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A REFINED ENUMERATION OF p-ARY LABELED TREES

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Abstract. Let $\mathcal{T}_n^{(p)}$ be the set of p-ary labeled trees on $\{1, 2, \ldots, n\}$. A maximal decreasing subtree of a p-ary labeled tree is defined by the maximal p-ary subtree from the root with all edges being decreasing. In this paper, we study a new refinement $\mathcal{T}_{n,k}^{(p)}$ of $\mathcal{T}_n^{(p)}$, which is the set of p-ary labeled trees whose maximal decreasing subtree has k vertices.

1. Introduction

Let p be a fixed integer greater than 1. A *p-ary tree* T is a tree such that

(i) either T is empty or has a distinguished vertex r which is called the root of T, and
(ii) $T - r$ consists of a weak ordered partition $(T_1, \ldots, T_p)$ of p-ary trees.

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A 2-ary(resp. 3-ary) tree is called binary(resp. ternary) tree. Figure 1 exhibits all the ternary tree with 3 vertices. A *full p-ary tree* is a p-ary tree, where each vertex has either 0 or p children. It is well known (see [6, 6.2.2 Proposition]) that the number of full p-ary trees with n internal vertices is given by the nth order-p Fuss-Catalan number [2, p. 361] $C_n^{(p)} = \frac{1}{pn+1}\binom{pn+1}{n}$. Clearly a full p-ary tree T with m internal vertices corresponds to a p-ary tree with m vertices by deleting all the leaves in T, so the number of p-ary trees with n vertices is also $C_n^{(p)}$.

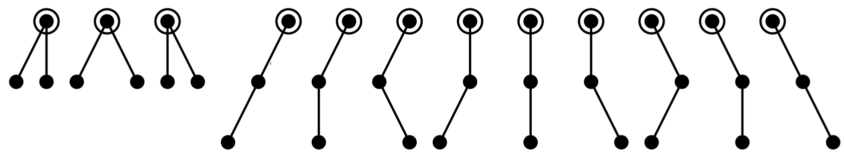

FIGURE 1. All 12 ternary trees with 3 vertices

A *p-ary labeled tree* is a p-ary tree whose vertices are labeled by distinct positive integers. In most cases, a p-ary labeled tree with n vertices is identified with a p-ary tree on the vertex set $[n] := \{1, 2, \ldots, n\}$. Let $\mathcal{T}_n^{(p)}$ be the set of p-ary labeled trees on $[n]$. Clearly the cardinality of $\mathcal{T}_n^{(p)}$ is given by

$$|\mathcal{T}_n^{(p)}| = n!\, C_n^{(p)} = (pn)_{(n-1)}, \tag{1}$$

where $m_{(k)} := m(m-1)\cdots(m-k+1)$ is a falling factorial.

For a given p-ary labeled tree T, a *maximal decreasing subtree* of T is defined as the maximal p-ary subtree from the root with all edges being decreasing, denoted by $\mathrm{MD}(T)$. Figure 2 illustrates the maximal decreasing subtree of a given ternary tree T. Let $\mathcal{T}_{n,k}^{(p)}$ be the set of p-ary labeled trees on $[n]$ with its maximal decreasing subtree having k vertices.

In this paper we present a formula for $|\mathcal{T}_{n,k}^{(p)}|$, which makes a refined enumeration of $\mathcal{T}_n^{(p)}$, or a generalization of equation (1). Note that similar refinements for rooted labeled trees and ordered labeled trees were done before (see [4,5]), but the p-ary case is much more complicated and has quite different features.

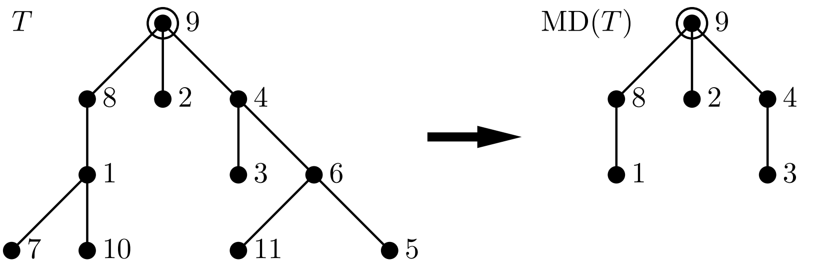

FIGURE 2. The maximal decreasing subtree of T

2. Main results

From now on we will consider labeled trees only. So we will omit the word "labeled". Recall that $\mathcal{T}_{n,k}^{(p)}$ is the set of p-ary trees on $[n]$, whose maximal decreasing subtree has k vertices. Let $\mathcal{Y}_{n,k}^{(p)}$ be the set of p-ary trees T on $[n]$, where T is given by attaching additional $(n-k)$ increasing leaves to a decreasing tree with k vertices. Let $\mathcal{F}_{n,k}^{(p)}$ be the set of (non-ordered) forests on $[n]$ consisting of k p-ary trees, where the k roots are not ordered. In Figure 3, the first two forests are the same, but the third one is a different forest in $\mathcal{F}_{4,2}^{(2)}$.

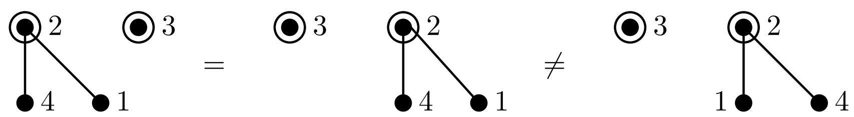

FIGURE 3. Forests in $\mathcal{F}_{4,2}^{(3)}$

Define the numbers

$$t(n,k) = \left|\mathcal{T}_{n,k}^{(p)}\right|,$$
$$y(n,k) = \left|\mathcal{Y}_{n,k}^{(p)}\right|,$$
$$f(n,k) = \left|\mathcal{F}_{n,k}^{(p)}\right|.$$

We will show that a p-ary tree can be "decomposed" into a p-ary tree in $\bigcup_{n,k} \mathcal{Y}_{n,k}^{(p)}$ and a forest in $\bigcup_{n,k} \mathcal{F}_{n,k}^{(p)}$. Thus it is important to count the numbers $y(n,k)$ and $f(n,k)$.

LEMMA 2.1. *For $0 \le k < n$, the number $y(n,k)$ satisfies the recursion:*

$$y(n+1,k+1) = \sum_{m=0}^{p} \binom{n}{m} p_{(m)} \, (kp - n + m + 1) \cdot y(n-m,k) \tag{2}$$

with the following boundary conditions:

$$y(n,n) = \prod_{j=0}^{n-1} (1 + (p-1)j) \qquad \text{for } n \ge 1 \tag{3}$$

$$y(n,k) = 0 \qquad \text{for } k < \max\left(\frac{n-1}{p}, 1\right). \tag{4}$$

Proof. Consider a tree Y in $\mathcal{Y}^{(p)}_{n+1,k+1}$. The tree Y with $(n+1)$ vertices consists of its maximal decreasing tree with $(k+1)$ vertices and $(n-k)$ increasing leaves. Note that the vertex 1 is always contained in $\mathrm{MD}(Y)$.

If the vertex 1 is a leaf of Y, consider the tree Y' obtained by deleting the leaf 1 from Y. The number of vertices in Y' and $\mathrm{MD}(Y')$ are n and k, respectively. So the number of possible trees Y' is $y(n,k)$. Since we cannot attach the vertex 1 to $(n-k)$ increasing leaves of Y', there are $kp-(n-1)$ ways of recovering Y. Thus the number of Y with the leaf 1 is

$$(kp - n + 1) \cdot y(n,k). \tag{5}$$

If the vertex 1 is not a leaf of Y, then the vertex 1 has increasing leaves $\ell_1, \ldots, \ell_m$, where $1 \le m \le p$. Consider the tree Y'' obtained by deleting $\ell_1, \ldots, \ell_m$ from Y. Clearly 1 is a leaf of Y'' and the number of vertices in Y'' and $\mathrm{MD}(Y'')$ are $n-m+1$ and $k+1$, respectively. Thus by (5), the number of possible trees Y'' is $(kp-(n-m)+1) \cdot y(n-m,k)$. To recover Y is to relabel all the vertices except 1 of Y'' with the label set $\{2,3,\ldots,n+1\} \setminus \{\ell_1,\ldots,\ell_m\}$ and to attach the leaves $\ell_1,\ldots,\ell_m$ to the vertex 1 of Y''. Clearly $\ell_1,\ldots,\ell_m$ is a subset of $\{2,3,\ldots,n+1\}$. It is obvious that a way of attaching $\ell_1,\ldots,\ell_m$ to vertex 1 can be regarded as an injection from $\ell_1,\ldots,\ell_m$ to $[p]$. Thus the number of Y without the leaf 1 is

$$\binom{n}{m}\binom{p}{m} m! \, (kp - (n-m) + 1) \cdot y(n-m,k). \tag{6}$$

Since m may be the number from 1 to p and substituting $m=0$ in (6) yields (5), we have the recursion (2).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	2	2							
3	0	2	10	6						
4	0	0	24	56	24					
5	0	0	24	256	360	120				
6	0	0	0	640	2672	2640	720			
7	0	0	0	720	11824	28896	21840	5040		
8	0	0	0	0	30464	196352	330624	201600	40320	
9	0	0	0	0	35840	857728	3177600	4032000	2056320	362880

TABLE 1. $y(n,k)$ with $p=2$

Since $\mathcal{Y}_{n,n}^{(p)}$ is the set of decreasing p-ary trees on $[n]$, the equation (3) holds (see [1]). For a arbitrary tree $Y \in \mathcal{Y}_{n,k}^{(p)}$, $\mathrm{MD}(Y)$ consists of k vertices. So $\mathrm{MD}(Y)$ has $pk-(k-1)$ locations to attach $n-k$ increasing leaves. Thus, if the inequality $pk-(k-1) < n-k$ holds, then $\mathcal{Y}_{n,k}^{(p)}$ should be empty. For $n \geq 1$ and $k=0$, it is obvious that $\mathcal{Y}_{n,k}^{(p)}$ is also empty. These give the equation (4). $\square$

The sequence $y(n,k)$ with $p=2$ is shown in Table 1.

Now we calculate $f(n,k)$ which is the number of forests on $[n]$ consisting of k p-ary trees, where the k components are not ordered. Here we use the convention that the empty product is 1.

LEMMA 2.2. *For $0 \leq k \leq n$, we have*

$$f(n,k) = \binom{n}{k} pk \prod_{i=1}^{n-k-1} (pn-i) \quad \text{if } n>k, \tag{7}$$

else $f(n,n)=1$.

Proof. Consider a forest F in $\mathcal{F}_{n,k}^{(p)}$. The forest F consists of (non-ordered) p-ary trees $T_1,\ldots,T_k$ with roots $r_1,r_2,\ldots,r_k$, where $r_1<r_2<\cdots<r_k$. The number of ways for choosing roots $r_1,r_2,\cdots,r_k$ from $[n]$ is equal to $\binom{n}{k}$. From the *reverse Prüfer algorithm (RP Algorithm)* in [3], the number of ways for adding $n-k$ vertices successively to k roots $r_1,r_2,\cdots,r_k$ is equal to

$$pk(pn-1)(pn-2)\cdots(pn-n+k+1)$$

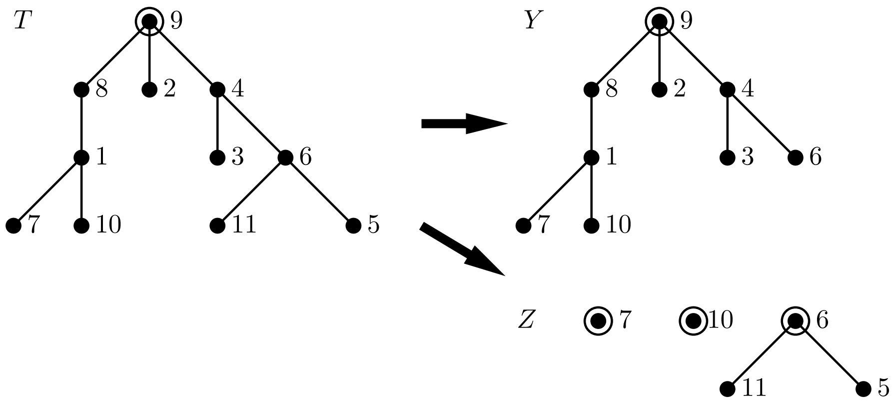

FIGURE 4. Decomposition of T into Y and Z

for $0 < k < n$, thus the equation (7) holds. For $0 = k < n$, $\mathcal{F}_{n,0}^{(p)}$ is empty, so $f(n,0) = 0$ included in (7). For $0 \le k = n$, $\mathcal{F}_{n,n}^{(p)}$ is the set of forests with no edges, so $f(n,n) = 1$. □

Since the number $y(n,k)$ is determined by the recurrence relation (2) in Lemma 2.1, we can count the number $t(n,k)$ with the following theorem.

THEOREM 2.3. *For $n \ge 1$, we have*

$$t(n,k) = \sum_{m=k}^{n} \binom{n}{m} \frac{m-k}{n-k} (pn-pk)_{(n-m)} \cdot y(m,k) \quad \text{if } 1 \le k < n, \tag{8}$$

else $t(n,n) = \prod_{j=0}^{n-1} (pj - j + 1)$, *where* $a_{(\ell)} := a(a-1)\cdots(a-\ell+1)$ *is a falling factorial.*

Proof. Given a p-ary tree T in $\mathcal{T}_{n,k}^{(p)}$, let Y be the subtree of T consisting of $\mathrm{MD}(T)$ and its increasing children. If Y has m vertices, then Y is a subtree of T with $(m-k)$ increasing leaves. Also, the induced subgraph Z of T generated by the $(n-k)$ vertices not belonging to $\mathrm{MD}(T)$ is a (non-ordered) forest consisting of $(m-k)$ p-ary trees whose roots are increasing leaves of Y. Figure 4 illustrates the subgraphs Y and Z of a given ternary tree T.

Now let us count the number of p-ary trees $T \in \mathcal{T}_{n,k}^{(p)}$ with $|V(Y)| = m$ where $V(Y)$ is the set of vertices in Y. First of all, the number of ways for selecting a set $V(Y) \subset [n]$ is equal to $\binom{n}{m}$. By attaching $(m-k)$

$n\backslash k$	0	1	2	3	4	5	6	7	$n!C_n$
0	1								1
1	0	1							1
2	0	2	2						4
3	0	14	10	6					30
4	0	152	104	56	24				336
5	0	2240	1504	816	360	120			5040
6	0	41760	27744	15184	6992	2640	720		95040
7	0	942480	621936	342768	162240	65856	21840	5040	2162160

TABLE 2. $t(n,k)$ with $p=2$

increasing leaves to a decreasing p-ary tree with k vertices, we can make a p-ary tree on $V(Y)$. So there are exactly $y(m,k)$ ways for making such a p-ary subtree on $V(Y)$. Since all the roots of Z are determined by Y, by the definition of $\mathcal{F}^{(p)}_{n,k}$ and Lemma 2.2, the number of ways for constructing the other parts on $V(T)\setminus V(\mathrm{MD}(T))$ is equal to

$$f(n-k,m-k)\Big/\binom{n-k}{m-k}=\frac{m-k}{n-k}(pn-pk)_{(n-m)}.$$

Since the range of m is $k\le m\le n$, the equation (8) holds.

Finally, $\mathcal{T}^{(p)}(n,n)$ is the set of decreasing p-ary trees on $[n]$, so

$$t(n,n)=y(n,n)=\prod_{j=0}^{n-1}(pj-j+1)$$

holds for $n\ge 1$. □

The sequence $t(n,k)$ with $p=2$ is listed in Table 2. Note that each row sum is equal to $n!C^{(p)}_n$ with $p=2$.

REMARK. Due to Lemma 2.1 and Theorem 2.3, we can calculate $t(n,k)$ for all n, k. In particular we express $t(n,k)$ as a linear combination of $y(k,k),y(k+1,k),\dots,y(n,k)$. However a closed form, a recurrence relation, or a (double) generating function of $t(n,k)$ have not been found yet.

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