

NEW CONTRACTIONS IN MODULAR METRIC SPACES AND APPLICATIONS

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ABSTRACT. This article discusses new contraction mappings with reference to modular metric spaces. It examines conditions for such operators to have invariant points. A nontrivial case is shown to illustrate the veracity of the established results. From applications point of consideration, it is observed that the ideas proposed herein allow for the presentation of a general criteria for solving a class of integral equations.

1. Introduction

Banach [1] proposed a well-known fixed point result that laid the foundation for metric fixed point theory. Banach noted that a self-map Ω , defined over the complete metric space (Ξ, ϱ) , possesses distinct fixed point if it satisfies the contraction inequality, that is, in the event that there is a constant $\kappa \in [0, 1)$ such that $\varrho(\Omega\xi, \Omega\zeta) \leq \kappa\varrho(\xi, \zeta)$ for all $\xi, \zeta \in \Xi$. A result with a new fixed point was proposed by Kannan [2], who took into account the contraction $\varrho(\Omega\xi, \Omega\zeta) \leq \kappa[\varrho(\xi, \Omega\xi) + \varrho(\zeta, \Omega\zeta)]$, for all $\xi, \zeta \in \Xi$, where $\kappa \in [0, \frac{1}{2})$. Since then, the Banach contraction principle has been expanded in several ways (for more details see [3], [4] and many others). On the other hand, one of the active development within the non-linear functional analysis discipline is the formulation of new spaces and the corresponding theorems of fixed point. As a generalization of metric spaces, for example, Modular spaces were proposed by Nakano [5] and then extensively explored by Koshi [6] and Yamamuro [7]. Furthermore, in 2008, by using properties of modular spaces, Chistyakov [8, 9] defined a new structure and presented the concept of modular metric spaces. A primary driving force for Chistyakov's recently defined modular is its physical interpretation. In particular, a modular within a set links the elements to a non-negative, sometimes infinitely valued field of velocities, while a metric defined on a non-empty set indicates the distance between any two points in the set. In metric spaces that are modular, Mongkolkeha et al. [10] investigated and demonstrated novel fixed-point existence theorems for contraction mappings. Surajit et al. [11] established the presence of fixed points of Hardy-Rogers contractions and produced numerical examples

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to support their findings. Also, a new class of generalized F -contraction is shown to have fixed points in [12], and the outcomes are then applied to an integral equation class's existence problems. Yogita and Shishir [13] developed a mixed G -monotone mapping on a modular metric space with a graph and established fixed point theorems for this new class of mappings. Also, [14] Generalized *Prešić*-type w -contractive mappings and highly w -contractive mappings in a modular metric space were introduced, along with fixed-point theorems for such contractions. In addition, [15] presented an established theorem of fixed points for four self-maps and developed the idea of partial modular metrics to remove the possibility of triangle inequality and non-zero self-distance discrepancies. Also, Somaye et al. [17] defined Hausdorff topology on modular spaces with metric and demonstrated a number of well-known metric space results, such as the Uniform Limit Theorem for Metric Modular Spaces and Baire's theorems. In addition, Rahimpour [18] employs the mixed monotone mapping property to demonstrate a number of fixed point theorems for partial order sets in modular metric spaces. Also, [19] presented a few fixed point outcomes of modular metric spaces with cyclic weak φ -contractions that are ω -compact and ω -complete, respectively. Furthermore, [20] the idea of weakly commuting mappings in modular metric spaces was extended to the setting of modular ω^G -metric spaces by three pairs of weakly commuting self-maps that share a single fixed point. Moreover, in the context of JS-metric spaces, Doru and Ariana [21] demonstrated the presence of fixed points of certain kinds of convex contractions. We observe from literature that the fixed point theorem of modular contraction types and related applications are not receiving sufficient attention. This work establishes fixed point results of various types of modular contraction inequality. To support the hypothesis formulated by the results presented here, an example is presented. The general usefulness of this research is indicated by examining an application to the integral equation problem.

2. Preliminaries

First, we provide an overview of the basic concepts in both modular metric spaces and modular spaces (for further information, see [8, 9, 22, 23]).

DEFINITION 2.1. Suppose that Ξ is a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A modular is a functional $\rho : \Xi \rightarrow [0, \infty]$ if for arbitrary $\xi, \zeta \in \Xi$, the following three requirements are met:

- (A 1) $\rho(\xi) = 0 \Leftrightarrow \xi = 0$;
- (A 2) $\rho(\alpha\xi) = \rho(\xi) \ \forall \text{ scalar } \alpha \text{ with } |\alpha| = 1$;
- (A 3) $\rho(\alpha\xi + \beta\zeta) \leq \rho(\xi) + \rho(\zeta)$ whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

Moreover, Chistyakov [8] claimed that if condition (A 3) is replaced by $\rho(\alpha\xi + \beta\zeta) \leq \rho^s(\xi) + \rho^s(\zeta)$, for $\alpha, \beta \geq 0$, $\alpha^s + \beta^s = 1$ with $s \in (0, 1]$, the modular ρ will thus be referred to as s -convex modular, and if $s = 1$, it will be referred to as a convex modular. If ρ is modular in Ξ , then the set defined by $\Xi_\rho = \{\xi \in \Xi : \rho(\lambda\xi) \rightarrow 0\}$ as $\lambda \rightarrow 0^+$ is called a modular space. Ξ_ρ is a vector subspace of Ξ . Hence, it can be equipped with an F -norm defined by setting $\|\xi_\rho\| = \inf\{\lambda > 0 : \rho(\frac{\xi}{\lambda}) \leq \lambda\}$, $\xi \in \Xi_\rho$. In addition, if ρ is convex, then the modular space Ξ_ρ coincides with $\Xi_\rho^* = \{\xi \in \Xi : \exists \lambda = \lambda(\xi) > 0 \text{ such that } \rho(\lambda\xi) < \infty\}$ and the functional $\|\xi_\rho\| = \inf\{\lambda > 0 : \rho(\frac{\xi}{\lambda}) \leq 1\}$, is an ordinary norm on Ξ_ρ^* which is equivalence to $\|\xi_\rho\|$ (see [24]).

The function $\omega : (0, \infty) \times \Xi \times \Xi \rightarrow [0, \infty]$ will be expressed as $\omega_\lambda(\xi, \zeta) = \omega(\lambda, \xi, \zeta)$ for all $\lambda > 0$ and $\xi, \zeta \in \Xi$ due to the disparity of the inputs.

DEFINITION 2.2. [8, Definition 2.1] Consider the nonempty set Ξ . For any $\xi, \zeta, \gamma \in \Xi$, a function $\omega : (0, \infty) \times \Xi \times \Xi \rightarrow [0, \infty]$ is considered a metric modular on Ξ , if the following conditions hold:

- (i) $\omega_\lambda(\xi, \zeta) = 0$ for all $\lambda > 0$ if and only if $\xi = \zeta$;
- (ii) $\omega_\lambda(\xi, \zeta) = \omega_\lambda(\zeta, \xi)$ for all $\lambda > 0$;
- (iii) $\omega_{\lambda+\mu}(\xi, \zeta) \leq \omega_\lambda(\xi, \gamma) + \omega_\mu(\gamma, \zeta)$ for all $\lambda, \mu > 0$.

According to Chistyakov [8], suppose we only have condition (i') $\omega_\lambda(\xi, \xi) = 0$ for all $\lambda > 0$ in place of condition (i) above, then ω is said to be a (metric) pseudomodular on Ξ . The main property of a (pseudo) modular ω on a set Ξ is the following: given $\xi, \zeta \in \Xi$, the function $0 < \lambda \rightarrow \omega_\lambda(\xi, \zeta) \in [0, \infty]$ is a non-increasing on $(0, \infty)$. In fact, if $0 < \mu < \lambda$, then (iii), (i') and (ii) imply $\omega_\lambda(\xi, \zeta) \leq \omega_{\lambda-\mu}(\xi, \xi) + \omega_\mu(\xi, \zeta) = \omega_\mu(\xi, \zeta)$. Thus, at every point where $\lambda > 0$, the right limit $\omega_{\lambda+0}(\xi, \zeta) := \lim_{\epsilon \rightarrow 0^+} \omega_{\lambda+\epsilon}(\xi, \zeta)$ and the left limit $\omega_{\lambda-0}(\xi, \zeta) := \lim_{\epsilon \rightarrow 0^+} \omega_{\lambda-\epsilon}(\xi, \zeta)$ exist in $[0, \infty]$ and the two inequalities that follow are true: $\omega_{\lambda+0}(\xi, \zeta) \leq \omega_\lambda(\xi, \zeta) \leq \omega_{\lambda-0}(\xi, \zeta)$.

If we replace pseudomodular with modular, the same claim remains true. The metric space Ξ_ω is modular if ω is metric modular in Ξ . By the property of modular and metric spaces, the following are defined:

DEFINITION 2.3. [10, Definition 2.4] Consider the modular metric space Ξ_ω .

- (i) Convergence of a sequence $(\xi_j)_{j \in \mathbb{N}}$ in Ξ_ω is defined as $j \rightarrow \infty$ for all $\lambda > 0$ if $\omega_\lambda(\xi_j, \xi) \rightarrow 0$;
- (ii) The Cauchy sequence $(\xi_j)_{j \in \mathbb{N}}$ in Ξ_ω is defined as $\omega_\lambda(\xi_k, \xi_j) \rightarrow 0$, as $k, j \rightarrow \infty$ for all $\lambda > 0$;
- (iii) If a convergent sequence of A always has a limit that belongs to A , then A is a closed subset of Ξ_ω ;
- (iv) To be considered complete, a subset A of Ξ_ω must have a convergent Cauchy sequence in A with a limit in A .

Let us define the notions of α -admissibility and triangular α -admissibility, which are going to be used in formulating the contractive inequalities which feature in the fixed point results.

DEFINITION 2.4. [16] Let us consider $\Xi \neq \emptyset$, and the mapping $\alpha : \Xi \times \Xi \rightarrow [0, \infty)$. An operator $\Lambda : \Xi \rightarrow \Xi$ is α -admissible if $\alpha(\xi, \zeta) \geq 1$ implies $\alpha(\Lambda\xi, \Lambda\zeta) \geq 1$, for all $\xi, \zeta \in \Xi$.

DEFINITION 2.5. [27] Let us consider $\Xi \neq \emptyset$, and the mapping $\alpha : \Xi \times \Xi \rightarrow [0, \infty)$. We say that $\Lambda : \Xi \rightarrow \Xi$ is triangular α -admissible if $\alpha(\xi, \zeta) \geq 1$ implies $\alpha(\Lambda\xi, \Lambda\zeta) \geq 1$, and $\alpha(\xi, \zeta) \geq 1$, $\alpha(\zeta, z) \geq 1$ imply $\alpha(\xi, z) \geq 1$ for all $\xi, \zeta, z \in \Xi$.

DEFINITION 2.6. [25] Let $\Lambda : \Xi \rightarrow \Xi$ be a self mapping. For $\epsilon > 0$, we say that $\xi \in \Xi$ is an ϵ -fixed point of Λ if $D(\xi, \Lambda\xi) < \epsilon$. As a notation, $F_\epsilon(\Lambda)$ is the set of all ϵ -fixed points of Λ . Moreover, it is said that Λ has the approximate fixed point property if Λ has an ϵ -fixed point for all $\epsilon > 0$.

LEMMA 2.7. [26, 1.2] Suppose that for a triangular α -orbital admissible mapping $\Lambda : \Xi \rightarrow \Xi$, $\exists \xi_0 \in \Xi$ s.t. $\alpha(\xi_0, \Lambda\xi_0) \geq 1$. Then $\alpha(\xi_j, \xi_k) \geq 1$, $\forall j, k \in \mathbb{N}$, where the sequence $\{\xi_j\}$ is defined by $\xi_{j+1} = \Lambda\xi_j$, $j \in \mathbb{N}$.

In order to formulate and prove theorems in this paper, we consider the following notations:

$$\delta_{j_0}(\omega_\lambda, \Omega, \xi_0) = \sup\{\omega_\lambda(\Omega^j \xi_0, \Omega^k \xi_0) : j, k \in \mathbb{N}, j, k \geq j_0\},$$

where $j_0 \in \mathbb{N}$, and

$$\delta(\omega_\lambda, \Omega, \xi_0) = \sup\{\omega_\lambda(\Omega^j \xi_0, \Omega^k \xi_0) : j, k \in \mathbb{N}\}.$$

Let us consider the orbit of an element ξ_0 by an operator $\Omega : \Xi_\omega \rightarrow \Xi_\omega$ using the symbol $O_\Omega(\xi_0) = \{\Omega^j \xi_0 : j \in \mathbb{N}\}$.

3. Using modular metrics to map fixed points of contraction in spaces

This section has shown the presence and singularity of a stationary point with a modular metric in spaces for contraction mapping.

THEOREM 3.1. Suppose $(\Xi_\omega, \omega_\lambda)$ is a modular metric space, $\Omega : \Xi_\omega \rightarrow \Xi_\omega$ a self-map and $\alpha : \Xi_\omega \times \Xi_\omega \rightarrow [0, \infty)$ a given mapping. Assume that the following requirements are satisfied:

- (i) Ω is a α -admissible mapping;
- (ii) there is $\xi_0 \in \Xi_\omega$ with $\delta(\omega, \Omega, \xi_0) < \infty$, such that $\alpha(\xi_0, \Omega \xi_0) \geq 1$;
- (iii) there are $\eta_1, \eta_2 \in [0, 1)$ with $\eta_1 + \eta_2 < 1$, such that

$$\alpha(\xi, \zeta)\omega_\lambda(\Omega^2 \xi, \Omega^2 \zeta) \leq \eta_1 \omega_\lambda(\Omega \xi, \Omega \zeta) + \eta_2 \omega_\lambda(\zeta, \Omega \zeta),$$

for all $\xi, \zeta \in O_\Omega(\xi_0)$, $\lambda > 0$.

Then the mapping Ω has an approximate fixed point property.

Proof. Let $\xi_0 \in \Xi_\omega$ be a point such that (ii) is satisfied. Then, by Property (ii) $\alpha(\xi_0, \Omega \xi_0) \geq 1$. Again, for $\xi_1 \in \Xi_\omega$, we have $\alpha(\xi_1, \Omega \xi_1) \geq 1$. In general, $\alpha(\xi_j, \Omega \xi_j) \geq 1$, for all $j \in \mathbb{N}$. Now, using Property (iii), we have

$$\begin{aligned} \omega_\lambda(\xi_2, \xi_3) &= \omega_\lambda(\Omega^2 \xi_0, \Omega^2 \xi_1) \leq \alpha(\xi_0, \xi_1)\omega_\lambda(\Omega^2 \xi_0, \Omega^2 \xi_1) \\ &\leq \eta_1 \omega_\lambda(\Omega \xi_0, \Omega \xi_1) + \eta_2 \omega_\lambda(\xi_1, \Omega \xi_1) \\ &= \eta_1 \omega_\lambda(\xi_1, \xi_2) + \eta_2 \omega_\lambda(\xi_1, \xi_2) \\ &= (\eta_1 + \eta_2)\omega_\lambda(\xi_1, \xi_2) = r\omega_\lambda(\xi_1, \xi_2), \text{ where } r = \eta_1 + \eta_2 < 1. \end{aligned}$$

Again,

$$\begin{aligned} \omega_\lambda(\xi_3, \xi_4) &= \omega_\lambda(\Omega^2 \xi_1, \Omega^2 \xi_2) \leq \alpha(\xi_1, \xi_2)\omega_\lambda(\Omega^2 \xi_1, \Omega^2 \xi_2) \\ &\leq \eta_1 \omega_\lambda(\Omega \xi_1, \Omega \xi_2) + \eta_2 \omega_\lambda(\xi_2, \Omega \xi_2) \\ &= \eta_1 \omega_\lambda(\xi_2, \xi_3) + \eta_2 \omega_\lambda(\xi_2, \xi_3) = (\eta_1 + \eta_2)\omega_\lambda(\xi_2, \xi_3) = r\omega_\lambda(\xi_2, \xi_3) \\ &\leq r[\omega_\lambda(\xi_1, \xi_2)] = r^2 \omega_\lambda(\xi_1, \xi_2). \end{aligned}$$

Inductively,

$$(1) \quad \omega_\lambda(\xi_j, \xi_{j+1}) \leq r^{j-1} \omega_\lambda(\xi_1, \xi_2), \text{ where } j \geq 2.$$

So, by taking limit as $j \rightarrow \infty$ in (1) gives $\lim_{j \rightarrow \infty} \omega_\lambda(\xi_j, \xi_{j+1}) = 0$. According to this definition, there exists $j_\epsilon \in \mathbb{N}$ such that for every $\epsilon > 0$,

$$\omega_\lambda(\Omega^j \xi_0, \Omega^{j+1} \xi_0) = \omega_\lambda(\Omega^j \xi_0, \Omega \Omega^j \xi_0) < \epsilon, \text{ for all } j \geq j_\epsilon.$$

Let $u = \Omega^{j_\epsilon} \xi_0$, so for every $\epsilon > 0$, we observe that $u \in O_\Omega(\xi_0)$ exists. In this way, $\omega_\lambda(u, \Omega u) < \epsilon$. $\square$

THEOREM 3.2. Suppose $(\Xi_\omega, \omega_\lambda)$ is a complete modular metric space, $\Omega : \Xi_\omega \longrightarrow \Xi_\omega$ a self-map and $\alpha : \Xi_\omega \times \Xi_\omega \longrightarrow [0, \infty)$ a given map. Assume that the following requirements are satisfied:

- (i) Ω is a triangular α -admissible mapping;
- (ii) there is $\xi_0 \in \Xi_\omega$ with $\delta(\omega, \Omega, \xi_0) < \infty$, such that $\alpha(\xi_0, \Omega\xi_0) \geq 1$;
- (iii) there are $\eta_1, \eta_2 \in [0, 1)$ with $\eta_1 + \eta_2 < 1$, such that

$$\alpha(\xi, \zeta)\omega_\lambda(\Omega^2\xi, \Omega^2\zeta) \leq \eta_1\omega_\lambda(\Omega\xi, \Omega\zeta) + \eta_2\omega_\lambda(\zeta, \Omega\zeta),$$

for all $\xi, \zeta \in O_\Omega(\xi_0)$, $\lambda > 0$;

- (iv) Ω is continuous;
- (v) for all $\xi, \zeta \in \Xi_\omega$, there exists $z \in \Xi_\omega$ such that $\alpha(\xi, z) \geq 1$ and $\alpha(\zeta, z) \geq 1$.

After that, the sequence $\{\Omega^j\xi_0\}$ converges to a point $\xi \in \Xi_\omega$. Additionally, ξ represents a unique fixed point of Ω .

Proof. Let $\xi_0 \in \Xi_\omega$ be arbitrary point. Then as in the previous result, the sequence $\{\xi_j\}$ converges to the point $\xi_0 \in \Xi_\omega$. Since Ω is triangular α -admissible, then by Lemma 2.7, it follows that $\alpha(\xi_j, \xi_k) \geq 1$, for all $j, k \in \mathbb{N}$, $k > j$. Also, from the previous result, for every $\epsilon, \lambda > 0$, there exists $j_\epsilon \in \mathbb{N}$ such that $\omega_\lambda(\xi_{j+2}, \xi_{j+3}) < \epsilon$, for all $j \geq j_\epsilon$. Consequently, we want show that the sequence $\{\xi_j\}$ is Cauchy. Using the triangle inequality suppose that $k, j \in \mathbb{N}$ and $k > j$. Observe that for $\frac{\lambda}{k-j} > 0$, there exists $n_{\frac{\lambda}{k-j}} \in \mathbb{N}$ such that $\omega_{\frac{\lambda}{k-j}}(\xi_j, \xi_{j+1}) < \frac{\epsilon}{k-j}$ for all $n \geq n_{\frac{\lambda}{k-j}}$. Now, we have

$$\begin{aligned} \omega_\lambda(\xi_j, \xi_k) &\leq \omega_{\frac{\lambda}{k-j}}(\xi_j, \xi_{j+1}) + \omega_{\frac{\lambda}{k-j}}(\xi_{j+1}, \xi_{j+2}) + \dots + \omega_{\frac{\lambda}{k-j}}(\xi_{k-1}, \xi_k) \\ &< \frac{\epsilon}{k-j} + \frac{\epsilon}{k-j} + \dots + \frac{\epsilon}{k-j} \\ &= \epsilon, \text{ for all } j, k \geq n_{\frac{\lambda}{k-j}}. \end{aligned}$$

This implies that the sequence $\{\Omega^j\xi_0\}$ is Cauchy. There is a point $\xi \in \Xi$ such that $\xi_j \rightarrow \xi$ as $j \rightarrow \infty$ according to the completeness of $(\Xi_\omega, \omega_\lambda)$. Also, since Ω is continuous it is clear that $\lim_{j \rightarrow \infty} \omega_\lambda(\Omega\xi_j, \Omega\xi) = 0$, for all $\lambda > 0$. Hence, $\Omega\xi = \xi$. Therefore, ξ is a stationary point of Ω . Consider ξ and ζ as two fixed points of Ω for uniqueness. Using Property (v), there exists $z \in \Xi_\omega$ such that

$$(2) \quad \alpha(\xi, z) \geq 1 \text{ and } \alpha(\zeta, z) \geq 1.$$

Since Ω is α -admissible, from (2) we get

$$(3) \quad \alpha(\xi, \Omega^n z) \geq 1 \text{ and } \alpha(\zeta, \Omega^n z) \geq 1, \text{ for all } n \in \mathbb{N}$$

Using condition (iii) and (3) we have

$$\begin{aligned} \omega_\lambda(\xi, \Omega^{n+2}z) &= \omega_\lambda(\Omega^2\xi, \Omega^2(\Omega^n z)) \leq \alpha(\xi, \Omega^n z)\omega_\lambda(\Omega^2\xi, \Omega^2(\Omega^n z)) \\ &\leq \eta_1\omega_\lambda(\Omega\xi, \Omega(\Omega^n z)) + \eta_2\omega_\lambda(\Omega^n z, \Omega(\Omega^n z)) \\ (4) \quad &= \eta_1\omega_\lambda(\xi, \Omega^{n+1}z) + \eta_2\omega_\lambda(\Omega^n z, \Omega^{n+1}z). \end{aligned}$$

Then, letting $n \longrightarrow +\infty$, we have

$$(5) \quad \Omega^n z \longrightarrow \xi.$$

Similarly, using inequality (3) and condition (iii), we get

$$(6) \quad \Omega^n z \longrightarrow \zeta \text{ as } n \longrightarrow +\infty.$$

Using (5) and (6), the uniqueness of the limit gives us $\xi = \zeta$. Thus, ξ is a unique fixed point of Ω . $\square$

An example is constructed in this section to bolster the assumptions of Theorem 3.2.

EXAMPLE 3.3. Let $\Xi_\omega = [0, 1]$ endowed with the metric $\omega_\lambda : (0, \infty) \times \Xi_\omega \times \Xi_\omega \rightarrow [0, \infty]$ be defined by,

$$\omega_\lambda(\xi, \zeta) = \begin{cases} \frac{|\xi - \zeta|}{\lambda} & \text{if } \xi \neq \zeta, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\xi, \zeta \in \Xi_\omega$ and $\lambda > 0$. Clearly, $(\Xi_\omega, \omega_\lambda)$ is a complete modular metric space. Choose

$$\alpha(\xi, \zeta) = \begin{cases} 1 & \text{if } \xi, \zeta \in [0, \infty], \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Define $\Omega : \Xi_\omega \rightarrow \Xi_\omega$ as $\Omega\xi = \frac{\xi}{2}$ for all $\xi \in \Xi_\omega$ and take $\eta_1 = \frac{1}{2}$, $\eta_2 = 0$. Then $\eta_1 + \eta_2 < 1$. Furthermore,

$$\begin{aligned} \omega_\lambda(\Omega^2\xi, \Omega^2\zeta) &= \frac{|\Omega^2\xi - \Omega^2\zeta|}{\lambda} = \frac{|\xi - \zeta|}{4\lambda} \\ &\leq \frac{|\xi - \zeta|}{2\lambda} = \eta_1\omega_\lambda(\Omega\xi, \Omega\zeta) + \eta_2\omega_\lambda(\zeta, \Omega\zeta). \end{aligned}$$

This indicates that Ω meets every requirement of Theorem 3.2. Because of this, Ω possesses a distinct fixed point $\xi = 0$. The graph of the mapping in Example 3.3 is shown in Figure 1. From Figure 1, we see that:

- (i) Every mapping moves its value near zero, the fixed point.
- (ii) No other element in $[0, 1]$ fulfils $\Omega(\xi) = \xi$, except $\xi = 0$.
- (iii) The successive images $\Omega(\xi), \Omega(\xi)^2, \dots$, shows convergence to the fixed point.

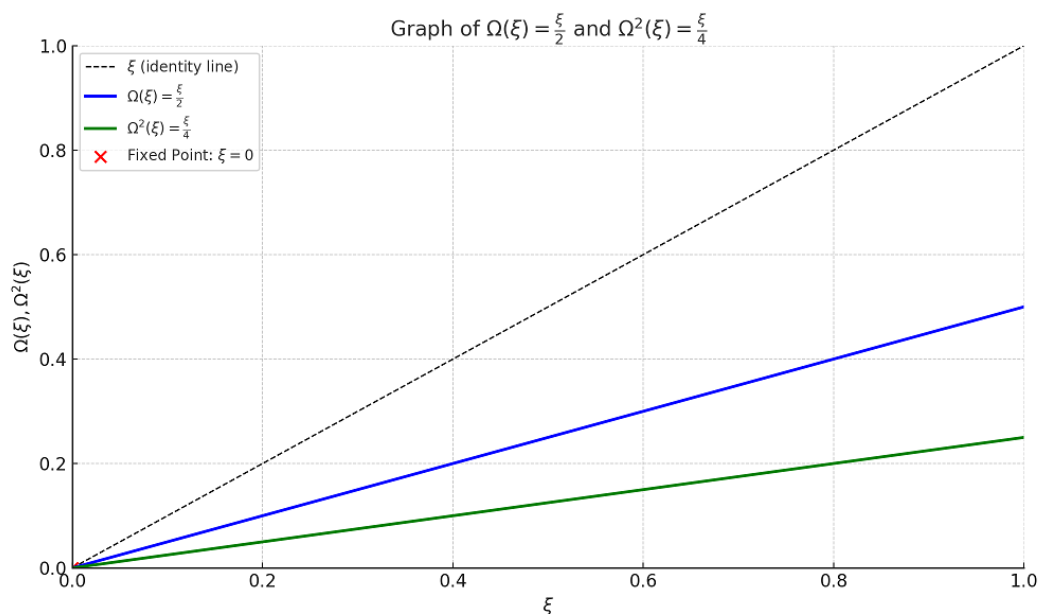


FIGURE 1. Graphical illustration of Example 3.3

EXAMPLE 3.4. Let $\Xi_\omega = [0, 1]$ endowed with the metric $\omega_\lambda : \Xi_\omega \times \Xi_\omega \rightarrow [0, \infty]$ defined by

$$\omega_\lambda(\xi, \zeta) = \begin{cases} \frac{|\xi - \zeta|}{\lambda} & \text{if } \xi = \zeta, \\ 0 & \text{otherwise.} \end{cases}$$

For all $\xi, \zeta \in \Xi_\omega$ with $\xi \leq \zeta$ and $\lambda > 0$. Clearly, $(\Xi_\omega, \omega_\lambda)$ is a complete modular metric space. Choose

$$\alpha(\xi, \zeta) = \begin{cases} 1 & \text{if } \xi, \zeta \in [0, \infty], \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Define $\Omega : \Xi_\omega \rightarrow \Xi_\omega$ as $\Omega\xi = \frac{\xi}{3}$ for all $\xi \in \Xi_\omega$ and take $\eta_1 = \eta_2 = \frac{1}{6}$. Then $\eta_1 + \eta_2 < 1$. Furthermore,

$$\begin{aligned} \omega_\lambda(\Omega^2\xi, \Omega^2\zeta) &= \frac{|\Omega^2\xi - \Omega^2\zeta|}{\lambda} = \frac{|\xi - \zeta|}{9\lambda} = \frac{|\xi - \zeta|}{18\lambda} + \frac{|\xi - \zeta|}{18\lambda} = \frac{|\frac{\xi}{3} - \frac{\zeta}{3}|}{6\lambda} + \frac{|\frac{\xi}{3} - \frac{\zeta}{3}|}{6\lambda} \\ &\leq \frac{|\frac{\xi}{3} - \frac{\zeta}{3}|}{6\lambda} + \frac{|\zeta - \frac{\zeta}{3}|}{6\lambda} \\ &\leq \frac{1}{6}\omega_\lambda(\Omega\xi, \Omega\zeta) + \frac{1}{6}\omega_\lambda(\zeta, \Omega\zeta) \\ &\leq \eta_1\omega_\lambda(\Omega\xi, \Omega\zeta) + \eta_2\omega_\lambda(\zeta, \Omega\zeta). \end{aligned}$$

This indicates that Ω meets every requirement of Theorem 3.2. Because of this, Ω has a unique fixed point $\xi = 0$.

4. Applications to the solution of certain integral equations

First, we apply the results to determine whether the solution to an integral equation exists and is unique. Specifically, we apply the function of Theorem 3.2 to the integral equation for an unknown ξ :

$$(7) \quad \xi(\tau) = h(\tau) + \int_n^m F(\tau, s)\Lambda(s, \xi(\tau))ds, \tau \in [n, m],$$

$F : [n, m] \times [n, m] \rightarrow [0, \infty)$, $h : [n, m] \rightarrow \mathbb{R}$, and $\Lambda : [n, m] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. Let $\Xi_\omega = C([n, m], \mathbb{R})$ and $\omega_\lambda : (0, \infty) \times \Xi_\omega \times \Xi_\omega \rightarrow [0, \infty]$ be defined by

$$\omega_\lambda(\xi, \zeta) = \max_{m \leq \tau \leq n} \frac{|\xi(\tau) - \zeta(\tau)|}{\lambda},$$

with $\lambda > 0$. Clearly, $(\Xi_\omega, \omega_\lambda)$ is a complete modular metric space. Let $\Omega : \Xi_\omega \rightarrow \Xi_\omega$ be defined by

$$(8) \quad \Omega\xi(\tau) = h(\tau) + \int_n^m F(\tau, s)\Lambda(s, \xi(\tau))ds, \tau \in [n, m].$$

Then, a fixed point ξ of Ω corresponds to a solution of equation (7).

THEOREM 4.1. *Presume the following conditions are satisfied:*

$$(i) \quad \max_{m \leq \tau \leq n} \int_n^m |F(\tau, s)|ds \leq \frac{1}{m - n};$$

(ii) There exists a function $\rho : [m, n] \rightarrow (0, 1]$ satisfying $\max_{\tau \in [m, n]} \rho(\tau) \leq \frac{1}{2}$ such that $\forall$

$$\xi, \zeta \in \Xi_\omega, |\Lambda(s, \xi) - \Lambda(s, \zeta)| \leq \rho(\tau) |\xi(\tau) - \zeta(\tau)|.$$

The solution to the integral equation 7 is then contained in Ξ_ω .

Proof. Note that for any $\xi, \zeta \in \Xi_\omega$,

$$\begin{aligned} \omega_\lambda(\Omega\xi, \Omega\zeta) &= \max_{m \leq \tau \leq n} \frac{|\Omega\xi(\tau) - \Omega\zeta(\tau)|}{\lambda} \\ &= \frac{1}{\lambda} \max_{m \leq \tau \leq n} \left\{ \left| [h(\tau) + \int_n^m F(\tau, s)\Lambda(s, \xi(\tau))ds] - [h(\tau) + \int_n^m F(\tau, s)\Lambda(s, \zeta(\tau))ds] \right| \right\} \\ &= \frac{1}{\lambda} \max_{m \leq \tau \leq n} \left\{ \left| \int_n^m F(\tau, s)\Lambda(s, \xi(\tau))ds - \int_n^m F(\tau, s)\Lambda(s, \zeta(\tau))ds \right| \right\} \\ &= \frac{1}{\lambda} \max_{m \leq \tau \leq n} \left\{ \left| \int_n^m F(\tau, s)\rho(s)[\Lambda(s, \xi(\tau)) - \Lambda(s, \zeta(\tau))]ds \right| \right\} \\ &\leq \frac{1}{\lambda} \left\{ \max_{m \leq \tau \leq n} \int_n^m |F(\tau, s)|ds \right\} \left\{ \max_{m \leq \tau \leq n} \rho(\tau) \right\} \left\{ \max_{m \leq \tau \leq n} \int_n^m |\Lambda(s, \xi(\tau)) - \Lambda(s, \zeta(\tau))|ds \right\} \\ &\leq \frac{1}{\lambda} \frac{1}{m-n} \int_n^m \frac{1}{2} \max_{m \leq \tau \leq n} |\xi(s) - \zeta(s)|ds \\ &\leq \frac{1}{2\lambda(m-n)} \int_n^m \max_{m \leq \tau \leq n} |\xi(\tau) - \zeta(\tau)|ds \\ (9) &= \frac{1}{2(m-n)} \int_n^m \max_{m \leq \tau \leq n} \frac{|\xi(\tau) - \zeta(\tau)|}{\lambda} ds = \frac{1}{2} \max_{m \leq \tau \leq n} \frac{|\xi(\tau) - \zeta(\tau)|}{\lambda}. \end{aligned}$$

Using equation (9), gives

$$\begin{aligned} \omega_\lambda(\Omega^2\xi, \Omega^2\zeta) &= \max_{m \leq \tau \leq n} \frac{|\Omega^2\xi(\tau) - \Omega^2\zeta(\tau)|}{\lambda} \\ &= \frac{1}{\lambda} \max_{m \leq \tau \leq n} \left\{ \left| [h(\tau) + \int_n^m F(\tau, s)\Lambda(s, \Omega\xi(\tau))ds] - [h(\tau) + \int_n^m F(\tau, s)\Lambda(s, \Omega\zeta(\tau))ds] \right| \right\} \\ &= \frac{1}{\lambda} \max_{m \leq \tau \leq n} \left\{ \left| \int_n^m F(\tau, s)\Lambda(s, \Omega\xi(\tau))ds - \int_n^m F(\tau, s)\Lambda(s, \Omega\zeta(\tau))ds \right| \right\} \\ &= \frac{1}{\lambda} \max_{m \leq \tau \leq n} \left\{ \left| \int_n^m F(\tau, s)\rho(s)[\Lambda(s, \Omega\xi(\tau)) - \Lambda(s, \Omega\zeta(\tau))]ds \right| \right\} \\ &\leq \frac{1}{\lambda} \left\{ \max_{m \leq \tau \leq n} \int_n^m |F(\tau, s)|ds \right\} \left\{ \max_{m \leq \tau \leq n} \rho(\tau) \right\} \left\{ \int_n^m |\Lambda(s, \Omega\xi(\tau)) - \Lambda(s, \Omega\zeta(\tau))|ds \right\} \\ &\leq \frac{1}{\lambda} \frac{1}{m-n} \int_n^m \frac{1}{2} \max_{m \leq \tau \leq n} |\Omega\xi(s) - \Omega\zeta(s)|ds \\ &\leq \frac{1}{2\lambda(m-n)} \int_n^m \max_{m \leq \tau \leq n} |\Omega\xi(\tau) - \Omega\zeta(\tau)|ds \\ &= \frac{1}{2(m-n)} \int_n^m \max_{m \leq \tau \leq n} \frac{|\Omega\xi(\tau) - \Omega\zeta(\tau)|}{\lambda} ds \\ &= \frac{1}{2} \max_{m \leq \tau \leq n} \frac{|\Omega\xi(\tau) - \Omega\zeta(\tau)|}{\lambda} = \frac{1}{2} \omega_\lambda(\Omega\xi, \Omega\zeta) \\ &\leq \eta_1 \omega_\lambda(\Omega\xi, \Omega\zeta) + \eta_2 \omega_\lambda(\zeta, \Omega\zeta), \end{aligned}$$

where $\eta_1 = \frac{1}{2}, \eta_2 = 0$. Then $\eta_1 + \eta_2 < 1$. Therefore, Theorem 3.2's hypotheses are all met. The unique fixed point $\xi \in \Xi_\omega$ of Ω is thus determined. This suggests that there is only one solution to the integral equation 7 in Ξ_ω . $\square$

5. Conclusion

This study has established forms of contractions and accompanying Theorems of fixed points in relation to modular metric spaces. Theorem 3.1 addressed the prerequisites for the presence of an approximate fixed point, Theorem 3.2 demonstrated the fixed point's existence and uniqueness for the given operator. The case 3.3 supports the hypothesis of the main finding. Furthermore, the feasibility and justification for existences of a solution to the class of integral equations were investigated by Theorem 4.1. It has been noted that the concept in these study enhances several related findings in the literature.

References

- [1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1) (1922), 133–181.
- [2] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **60** (1968), 71–76.
- [3] M. S. Shagari, R. Chiroma, T. Alotaibi, M. Noorwali, A. Saliu, and A. Azam, *Notes on quasi contractive operators and their applications*, Res. Math. **11** (1) (2024), 2424598.
<https://doi.org/10.1080/27684830.2024.2424598>
- [4] S. Yahaya, M. S. Shagari, and I. A. Fulatan, *Fixed points of bilateral multivalued contractions*, Filomat **38** (8) (2024), 2835–2846.
- [5] H. Nakano, *Modulare Semi-Ordered Linear Spaces*, Tokyo Math. Book Ser., Vol. 1, Maruzen Co., Tokyo (1950).
- [6] S. Koshi and T. Shimogaki, *On F-norms of quasi-modular spaces*, J. Fac. Sci. Hokkaido Univ. Ser. I **15** (3-4) (1961), 202–218.
- [7] S. Yamamuro, *On conjugate spaces of Nakano spaces*, Trans. Amer. Math. Soc. **90** (1959), 291–311.
- [8] V. V. Chistyakov, *Modular metric spaces I: Basic concepts*, Nonlinear Anal. **72** (1) (2010), 1–14.
<https://doi.org/10.1016/j.na.2009.04.057>
- [9] V. V. Chistyakov, *Modular metric spaces II: Application to superposition operators*, Nonlinear Anal. **72** (1) (2010), 15–30.
<https://doi.org/10.1016/j.na.2009.04.018>
- [10] C. Mongkolkeha, W. Sintunavarat, and P. Kumam, *Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory Appl. **2011** (2011), Article ID 93.
<https://doi.org/10.1186/1687-1812-2011-93>
- [11] S. Kumar, L. D. Sharma, A. Chanda, and Z. D. Mitrović, *On some non-linear contractions in modular metric spaces*, Filomat **34** (12) (2020), 3971–3980.
- [12] M. Younis, D. Singh, S. Radenović, and M. Illati, *Convergence theorems for generalized contractions and applications*, Filomat **34** (3) (2020), 945–964.
- [13] Y. Sharma and S. Jain, *Coupled fixed point theorems in modular metric spaces endowed with a graph*, Kyungpook Math. J. **61** (2) (2021), 441–453.
<https://doi.org/10.5666/KMJ.2021.61.2.441>
- [14] F. S. Al-shammari, R. K. P., R. Rajagopalan, and R. George, *Generalised Presic type operators in modular metric space and an application to integral equations of Carathéodory type functions*, J. Math. **2021** (2021), Article ID 7915448, 20 pages.
<https://doi.org/10.1155/2021/7915448>
- [15] D. Das, S. Nandi, Y. M. Singh, M. S. Khan, and S. Sessa, *Fixed point results on partial modular metric space*, Axioms **11** (2) (2022), Article ID 62.
<https://doi.org/10.3390/axioms11020062>
- [16] H. H. Alsulami, S. Chandok, M. A. Taoudi, and I. M. Erhan, *Some fixed point theorems for (α, ψ) -rational type contractive mappings*, Fixed Point Theory Appl. **2015** (2015), Article ID 97.
<https://doi.org/10.1186/s13663-015-0332-3>

- [17] S. G. Talebi, A. Bodaghi, and A. N. Motlagh, *Some results in metric modular spaces*, Int. J. Nonlinear Anal. Appl. **13** (2) (2022), 983–988.
<https://doi.org/10.22075/ijnaa.2022.25676.3091>
- [18] H. Rahimpour, *Some contraction fixed point theorems in partially ordered modular metric spaces*, Int. J. Nonlinear Anal. Appl. **14** (9) (2023), 273–282.
<https://doi.org/10.22075/ijnaa.2021.25264.2971>
- [19] H. Rahimpour and I. Nikoufar, *Some results for cyclic weak contractions in modular metric space*, J. Linear Topol. Algebra **12** (1) (2023), 57–66.
- [20] G. A. Okeke and D. Francis, *Some common fixed point theorems in generalized modular metric spaces with applications*, Sci. Afr. **23** (2024), e02018.
<https://doi.org/10.1016/j.sciaf.2023.e02018>
- [21] D. Păcurar and A. Păcurar, *Convex contractive mappings in generalized metric spaces*, U.P.B. Sci. Bull. Ser. A **86** (2) (2024), 21–30.
- [22] W. A. J. Luxemburg, *Banach Function Spaces*, Ph.D. Thesis, Delft University of Technology, Assen, The Netherlands (1955).
- [23] J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. **18** (1) (1959), 49–65.
- [24] J. Musielak and W. Orlicz, *Some remarks on modular spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **7** (1959), 661–668.
- [25] M. Berinde, *Approximate fixed point theorems*, Stud. Univ. Babeş-Bolyai Math. **51** (1) (2006), 11–25.
- [26] E. Karapınar and A. Fulga, *An admissible hybrid contraction with an Ulam type stability*, Demonstr. Math. **52** (1) (2019), 428–436.
<https://doi.org/10.1515/dema-2019-0037>
- [27] X. Z. Wu and L. Hao, *Fixed point theorems for generalized α - ψ type contractive mappings in b -metric spaces and applications*, J. Math. Comput. Sci. **18** (1) (2018), 49–62.
<https://doi.org/10.22436/jmcs.018.01.06>

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