xi f'(\xi))'}{f'(\xi)} \prec \mathcal{G}(\xi),\ \xi \in \mathbb{D} \right\}.$$

For the bi-univalent functions, we consider the following definition:

Definition 1.3. The family $\overline{\mathcal{BS}}_{\tau}^{\kappa}[\mathcal{G}]$ involves the function $f(\xi)$ if

$$e^{i\tau}\frac{\xi(\mathscr{Q}_{\kappa} f(\xi))'}{\mathscr{Q}_{\kappa} f(\xi)} \prec \mathcal{G}(\xi)\cos\tau + i\sin\tau,\ |\tau| < \frac{\pi}{2} \tag{13}$$

and

$$e^{i\tau}\frac{\chi(\mathscr{Q}_{\kappa} f^{-1}(\chi))'}{\mathscr{Q}_{\kappa} f^{-1}(\chi)} \prec \mathcal{G}(\chi)\cos\tau + i\sin\tau,\ |\tau| < \frac{\pi}{2}. \tag{14}$$

Definition 1.4. The family $\overline{\mathcal{BC}}_{\tau}^{\kappa}[\mathcal{G}]$ involves the function $f(\xi)$ if

$$e^{i\tau}\frac{(\xi(\mathscr{Q}_{\kappa} f(\xi))')'}{(\mathscr{Q}_{\kappa} f(\xi))'} \prec \mathcal{G}(\xi)\cos\tau + i\sin\tau,\ |\tau| < \frac{\pi}{2} \tag{15}$$

and

$$e^{i\tau}\frac{(\chi(\mathscr{Q}_{\kappa} f^{-1}(\chi))')'}{(\mathscr{Q}_{\kappa} f^{-1}(\chi))'} \prec \mathcal{G}(\chi)\cos\tau + i\sin\tau,\ |\tau| < \frac{\pi}{2}. \tag{16}$$

The following Lemmas are based on the inferred estimation in [13,44], making them an important tool for restricting and analyzing complex expressions in this work.

Lemma 1.5. [13] Let $\mathfrak{p} \in \mathbb{P}$ be a Carathéodory function as in (2). Then, the result $|\mathfrak{p}_j| \leqq 2$ is sharp for $f(\xi) = \frac{1+\xi}{1-\xi}$.

Lemma 1.6. [44] If $n, m \in R$ and $\xi_1, \xi_2 \in \mathbb{C}$. Assuming $|\xi_1| < L$ and $|\xi_2| < L$, then

$$|(n+m)\xi_1 + (n-m)\xi_2| \leq \begin{cases} 2|n|L, & \text{for} \quad |n| \geq |m| \\ 2|m|L, & \text{for} \quad |n| \leq |m| \end{cases} \tag{17}$$

To determine the bound Fekete-Szegö inequality, if the parameters $n$ or $m$ being in a complex form, this prevents the fulfillment of traditional conditions of Lemma 1.6. To overcome this limitation, we propose a generalized formula, which is later used to derive the Fekete-Szegö inequality.

Lemma 1.7. *If $m \in R$ and $n, \xi_1, \xi_2 \in \mathbb{C}$ with $n = a + ib$. Assuming $|\xi_1| < L$ and $|\xi_2| < L$, then*

$$|(n+m)\xi_1 + (n-m)\xi_2| \leqq \begin{cases} 2(|a|+|b|)L, & \textit{for} \quad |a| \geqq |m| \\ 2(|m|+|b|)L, & \textit{for} \quad |a| \leqq |m| \end{cases} \tag{18}$$

*Proof.* By similar technique of Lemma 1.6 in [44], we easily calculate

$$\begin{aligned}|(n+m)\xi_1+(n-m)\xi_2| &= |(a+ib+m)\xi_1+(a+ib-m)\xi_2| \\ &\leqq |(a+m)\xi_1+(a-m)\xi_2|+|b||\xi_1+\xi_2| \\ &\leqq \begin{cases} 2|a|L+2|b|L, & \text{for} \quad |a|\geqq |m| \\ 2|m|L+2|b|L, & \text{for} \quad |a|\leqq |m| \end{cases} \\ &= \begin{cases} 2(|a|+|b|)L, & \text{for} \quad |a|\geqq |m| \\ 2(|m|+|b|)L, & \text{for} \quad |a|\leqq |m| \end{cases}.\end{aligned}$$

□

The next part begins by deriving the initial bounds of the coefficients $|a_j|$, as these values represent the basis upon which Fekete-Szegö inequality is built, and are essential in determining its final formulation and analytical limits.

## 2. Essential Findings

THEOREM 2.1. *For $f(\xi)\in\overline{\mathcal{BS}}_{\tau}^{\kappa}[\mathcal{G}]$, we have*

$$|a_2|\leqq min\{E_1,E_2\}$$

*and*

$$|a_3|\leqq min\{B_1,B_2\},$$

*where $\sigma_{\kappa,2}$, $\sigma_{\kappa,3}$ recalled in* (10),

$$E_1:=\sqrt{\frac{|e^{-i\tau}|\cos(\tau)}{2\left|\left\{2\sigma_{\kappa,3}+(\frac{7}{3}e^{i\tau}sec(\tau)-1)\sigma_{\kappa,2}^2\right\}\right|}},\ E_2:=\frac{\cos(\tau)}{2|e^{i\tau}||\sigma_{\kappa,2}|},$$

*and*

$$B_1:=\frac{|e^{-i\tau}|\cos(\tau)}{8|\sigma_{\kappa,3}|}+\frac{|e^{-2i\tau}|\cos^2(\tau)}{16\sigma_{\kappa,2}^2},\ B_2:=\frac{|e^{-i\tau}|\cos(\tau)}{8|\sigma_{\kappa,3}|}+E_1^2.$$

*Proof.* Since $f(\xi)\in\overline{\mathcal{BS}}_{\tau}^{\kappa}[\mathcal{G}]$, by definition there exist analytic functions $\eta,\delta$ with $\eta(0)=\delta(0)=0$ and $|\eta(\xi)|<1$, $|\delta(\chi)|<1$ on $\mathbb{D}$ such that

$$e^{i\tau}\frac{\xi(\mathscr{Q}_\kappa f(\xi))'}{\mathscr{Q}_\kappa f(\xi)}=\mathcal{G}(\eta(\xi))\cos\tau+i\sin\tau, \tag{19}$$

$$e^{i\tau}\frac{\chi(\mathscr{Q}_\kappa f^{-1}(\chi))'}{\mathscr{Q}_\kappa f^{-1}(\chi)}=\mathcal{G}(\delta(\chi))\cos\tau+i\sin\tau. \tag{20}$$

Therefore, the function

$$\mathfrak{p}(\xi)=\frac{1+\eta(\xi)}{1-\eta(\xi)}=\mu_1\xi+\mu_2\xi^2+\mu_3\xi^3+\cdots$$

and

$$\mathfrak{q}(\chi)=\frac{1+\delta(\chi)}{1-\delta(\chi)}=\ell_1\chi+\ell_2\chi^2+\ell_3\chi^3+\cdots$$

belong to the Carathéodory function category.

From this, the expansions of the two functions $\eta(\xi)$ and $\delta(\chi)$ can be easily found as follows:

$$\eta(\xi)=\frac{\mathfrak{p}(\xi)-1}{\mathfrak{p}(\xi)+1}=\frac{1}{2}\mu_1\xi+\frac{1}{2}\left(\mu_2-\frac{1}{2}\mu_1^2\right)\xi^2+\cdots,$$
$$\delta(\chi)=\frac{\mathfrak{q}(\chi)-1}{\mathfrak{q}(\chi)+1}=\frac{1}{2}\ell_1\chi+\frac{1}{2}\left(\ell_2-\frac{1}{2}\ell_1^2\right)\chi^2+\cdots. \tag{21}$$

Also the expansion of both sides of (19) and (20) are given by

(22)
$$\cos(\tau)\frac{\eta(\xi)}{\log\left(\eta(\xi)+1\right)}+i\sin(\tau)=\cos(\tau)+i\sin(\tau)+\frac{1}{4}\mu_1\cos(\tau)\xi$$
$$-\frac{1}{48}\left(\left(7\mu_1^2-12\mu_2\right)\cos(\tau)\right)\xi^2+\frac{1}{192}\left(17\mu_1^3-56\mu_2\mu_1+48\mu_3\right)\cos(\tau)\xi^3+\cdots,$$
$$\cos(\tau)\frac{\delta(\chi)}{\log\left(\delta(\chi)+1\right)}+i\sin(\tau)=\cos(\tau)+i\sin(\tau)+\frac{1}{4}\ell_1\cos(\tau)\chi$$
$$-\frac{1}{48}\left(\left(7\ell_1^2-12\ell_2\right)\cos(\tau)\right)\chi^2+\frac{1}{192}\left(17\ell_1^3-56\ell_2\ell_1+48\ell_3\right)\cos(\tau)\chi^3+\cdots,$$

and

$$e^{i\tau}\frac{\xi(\mathscr{Q}_\kappa f(\xi))'}{\mathscr{Q}_\kappa f(\xi)}=e^{i\tau}+e^{i\tau}\sigma_{\kappa,2}a_2\xi-e^{i\tau}\left(\sigma_{\kappa,2}^2a_2^2-2\sigma_{\kappa,3}a_3\right)\xi^2$$
$$+e^{i\tau}\left(\sigma_{\kappa,2}^3a_2^3-3\sigma_{\kappa,3}\sigma_{\kappa,2}a_2a_3+3\sigma_{\kappa,4}a_4\right)\xi^3+\cdots,$$
$$e^{i\tau}\frac{\chi(\mathscr{Q}_\kappa f^{-1}(\chi))'}{\mathscr{Q}_\kappa f^{-1}(\chi)}=e^{i\tau}-e^{i\tau}\sigma_{\kappa,2}a_2\chi-e^{i\tau}\left(\sigma_{\kappa,2}^2a_2^2-2\sigma_{\kappa,3}[2a_2^2-a_3]\right)\chi^2$$
$$-e^{i\tau}(a_2^3\left(\sigma_{k,2}^3-6\sigma_{k,3}\sigma_{k,2}+15\sigma_{k,4}\right)+3a_3a_2\left(\sigma_{k,2}\sigma_{k,3}-5\sigma_{k,4}\right)$$
$$+3a_4\sigma_{k,4})\chi^3+\cdots.$$

Expanding both sides of (19) and (20) and comparing coefficients yield the system

$$e^{i\tau}\sigma_{\kappa,2}a_2=\frac{1}{4}\cos(\tau)\mu_1, \tag{23}$$

$$e^{i\tau}\left(2\sigma_{\kappa,3}a_3-\sigma_{\kappa,2}^2a_2^2\right)=\frac{1}{48}\left(12\mu_2-7\mu_1^2\right)\cos(\tau), \tag{24}$$

$$-e^{i\tau}\sigma_{\kappa,2}a_2=\frac{1}{4}\cos(\tau)\ell_1, \tag{25}$$

$$e^{i\tau}\left((4\sigma_{\kappa,3}-\sigma_{\kappa,2}^2)a_2^2-2\sigma_{\kappa,3}a_3\right)=\frac{1}{48}\left(12\ell_2-7\ell_1^2\right)\cos(\tau). \tag{26}$$

From (23) and (25) we get $\mu_1=-\ell_1$ and

$$32e^{2i\tau}\sigma_{\kappa,2}^2a_2^2=\cos^2(\tau)(\mu_1^2+\ell_1^2). \tag{27}$$

Adding (24) and (26) leads to

$$e^{i\tau}(4\sigma_{\kappa,3}-2\sigma_{\kappa,2}^2)a_2^2=\cos(\tau)\left(\frac{1}{4}\left(\mu_2+\ell_2\right)-\frac{7}{48}\left(\mu_1^2+\ell_1^2\right)\right). \tag{28}$$

Substitute $(\mu_1^2+\ell_1^2)$ from (27) into (28) to express $a_2^2$:

$$2e^{i\tau}\left\{2\sigma_{\kappa,3}+(\frac{7}{3}e^{i\tau}sec(\tau)-1)\sigma_{\kappa,2}^2\right\}a_2^2=\frac{1}{4}\cos(\tau)\left(\mu_2+\ell_2\right), \tag{29}$$

which yields

$$a_2 = \sqrt{\frac{e^{-i\tau}\cos(\tau)\left(\mu_2+\ell_2\right)}{8\left\{2\sigma_{\kappa,3}+(\frac{7}{3}e^{i\tau}sec(\tau)-1)\sigma_{\kappa,2}^2\right\}}}.$$

By Lemma 1.5, we deduce

$$|a_2| \leqq \sqrt{\frac{|e^{-i\tau}|\cos(\tau)}{2\left|\left\{2\sigma_{\kappa,3}+(\frac{7}{3}e^{i\tau}sec(\tau)-1)\sigma_{\kappa,2}^2\right\}\right|}} =: E_1.$$

Another bound comes directly from (23), which gives

$$|a_2| \leqq \frac{\cos(\tau)}{2|e^{i\tau}||\sigma_{\kappa,2}|} =: E_2.$$

Thus,

$$|a_2| \leqq min\{E_1, E_2\}.$$

To estimate $a_3$, subtract (26) from (24) (and use $\mu_1^2 = \ell_1^2$) to obtain

$$4e^{i\tau}\sigma_{\kappa,3}\left(a_3-a_2^2\right) = \frac{1}{4}\cos(\tau)\left(\mu_2-\ell_2\right). \tag{30}$$

Solving for $a_3$ (using earlier relations to substitute the quadratic terms) yields

$$a_3 = \frac{e^{-i\tau}\cos(\tau)}{16\sigma_{\kappa,3}}(\mu_2-\ell_2)+a_2^2. \tag{31}$$

Now, by (27), we obtain

$$a_3 = \frac{e^{-i\tau}\cos(\tau)}{16\sigma_{\kappa,3}}(\mu_2-\ell_2)+\frac{e^{-2i\tau}\cos^2(\tau)(\mu_1^2+\ell_1^2)}{32\sigma_{\kappa,2}^2}.$$

Taking absolute values and using the bounds $|\mu_1^2+\ell_1^2| \leq 2$ and $|\mu_2-\ell_2| \leq 2$ (from Lemma 2.1 and basic estimates) gives

$$|a_3| \leqq \frac{|e^{-i\tau}|\cos(\tau)}{8|\sigma_{\kappa,3}|}+\frac{|e^{-2i\tau}|\cos^2(\tau)}{16\sigma_{\kappa,2}^2} := B_1$$

and if the value of $a_2^2$ is in the (29), then

$$|a_3| \leqq \frac{|e^{-i\tau}|\cos(\tau)}{8|\sigma_{\kappa,3}|}+\frac{|e^{-i\tau}|\cos(\tau)}{4\left|\left\{2\sigma_{\kappa,3}+(\frac{7}{3}e^{i\tau}sec(\tau)-1)\sigma_{\kappa,2}^2\right\}\right|} := B_2,$$

which are stated the upper bound of $|a_3|$.

This completes the proof. □

Theorem 2.2. *For $f(\xi) \in \overline{\mathcal{BC}}_{\tau}^{\kappa}[\mathcal{G}]$, we have*

$$|a_2| \leq min\left\{E_3, E_4\right\}$$

*and*

$$|a_3| \leqq \left\{B_3, B_4\right\},$$

*where $\sigma_{\kappa,2}$ and $\sigma_{\kappa,3}$ recalled in* (10),

$$E_3 := \sqrt{\frac{3|e^{-i\tau}|\cos(\tau)}{4\left|9\sigma_{\kappa,3}+2\left(7e^{i\tau}\sec(\tau)-3\right)\sigma_{\kappa,2}^2\right|}},\ E_4 := \frac{\cos(\tau)}{4|e^{i\tau}||\sigma_{\kappa,2}|},$$

*and*

$$B_3 := \frac{|e^{-2i\tau}|\cos^2(\tau)}{64\sigma_{\kappa,2}^2} + \frac{|e^{-i\tau}|\cos(\tau)}{24|\sigma_{\kappa,3}|},\ B_4 := \frac{|e^{-i\tau}|\cos(\tau)}{24|\sigma_{\kappa,3}|} + E_3^2.$$

*Proof.* Since $f(\xi) \in \overline{\mathcal{BC}}_\tau^\kappa[\mathcal{G}]$, by definition there exist analytic functions $\eta, \delta$ with $\eta(0) = \delta(0) = 0$ and $|\eta(\xi)| < 1$, $|\delta(\chi)| < 1$ on $\mathbb{D}$ such that

$$e^{i\tau}\frac{(\xi(\mathcal{Q}_\kappa f(\xi))')'}{(\mathcal{Q}_\kappa f(\xi))'} = \mathcal{G}(\eta(\xi))\cos\tau + i\sin\tau, \tag{32}$$

$$e^{i\tau}\frac{(\xi(\mathcal{Q}_\kappa f^{-1}(\chi))')'}{(\mathcal{Q}_\kappa f^{-1}(\chi))'} = \mathcal{G}(\delta(\chi))\cos\tau + i\sin\tau. \tag{33}$$

where

$$\eta(\xi) = \frac{\mu_1\xi}{2} + \left(\frac{\mu_2}{2} - \frac{\mu_1^2}{4}\right)\xi^2 + \left(\frac{\mu_1^3}{8} - \frac{\mu_2\mu_1}{2} + \frac{\mu_3}{2}\right)\xi^3 + \cdots$$

and

$$\delta(\chi) = \frac{\ell_1\chi}{2} + \left(\frac{\ell_2}{2} - \frac{\ell_1^2}{4}\right)\chi^2 + \left(\frac{\ell_1^3}{8} - \frac{\ell_2\ell_1}{2} + \frac{\ell_3}{2}\right)\chi^3 + \cdots$$

are the Schwarz functions.

The expansion of both sides of (32) and (33) are given by in (22) and

$$\begin{aligned} e^{i\tau}\frac{(\xi(\mathcal{Q}_\kappa f(\xi))')'}{(\mathcal{Q}_\kappa f(\xi))'} &= e^{i\tau} + 2e^{i\tau}\sigma_{\kappa,2}a_2\xi - 2e^{i\tau}\left(2\sigma_{\kappa,2}^2a_2^2 - 3\sigma_{\kappa,3}a_3\right)\xi^2 \\ &\quad + 2e^{i\tau}\left(4\sigma_{\kappa,2}^3a_2^3 - 9\sigma_{\kappa,3}\sigma_{\kappa,2}a_2a_3 + 6\sigma_{\kappa,4}a_4\right)\xi^3 + \cdots, \\ e^{i\tau}\frac{(\chi(\mathcal{Q}_\kappa f^{-1}(\chi))')'}{(\mathcal{Q}_\kappa f^{-1}(\chi))'} &= e^{i\tau} - 2e^{i\tau}\sigma_{\kappa,2}a_2\chi - 2e^{i\tau}\left(2\sigma_{\kappa,2}^2a_2^2 - 3\sigma_{\kappa,3}[2a_2^2 - a_3]\right)\chi^2 \\ &\quad - 2e^{i\tau}(2a_2^3\left(2\sigma_{\kappa,2}^3 - 9\sigma_{\kappa,3}\sigma_{\kappa,2} + 15\sigma_{\kappa,4}\right) + 3a_3a_2 \\ &\qquad .\,(3\sigma_{\kappa,2}\sigma_{\kappa,3} - 10\sigma_{\kappa,4}) + 6a_4\sigma_{\kappa,4})\chi^3 + \cdots. \end{aligned}$$

Thus,

$$2e^{i\tau}\sigma_{\kappa,2}a_2 = \frac{1}{4}\cos(\tau)\mu_1, \tag{34}$$

$$e^{i\tau}\left(6\sigma_{\kappa,3}a_3 - 4\sigma_{\kappa,2}^2a_2^2\right) = \frac{1}{48}\left(12\mu_2 - 7\mu_1^2\right)\cos(\tau), \tag{35}$$

$$-2e^{i\tau}\sigma_{\kappa,2}a_2 = \frac{1}{4}\cos(\tau)\ell_1, \tag{36}$$

$$e^{i\tau}\left(4\left(3\sigma_{\kappa,3} - \sigma_{\kappa,2}^2\right)a_2^2 - 6a_3\sigma_{\kappa,3}\right) = \frac{1}{48}\left(12\ell_2 - 7\ell_1^2\right)\cos(\tau). \tag{37}$$

From (34) and (36) we get $\mu_1 = -\ell_1$ and

$$128e^{2i\tau}\sigma_{\kappa,2}^2a_2^2 = \cos^2(\tau)(\mu_1^2 + \ell_1^2). \tag{38}$$

Following the technique used in Theorem 2.1 generates the values of $|a_2|$ and $|a_3|$. □

For $\kappa = 0$, the direct results hold:

Corollary 2.3. *For $f(\xi) \in \overline{\mathcal{BS}}_\tau^0[\mathcal{G}]$, we have*

$$|a_2| \leqq min\left\{\sqrt{\frac{|e^{-i\tau}|\cos(\tau)}{2\left|1 + \frac{7}{3}e^{i\tau}sec(\tau)\right|}},\ \frac{\cos(\tau)}{2|e^{i\tau}|}\right\}$$

*and*

$$|a_3| \leqq min\left\{\frac{|e^{-i\tau}|\cos(\tau)}{8}+\frac{|e^{-2i\tau}|\cos^2(\tau)}{16},\ \frac{|e^{-i\tau}|\cos(\tau)}{8}+\frac{|e^{-i\tau}|\cos(\tau)}{2\left|1+\frac{7}{3}e^{i\tau}sec(\tau)\right|}\right\}.$$

COROLLARY 2.4. *For $f(\xi)\in\overline{\mathcal{BC}}_{\tau}^{0}[\mathcal{G}]$, we have*

$$|a_2| \leqq min\left\{\frac{1}{2}\sqrt{\frac{3|e^{-i\tau}|\cos(\tau)}{|3+14e^{i\tau}\sec(\tau)|}},\ \frac{\cos(\tau)}{4|e^{i\tau}|}\right\}$$

*and*

$$|a_3| \leqq min\left\{\frac{|e^{-2i\tau}|\cos^2(\tau)}{64}+\frac{|e^{-i\tau}|\cos(\tau)}{24},\ \frac{|e^{-i\tau}|\cos(\tau)}{24}+\frac{1}{4}\frac{3|e^{-i\tau}|\cos(\tau)}{|3+14e^{i\tau}\sec(\tau)|}\right\}.$$

Also, for $\tau=0$ and $\kappa=0$, the direct results hold:

COROLLARY 2.5. *[24] For $f(\xi)\in\overline{\mathcal{BS}}_{0}^{0}[\mathcal{G}]$, we have*

$$|a_2| \leqq min\{\sqrt{\frac{3}{20}},\frac{1}{2}\} = \sqrt{\frac{3}{20}} \quad and \quad |a_3| \leqq \frac{1}{4}.$$

Similar way, we deduce the following estimates:

COROLLARY 2.6. *[37] For $f(\xi)\in\overline{\mathcal{BC}}_{0}^{0}[\mathcal{G}]$, we have*

$$|a_2| \leqq \frac{1}{4} \quad and \quad |a_3| \leqq \frac{1}{12}.$$

## 3. Fekete-Szegö Inequality

The following investigation includes deriving Fekete-Szegö inequality of bi-univalent functions.

THEOREM 3.1. *For $f(\xi)\in\overline{\mathcal{BS}}_{\tau}^{\kappa}[\mathcal{G}]$ and let $\varrho\in\mathbb{R}$, we have*

$$|a_3-\varrho a_2^2| \leqq \begin{cases} \frac{cos(\tau)}{2|\sigma_{\kappa,3}|}\left[\frac{1}{2}+\frac{7\sigma_{\kappa,2}^2|\sigma_{\kappa,3}tan(\tau)||1-\varrho|}{3(2\sigma_{\kappa,3}+\frac{4}{3}\sigma_{\kappa,2}^2)^2+(\frac{49}{3}\sigma_{\kappa,2}^4tan^2(\tau))}\right], & |1-\varrho|\leqq H_1 \\ \frac{cos(\tau)}{6|\sigma_{\kappa,3}|}\left[\frac{(\frac{7}{3}\sigma_{\kappa,2}^2|\sigma_{\kappa,3}tan(\tau)|+|2\sigma_{\kappa,2}^2+\frac{4}{3}\sigma_{\kappa,2}^2\sigma_{\kappa,3}|)|1-\varrho|}{(2\sigma_{\kappa,3}+\frac{4}{3}\sigma_{\kappa,2}^2)^2+(\frac{49}{9}\sigma_{\kappa,2}^4tan^2(\tau))}\right], & |1-\varrho|\geqq H_1, \end{cases}$$

*where*

$$H_1=\frac{(2\sigma_{\kappa,3}+\frac{4}{3}\sigma_{\kappa,2}^2)^2+(\frac{49}{9}\sigma_{\kappa,2}^4tan^2(\tau))}{|4\sigma_{\kappa,3}^2+\frac{8}{3}\sigma_{\kappa,2}^2\sigma_{\kappa,3}|}.$$

*Proof.* It follows from (29) and (31) that

$$\begin{aligned} a_3-\varrho a_2^2 &= (a_3-a_2^2)+(1-\varrho)a_2^2 \\ &= \frac{\cos(\tau)(\mu_2-\ell_2)}{16e^{i\tau}\sigma_{\kappa,3}}+(1-\varrho)\frac{\cos(\tau)\,(\mu_2+\ell_2)}{8e^{i\tau}[2\sigma_{\kappa,3}+(\frac{7}{3}e^{i\tau}sec(\tau)-1)\sigma_{\kappa,2}^2]} \\ &= \frac{\cos(\tau)}{8e^{i\tau}\sigma_{\kappa,3}}\left[\frac{(\mu_2-\ell_2)}{2}+(1-\varrho)\frac{\sigma_{\kappa,3}(\mu_2+\ell_2)}{[2\sigma_{\kappa,3}+(\frac{7}{3}e^{i\tau}sec(\tau)-1)\sigma_{\kappa,2}^2]}\right] \\ &= \frac{\cos(\tau)}{8e^{i\tau}\sigma_{\kappa,3}}\left[[e(\varrho;\tau)+\frac{1}{2}]\mu_2+[e(\varrho;\tau)-\frac{1}{2}]\ell_2\right], \end{aligned}$$

where

$$e(\varrho;\tau) := \frac{\sigma_{\kappa,3}(1-\varrho)}{2\sigma_{\kappa,3} + (\frac{7}{3}e^{i\tau}sec(\tau)-1)\sigma_{\kappa,2}^2}.$$

So, by Lemma 1.7, we get

$$|a_3 - \varrho a_2^2| \leqq \frac{cos(\tau)}{8|\sigma_{\kappa,3}|}\begin{cases} 4(|\frac{1}{2}| + |Im(e(\varrho;\tau))|), & Re(e(\varrho;\tau)) \leqq \frac{1}{2} \\ 4(|Re(e(\varrho;\tau))| + |Im(e(\varrho;\tau))|), & Re(e(\varrho;\tau)) \geqq \frac{1}{2} \end{cases}$$

with

$$Re(e(\varrho,\tau)) = \frac{(2\sigma_{\kappa,3}^2 + \frac{4}{3}\sigma_{\kappa,2}^2\sigma_{\kappa,3})(1-\varrho)}{(2\sigma_{\kappa,3}+\frac{4}{3}\sigma_{\kappa,2}^2)^2 + (\frac{49}{9}\sigma_{\kappa,2}^4 tan^2(\tau))}$$

and

$$Im(e(\varrho;\tau)) = \frac{\frac{7}{3}\sigma_{\kappa,2}^2\sigma_{\kappa,3}tan(\tau)(1-\varrho)}{(2\sigma_{\kappa,3}+\frac{4}{3}\sigma_{\kappa,2}^2)^2 + (\frac{49}{9}\sigma_{\kappa,2}^4 tan^2(\tau))}.$$

This proof is complete after some calculations. □

For the convex bi-univalent family, we have

THEOREM 3.2. *For $f(\xi) \in \overline{\mathcal{BC}}_\tau^\kappa[\mathcal{G}]$ and $\varrho \in \mathbb{R}$, we have*

$$|a_3 - \varrho a_2^2| \leqq \begin{cases} \frac{cos(\tau)}{6|\sigma_{\kappa,3}|}\left[\frac{1}{2} + \frac{56\sigma_{\kappa,2}^2|\sigma_{\kappa,3}tan(\tau)||1-\varrho|}{(6\sigma_{\kappa,3}+\frac{8}{3}\sigma_{\kappa,2}^2)^2+(\frac{3136}{9}\sigma_{\kappa,2}^4tan^2(\tau))}\right], & |1-\varrho| \leqq H_2 \\ \frac{cos(\tau)}{6|\sigma_{\kappa,3}|}\left[\frac{(|3\sigma_{\kappa,3}(6\sigma_{\kappa,3}+\frac{8}{3}\sigma_{\kappa,2}^2)|+56\sigma_{\kappa,2}^2|\sigma_{\kappa,3}\tan(\tau)|)|1-\varrho|}{(6\sigma_{\kappa,3}+\frac{8}{3}\sigma_{\kappa,2}^2)^2+(\frac{3136}{9}\sigma_{\kappa,2}^4tan^2(\tau))}\right], & |1-\varrho| \geqq H_2, \end{cases}$$

*where*

$$H_2 = \frac{(6\sigma_{\kappa,3}+\frac{8}{3}\sigma_{\kappa,2}^2)^2 + (\frac{3136}{9}\sigma_{\kappa,2}^4 tan^2(\tau))}{6(|6\sigma_{\kappa,3}^2 + \frac{8}{3}\sigma_{\kappa,2}^2\sigma_{\kappa,3}|)}.$$

*Proof.* It follows from (35), (37) and (38) that

$$e^{i\tau}\left(12\sigma_{\kappa,3} - 8\sigma_{\kappa,2}^2 + \frac{112}{3}e^{i\tau}\sigma_{\kappa,2}^2 sec(\tau)\right)a_2^2 = \frac{1}{4}cos(\tau)\left(\mu_2+\ell_2\right). \tag{39}$$

Now, subtracting (37) from (35), we have

$$12e^{i\tau}\sigma_{\kappa,3}(a_3 - a_2^2) = \frac{1}{4}cos(\tau)\left(\mu_2 - \ell_2\right). \tag{40}$$

We not from (39) and (40) that

$$\begin{aligned} a_3 - \varrho a_2^2 &= (a_3 - a_2^2) + (1-\varrho)a_2^2 \\ &= \frac{\cos(\tau)(\mu_2-\ell_2)}{48e^{i\tau}\sigma_{\kappa,3}} + (1-\varrho)\frac{\cos(\tau)\left(\mu_2+\ell_2\right)}{16e^{i\tau}[3\sigma_{\kappa,3} + \frac{28}{3}e^{i\tau}sec(\tau) - 8\sigma_{\kappa,2}^2]} \\ &= \frac{\cos(\tau)}{24e^{i\tau}\sigma_{\kappa,3}}\left[\frac{(\mu_2-\ell_2)}{2} + (1-\varrho)\frac{3\sigma_{k,3}\left(\mu_2+\ell_2\right)}{2\left(3\sigma_{\kappa,3} - 8\sigma_{\kappa,2}^2 + \frac{28}{3}e^{i\tau}\sigma_{k,2}^2 sec(\tau)\right)}\right] \\ &= \frac{\cos(\tau)}{24e^{i\tau}\sigma_{\kappa,3}}\left[[e_1(\varrho;\tau)+\frac{1}{2}]\mu_2 + [e_1(\varrho;\tau)-\frac{1}{2}]\ell_2\right], \end{aligned}$$

where

$$e_1(\varrho;\tau) := \frac{3\sigma_{\kappa,3}(1-\varrho)}{2\left(3\sigma_{\kappa,3} - 8\sigma_{\kappa,2}^2 + \frac{28}{3}e^{i\tau}\sigma_{\kappa,2}^2 sec(\tau)\right)}.$$

So, by Lemma 1.7, we get

$$|a_3 - \varrho a_2^2| \leqq \frac{cos(\tau)}{24|\sigma_{\kappa,3}|}\begin{cases} 4(|\frac{1}{2}| + |Im(e_1(\varrho;\tau))|), & Re(e_1(\varrho;\tau)) \leqq \frac{1}{2} \\ 4(|Re(e_1(\varrho;\tau))| + |Im(e_1(\varrho;\tau))|), & Re(e_1(\varrho;\tau)) \geqq \frac{1}{2} \end{cases}$$

with

$$Re(e_1(\varrho,\tau)) = \frac{3(6\sigma_{\kappa,3}^2 + \frac{8}{3}\sigma_{\kappa,2}^2\sigma_{\kappa,3})(1-\varrho)}{(6\sigma_{\kappa,3} + \frac{8}{3}\sigma_{\kappa,2}^2)^2 + (\frac{3136}{9}\sigma_{\kappa,2}^4 tan^2(\tau))}$$

and

$$Im(e(\varrho;\tau)) = \frac{56\sigma_{\kappa,2}^2\sigma_{\kappa,3}tan(\tau)(1-\varrho)}{(6\sigma_{\kappa,3} + \frac{8}{3}\sigma_{\kappa,2}^2)^2 + (\frac{3136}{9}\sigma_{\kappa,2}^4 tan^2(\tau))}.$$

Therefore the proof is complete. □

## 4. Concluding and valuable remark

To derive the Fekete-Szegö inequality, when the coefficients $m, n$ appear as real numbers, lemma provided by Zaprawa [ [44], pp.2] can be used as a suitable way to restrict the bound of inequality. However, if one of these coefficients $n$ appears in complex form, that lemma becomes inapplicable. Therefore, Lemma 1.7 was introduced as a suitable generalization, as it allows for the estimation of Fekete-Szegö inequality when the coefficients are complex numbers. On the other hand, we presented categories of Spirelike analytical functions associated with Gregory coefficients and Skellam distribution series. Therefore, this study can serve as a starting point for future studies and applications such as Hankel determinants, differential subordination, and harmonic functions.

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