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DIFFERENTIAL PRE-LIE BIALGEBRAS AND ADMISSIBLE $\mathcal{S}$-EQUATIONS

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Abstract. The aim of this paper is to introduce the notion of differential pre-Lie bialgebra $(A, \delta, \zeta, \Phi, \Psi)$ and their admissibility conditions in terms of dual representations $(l_\circ^* - r_\circ^*, -r_\circ^*, \beta^*, V^*)$. Next, we show that differential pre-Lie bialgebras are characterized by matched pairs and Manin triples of differential pre-Lie algebras. Furthermore, the coboundary case leads to the introduction of the admissible $\mathcal{S}$-equation in differential pre-Lie algebras, whose symmetric solutions are used to construct differential pre-Lie bialgebras.

1. Introduction

For a given algebraic structure determined by a set of products of different arities and a set of relations among the operations, a bialgebra structure on this algebra is obtained by a corresponding set of coproducts together with a set of compatibility conditions between the products and coproducts. A good compatibility condition is prescribed by a rich structure theory and effective constructions. The central importance of bialgebras rests on their connection with other structures arising from mathematics and physics.

The concept of a pre-Lie algebra (also called left-symmetric algebras, quasi-associative algebras, Vinberg algebras and so on) has been introduced by M. Gerstenhaber in deformation theory of rings and algebras [6]. Pre-Lie algebras are appeared in many fields in mathematics and mathematical physics, such as symplectic structures on Lie groups, integrable systems, cohomology and deformations, vertex algebras and quantum field theory. For more details, see [2, 4, 5, 13, 16] and the references therein.

Recall that a pre-Lie algebra is a vector space A with a bilinear operation $\circ$ satisfying the pre-Lie identity

$$(x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z), \tag{1.1}$$

for all $x, y, z \in A$. Moreover, pre-Lie algebras are closely related to Lie algebras, in the sense, if $(A, \circ)$ is a pre-Lie algebra, the bilinear operation

$$[x, y] = x \circ y - y \circ x, \tag{1.2}$$

defines a Lie algebra, which is called the sub-adjacent Lie algebra of $(A, \circ)$ and $(A, \circ)$ is called a compatible pre-Lie algebra structure on $A^c = (A, [\cdot, \cdot])$.

The notion of pre-Lie bialgebra was introduced by Bai in [1]. This structure is equivalent to that of para-Kähler Lie algebra, which is, in fact, a symplectic algebra with a decomposition into a direct sum of two Lagrangian subalgebras. See [3, 7, 9, 10] for

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more details about applications in mathematical physics. The main importance of pre-Lie bialgebras is reflected by its close relationship with many other fundamental notions. First pre-Lie bialgebras are characterized by matched pairs and Manin triples of pre-Lie algebras. In fact, there is a one-one correspondence between Manin triples of pre-Lie algebras and pre-Lie bialgebras. The same holds for pre-Lie bialgebras and matched pairs of pre-Lie algebras associated to regular representations. The notion of coboundary pre-Lie bialgebra guides to an analogue of the classical Yang-Baxter equation (CYBE), called a '$\mathcal{S}$-equation'. A symmetric solution of this equation gives a pre-Lie bialgebra.

Algebraic systems with derivations and their generalizations are a popular object of study nowadays. In particular, the algebras of derivations and generalized derivations are important in the study of algebraic systems of Lie type. Differential of algebras have been intensively investigated by many authors, for more details about differential algebras in such approach can be found in [8, 11, 12, 14, 18–21]. Well, a linear map $\Phi : A \to A$ is called derivation on a pre-Lie algebra $(A, \circ)$ if the Leibniz rule is satisfied, i.e.,

$$\Phi(x \circ y) = \Phi(x) \circ y + x \circ \Phi(y), \quad \forall x, y \in A. \tag{1.3}$$

In this paper and given the joint roles played by bialgebras and derivations and their applications, it is possible to study their composed structures. In this direction, some progresses have been made, so there has been quite much study equipping free differential algebras with a bialgebra, for more details see [17]. In the following diagram, the horizontal arrows are equipping a given structure with suitable derivations and the vertical arrows are equipping a given structure by a suitable bialgebra structure.

$$\begin{tikzcd} \text{pre-Lie algebras} \arrow[r, "\text{derivatization}"] \arrow[d, "\text{bialgebraization}"'] & \begin{array}{c}\text{differential}\\ \text{pre-Lie algebras}\end{array} \arrow[d, "\text{bialgebraization}"] \\ \text{pre-Lie bialgebras} \arrow[r, "\text{derivatization}"'] & \begin{array}{c}\text{differential}\\ \text{pre-Lie bialgebras}\end{array} \end{tikzcd} \tag{1.4}$$

The paper is organized as follows. In Section 2, we introduce the definition of an admissible operator Λ (Definition 2.17), which is a result of Lemma 2.16. This Lemma describes the following conditions: if $(l_\circ, r_\circ, V)$ is a representation of the pre-Lie algebra $(A, \circ)$ and if $\Lambda : V \to V$ be a linear map, then the quadruple $(l_\circ^* - r_\circ^*, -r_\circ^*, \Lambda^*, V^*)$ is a representation of $(A, \circ, \Phi)$ if and only if the linear operator Λ satisfies the equations:

$$l_\circ(x)\Lambda(v) - l_\circ(\Phi(x))v - \Lambda(l_\circ(x)v) = 0,$$
$$r_\circ(x)\Lambda(v) - r_\circ(\Phi(x))v - \Lambda(r_\circ(x)v) = 0.$$

Then, we conclude that there is a tight correlation between a linear operator admissible Ψ and some compatibilities condition (Corollary 2.18). From here, we were able to find another definition of differential pre-Lie bialgebras $(A, \delta, \zeta, \Phi, \Psi)$ as long as Φ and Ψ^* satisfy certain admissibility condition (Proposition 2.19). Indeed, in Theorem 2.22, we consider that admissible quadruples $(A \ltimes_{l,r} V, \Phi + \Theta)$ and $(A \ltimes_{l^*-r^*,-r^*} V^*, \Phi + \Lambda^*)$ for semi-direct products of differential pre-Lie algebras and we study the equivalence between the proprieties 1, 2 and 3.

In section 3, we give the general notion of matched pair of differential pre-Lie algebras before specializing to the case when the underlying linear spaces of the two differential algebras are dual to each other. Furthermore, we consider the dual representation of the regular representation and we provide an equivalent between matched pair of differential pre-Lie algebras $(A, A^*, \Phi, \Psi^*, L_\circ^* - R_\circ^*, -R_\circ^*, L_\bullet^* - R_\bullet^*, -R_\bullet^*)$ and differential pre-Lie bialgebras $(A, \delta, \zeta, \Phi, \Psi)$ (Theorem 3.5). Another interesting phenomenon in our construction is to give the general notion of a standard Manin triple of differential pre-Lie algebra

$(A \bowtie A^*, A, A^*, \Phi + \Psi^*, B_s)$. Thus provides another equivalent condition of differential pre-Lie bialgebras (Theorem 3.18). Explicitly, differential pre-Lie bialgebras are characterized by generalizing matched pairs and Manin triples of differential pre-Lie algebras (Theorem 3.19).

In section 4, we study the coboundary differential pre-Lie bialgebras, leading to the notion of admissible $\mathcal{S}$-equation in differential pre-Lie algebras (Definition 4.6), whose symmetric solutions can be used to construct differential pre-Lie bialgebras (Proposition 4.7).

Throughout this paper $\mathbb{K}$ is a field of characteristic 0. All vector spaces, tensor products, and linear homomorphisms are over $\mathbb{K}$.

2. Differential pre-Lie bialgebras structures and their admissibility conditions

In this section, we recall the main definitions and facts related to pre-Lie algebras and pre-Lie bialgebras. Next, we introduce the notion of differential pre-Lie bialgebras and the notion of an admissible quadruple of a differential pre-Lie algebra. Their related relevant properties are also given.

2.1. Basic definitions.

Definition 2.1. A Lie algebra is a pair $(A, [\cdot,\cdot])$ consisting of a linear space A over a field $\mathbb{K}$ and a bilinear map $[\cdot,\cdot]$: $A \otimes A \to A$ satisfying for all $x, y, z \in A$,

$$[x,y] = -[y,x], \qquad \text{(Skew-symmetry identity)} \tag{2.1}$$

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0. \qquad \text{(Jacobi identity)} \tag{2.2}$$

Definition 2.2. Let $(A, [\cdot,\cdot])$ be a Lie algebra and V be a vector space. Let $\rho : A \to End(V)$ be a linear map. The pair (ρ, V) is called a representation of $(A, [\cdot,\cdot])$, if for all $x, y \in A$:

$$\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x). \tag{2.3}$$

Note that, if $(A, [\cdot,\cdot])$ is a Lie algebra. For any two representations (ρ_1, V_1) and (ρ_2, V_2) of A, it is easy to know that $(\rho_1 \otimes \mathrm{id} + \mathrm{id} \otimes \rho_2, V_1 \otimes V_2)$ is also a representation of $(A, [\cdot,\cdot])$, where for any $x \in A$, $v_1 \in V_1$ and $v_2 \in V_2$,

$$(\rho_1 \otimes \mathrm{id} + \mathrm{id} \otimes \rho_2)(x)(v_1 \otimes v_2) = \rho_1(x)v_1 \otimes v_2 + v_1 \otimes \rho_2(x)v_2. \tag{2.4}$$

Definition 2.3. [1] Let $(A, \circ)$ be a pre-Lie algebra and V be a vector space. Let $l_\circ, r_\circ : A \to End(V)$ be two linear maps. The triple $(l_\circ, r_\circ, V)$ is called a representation of A if for all $x, y \in A$:

$$l_\circ(x \circ y) - l_\circ(x)l_\circ(y) = l_\circ(y \circ x) - l_\circ(y)l_\circ(x), \tag{2.5}$$

$$r_\circ(y)l_\circ(x) - l_\circ(x)r_\circ(y) = r_\circ(y)r_\circ(x) - r_\circ(x \circ y). \tag{2.6}$$

For a vector space V and linear maps $l_\circ, r_\circ : A \to End(V)$, the triple $(l_\circ, r_\circ, V)$ is a representation of a pre-Lie algebra $(A, \circ)$ if and only if the direct sum of linear spaces, $A \oplus V$, is turned into a pre-Lie algebra by defining multiplication in $A \oplus V$ for all $x_1, x_2 \in A, v_1, v_2 \in V$ by

$$(x_1 + v_1) \circ' (x_2 + v_2) = x_1 \circ x_2 + (l_\circ(x_1)v_2 + r_\circ(x_2)v_1). \tag{2.7}$$

We denote such a pre-Lie algebra by $(A \oplus V, \circ')$, or $A \ltimes_{l_\circ, r_\circ} V$ and called it the semi-direct product of A by V.

Remark 2.4. [1] Recall that, if $(l_\circ, r_\circ, V)$ be a representation of a pre-Lie algebra $(A, \circ)$. Then:

1. $(l_\circ, V)$ and $(\rho = l_\circ - r_\circ, V)$ are representations of the sub-adjacent Lie algebra $A^c = (A, [\cdot,\cdot])$. In particular $(L_\circ, A)$ and $(\mathrm{ad} = L_\circ - R_\circ, A)$ are representations of the sub-adjacent Lie algebra A^c.
2. $(l_\circ^* - r_\circ^*, -r_\circ^*, V^*)$ is a representation of $(A, \circ)$, where $l_\circ^*, r_\circ^* : A \to End(V^*)$ such that $\forall\, x \in A,\ v \in V,\ u^* \in V^*$:

$$\langle l_\circ^*(x)u^*, v\rangle = -\langle l_\circ(x)v, u^*\rangle, \quad \langle r_\circ^*(x)u^*, v\rangle = -\langle r_\circ(x)v, u^*\rangle. \tag{2.8}$$

In particular, $(L_\circ^* - R_\circ^*, -R_\circ^*, A^*)$ is a representation of $(A, \circ)$.

Let V_1, V_2 be two linear spaces. For a linear map $\Phi : V_1 \to V_2$, we denote by $\Phi^* : V_2^* \to V_1^*$ the transpose map given by

$$\langle v, \Phi^*(u^*)\rangle = \langle \Phi(v), u^*\rangle \text{ for all } v \in V_1, u^* \in V_2^*. \tag{2.9}$$

Definition 2.5. [17] A pre-Lie coalgebra is a pair (A, δ) consisting of a vector space A over a field $\mathbb{K}$ and a linear comultiplication δ: $A \to A \otimes A$ satisfying

$$(\mathrm{id}^{\otimes 3} - \tau_{12})(\delta \otimes \mathrm{id} - \mathrm{id} \otimes \delta)\delta = 0, \tag{2.10}$$

where for any $x, y, z \in A$, $\tau_{12}(x \otimes y \otimes z) = y \otimes x \otimes z$. This amounts to the commutativity of the following diagram:

$$\begin{tikzcd} A \arrow[rr, "\delta_C"] \arrow[dr, "\delta_C"'] & & A \otimes A \otimes A \arrow[dl, "\tau_{12}"] \\ & A \otimes A \otimes A & \end{tikzcd}$$

where $\delta_C = (\delta \otimes \mathrm{id} - \mathrm{id} \otimes \delta)\delta$.

For a Lie algebra $(A, [\cdot,\cdot])$ and a representation (V, ρ), recall that a 1-cocycle Υ associated to (V, ρ) is a linear map from A to V satisfying for all $x, y \in A$:

$$\Upsilon([x, y]) = \rho(x)\Upsilon(y) - \rho(y)\Upsilon(x). \tag{2.11}$$

Definition 2.6. [1] A pre-Lie bialgebra on A is a triple (A, δ, ζ) consists of the following data:

1. A pre-Lie algebra $(A, \circ)$ and a linear map $\zeta : A^* \to A^* \otimes A^*$ such that for all $x, y \in A$, $\eta \in A^*$, we have $\langle \zeta(\eta), x \otimes y\rangle = \langle \eta, x \circ y\rangle$;
2. A pre-Lie algebra $(A^*, \bullet)$ and a linear map $\delta : A \to A \otimes A$ such that for all $x \in A$, $\xi, \eta \in A^*$, we have $\langle \delta(x), \xi \otimes \eta\rangle = \langle x, \xi \bullet \eta\rangle$
satisfying the following properties:
   (a) δ is a 1-cocycle associated to the representation $(A \otimes A; L_\circ \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_{[,]})$ over the sub-adjacent Lie algebra $A^c = (A, [\cdot,\cdot])$;
   (b) ζ is a 1-cocycle associated to the representation $(A^* \otimes A^*; L_\bullet \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_{\{,\}})$ over the sub-adjacent Lie algebra $A^{*c} = (A^*, \{\cdot,\cdot\})$.

Example 2.7. Let $A = \mathrm{span}\{e_1, e_2\}$ over a field $\mathbb{K}$ with

$$e_1 \circ e_1 = 0, \quad e_1 \circ e_2 = e_2, \quad e_2 \circ e_1 = 0, \quad e_2 \circ e_2 = 0,$$

$$\delta(e_1) = 0, \quad \delta(e_2) = e_1 \otimes e_2, \quad \zeta(e_1^*) = 0, \quad \zeta(e_2^*) = e_1^* \otimes e_2^*.$$

Then $(A, \circ, \delta, \zeta)$ is a pre-Lie bialgebra.

2.2. Differential pre-Lie bialgebras in terms of admissible quadruples.

Definition 2.8. A differential pre-Lie algebra is a triple $(A, \circ, \Phi)$, consisting of a pre-Lie algebra $(A, \circ)$ and a derivation Φ on $(A, \circ)$.

Definition 2.9. A differential pre-Lie coalgebra is a triple (A, δ, Ψ) where (A, δ) is a pre-Lie coalgebra and $\Psi : A \to A$ is a linear map such that

$$\delta\Psi = (\Psi \otimes id + id \otimes \Psi)\delta. \tag{2.12}$$

Remark 2.10. For any finite dimensional vector space A and linear maps $\Psi : A \to A$, $\delta : A \to A \otimes A$, if (A, δ, Ψ) is a differential pre-Lie coalgebra, then $(A^*, \delta^* = \bullet, \Psi^*)$ is a differential pre-Lie algebra.

Definition 2.11. A differential pre-Lie bialgebra is a quintuple $(A, \delta, \zeta, \Phi, \Psi)$ consisting of a vector space A and linear maps

$$\delta : A \to A \otimes A, \quad \zeta : A^* \to A^* \otimes A^*, \quad \Phi, \Psi : A \to A$$

such that

1. (A, δ, ζ) is a pre-Lie bialgebra,
2. $(A, \circ = \zeta^*, \Phi)$ is a differential pre-Lie algebra,
3. (A, δ, Ψ) is a differential pre-Lie coalgebra,
4. the following compatibility conditions hold for all $x, y \in A$,

$$x \circ \Psi(y) - \Phi(x) \circ y - \Psi(x \circ y) = 0, \tag{2.13}$$

$$\Psi(x) \circ y - x \circ \Phi(y) - \Psi(x \circ y) = 0, \tag{2.14}$$

$$(id \otimes \Phi)\delta - (\Psi \otimes id)\delta - \delta\Phi = 0, \tag{2.15}$$

$$(\Phi \otimes id)\delta - (id \otimes \Psi)\delta - \delta\Phi = 0. \tag{2.16}$$

We introduce now the notion of representation of differential pre-Lie algebras.

Example 2.12. Consider a two dimensional $\mathbb{K}$-linear space A with basis $\{e_1, e_2\}$. Define

$$e_2 \circ e_1 = e_1, \quad e_2 \circ e_2 = e_2, \quad \text{others } = 0,$$

and linear maps $\Phi, \Psi : A \to A$ by

$$\Phi(e_1) = e_1, \ \Phi(e_2) = 0, \qquad \Psi(e_1) = 0, \ \Psi(e_2) = e_2.$$

Let $r = e_2 \otimes e_2$ and $\delta(x) = (L_\circ(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_{[,]}(x))r$. Then

$$\delta(e_1) = 0, \quad \delta(e_2) = e_2 \otimes e_2, \quad \zeta = \circ^*.$$

Hence $(A, \delta, \zeta, \Phi, \Psi)$ is a differential pre-Lie bialgebra.

Definition 2.13. A representation of a differential pre-Lie algebra $(A, \circ, \Phi)$ is a quadruple $(l_\circ, r_\circ, \Theta, V)$ where $(l_\circ, r_\circ, V)$ is a representation of $(A, \circ)$ and $\Theta : V \to V$ is a linear map such that for all $x \in A, v \in V$:

$$\Theta(l_\circ(x)v) = l_\circ(\Phi(x))v + l_\circ(x)\Theta(v), \tag{2.17}$$

$$\Theta(r_\circ(x)v) = r_\circ(x)\Theta(v) + r_\circ(\Phi(x))v. \tag{2.18}$$

Two representations (l_1, r_1, Θ_1, V) and (l_2, r_2, Θ_2, W) of a differential pre-Lie algebra $(A, \circ, \Phi)$ are called equivalent if there exists an isomorphism $\varphi : V \to W$ such that for any $x \in A, v \in V$

$$\varphi(l_1(x)v) = l_2(x)\varphi(v), \quad \varphi(r_1(x)v) = r_2(x)\varphi(v), \quad \varphi(\Theta_1(v)) = \Theta_2(\varphi(v)). \tag{2.19}$$

The following proposition constructs a differential pre-Lie algebra structure on a semi-direct product associated to a representation.

PROPOSITION 2.14. *Let $(l_\circ, r_\circ, \Theta, V)$ a representation of a differential pre-Lie algebra $(A, \circ, \Phi)$. Then $(A \oplus V, \circ', \Phi + \Theta)$, is turned into a differential pre-Lie algebra by defining the multiplication in $A \oplus V$ for all $x_1, x_2 \in A, v_1, v_2 \in V$ by*

$$(x_1 + v_1) \circ' (x_2 + v_2) \quad := \quad x_1 \circ x_2 + (l_\circ(x_1)v_2 + r_\circ(x_2)v_1), \tag{2.20}$$

$$(\Phi + \Theta)(x_1 + v_1) \quad := \quad \Phi(x_1) + \Theta(v_1). \tag{2.21}$$

Proof. It is clear that $(A, \circ')$ is a pre-Lie algebra. Besides, for any $x_1, x_2 \in A,\ v_1, v_2 \in V$

$$\begin{aligned}
&(\Phi + \Theta)((x_1 + v_1) \circ' (x_2 + v_2)) - (\Phi + \Theta)(x_1 + v_1) \circ' (x_2 + v_2) \\
&- (x_1 + v_1) \circ' (\Phi + \Theta)(x_2 + v_2) \\
&= (\Phi + \Theta)(x_1 \circ x_2 + l_\circ(x_1)v_2 + r_\circ(x_2)v_1) - (\Phi(x_1) + \Theta(v_1)) \circ' (x_2 + v_2) \\
&- (x_1 + v_1) \circ' (\Phi(x_2) + \Theta(v_2)) \\
&= \Phi(x_1 \circ x_2) + \Theta(l_\circ(x_1)v_2) + \Theta(r_\circ(x_2)v_1) - \Phi(x_1) \circ x_2 - l_\circ(\Phi(x_1))v_2 \\
&- r_\circ(x_2)\Theta(v_1) - x_1 \circ \Phi(x_2) - l_\circ(x_1)\Theta(v_2) - r_\circ(\Phi(x_2))v_1 \\
&= \underbrace{\Big(\Phi(x_1 \circ x_2) - \Phi(x_1) \circ x_2 - x_1 \circ \Phi(x_2)\Big)}_{=0 \text{ by } (1.3)} \\
&+ \underbrace{\Big(\Theta(l_\circ(x_1)v_2) - l_\circ(\Phi(x_1))v_2 - l_\circ(x_1)\Theta(v_2)\Big)}_{=0 \text{ by } (2.17)} \\
&+ \underbrace{\Big(\Theta(r_\circ(x_2)v_1) - r_\circ(x_2)\Theta(v_1) - r_\circ(\Phi(x_2))v_1\Big)}_{=0 \text{ by } (2.18)} = 0.
\end{aligned}$$

Hence $\Phi + \Theta$ is a derivation of pre-Lie algebra $(A \oplus V, \circ')$. Then the conclusion holds. □

The resulting differential pre-Lie algebra is denoted by $(A \ltimes_{l_\circ, r_\circ} V, \circ', \Phi + \Theta)$ and called the semi-direct product of $(A, \circ, \Phi)$ by its representation $(l_\circ, r_\circ, \Theta, V)$.

EXAMPLE 2.15. If $(A, \circ, \Phi)$ is a differential pre-Lie algebra, then $(L_\circ, R_\circ, \Phi, A)$ is a representation of differential pre-Lie algebra $(A, \circ, \Phi)$, where $L_\circ(x)y = R_\circ(y)x = x \circ y$ for all $x, y \in A$, called the regular representation of $(A, \circ, \Phi)$.

The following lemma characterizes when the dual representation defines again a representation of a differential pre-Lie algebra.

LEMMA 2.16. *Let $(A, \circ, \Phi)$ be a differential pre-Lie algebra. Let $(l_\circ, r_\circ, V)$ be a representation of the pre-Lie algebra $(A, \circ)$ and $\Lambda : V \to V$ be a linear map. Then the quadruple $(l_\circ^* - r_\circ^*, -r_\circ^*, \Lambda^*, V^*)$ is a representation of $(A, \circ, \Phi)$ if and only if for all $x \in A, v \in V$ the linear operator Λ satisfies*

$$l_\circ(x)\Lambda(v) - l_\circ(\Phi(x))v - \Lambda(l_\circ(x)v) = 0, \tag{2.22}$$

$$r_\circ(x)\Lambda(v) - r_\circ(\Phi(x))v - \Lambda(r_\circ(x)v) = 0. \tag{2.23}$$

Proof. Since $(l_\circ^* - r_\circ^*, -r_\circ^*, V^*)$ is a representation of $(A, \circ)$, we just need to determine when Λ satisfies the equations (2.17) and (2.18) where $\Theta, l_\circ$ and $r_\circ$ are replaced by $\Lambda^*, l_\circ^* - r_\circ^*$ and $-r_\circ^*$ respectively. For all $x \in A, v \in V$ and $u^* \in V^*$ we have

$$\begin{aligned}
&\langle \Lambda^*((l_\circ^* - r_\circ^*)(x)u^*) - (l_\circ^* - r_\circ^*)(\Phi(x))u^* - (l_\circ^* - r_\circ^*)(x)\Lambda^*(u^*), v\rangle \\
\overset{\text{(by (2.9) and (2.8))}}{=} &\langle u^*, -(l_\circ - r_\circ)(x)\Lambda(v) + (l_\circ - r_\circ)(\Phi(x))v + \Lambda((l_\circ - r_\circ)(x)v)\rangle.
\end{aligned}$$

Then (2.17) is equivalent to the difference (2.23)-(2.22). Furthermore, we have

$$\begin{aligned}&\langle \Lambda^*((-r_\circ^*)(x)u^*) - (-r_\circ^*)(x)\Lambda^*(u^*) - (-r_\circ^*)(\Phi(x))u^*, v\rangle\\ \text{(by (2.9) and (2.8))} &= \langle u^*, r_\circ(x)\Lambda(v) - \Lambda(r_\circ(x)v) - r_\circ(\Phi(x))v\rangle.\end{aligned}$$

Then (2.18) is equivalent to (2.23). This finishes the proof. □

DEFINITION 2.17. Let $(A, \circ, \Phi)$ be a differential pre-Lie algebra. Let $(l_\circ, r_\circ, V)$ be a representation of the pre-Lie algebra $(A, \circ)$ and let $\Lambda : V \to V$ be a linear map. If the conditions (2.22) and (2.23) are satisfied, we say that Λ is admissible to the differential pre-Lie algebra $(A, \circ, \Phi)$ on $(l_\circ, r_\circ, V)$. When $(l_\circ, r_\circ, V)$ is taken to be the regular representation $(L_\circ, R_\circ, A)$ of the pre-Lie algebra $(A, \circ)$, we say that Λ is admissible to $(A, \circ, \Phi)$ or simply $(A, \circ, \Phi)$ is Λ-admissible.

The following corollary gives a practical criterion for admissibility in terms of the compatibility equations.

COROLLARY 2.18. *Let $(A, \circ, \Phi)$ be a differential pre-Lie algebra. A linear operator Ψ on A is admissible to $(A, \circ, \Phi)$ if and only if (2.13)-(2.14) hold.*

Proof. We just replace $\Lambda, l_\circ$ and $r_\circ$ by $\Psi, L_\circ$ and $R_\circ$ respectively in equations (2.22) and (2.23). □

Furthermore, the definition of a differential pre-Lie bialgebra can be rephrased as

PROPOSITION 2.19. *The quintuple $(A, \delta, \zeta, \Phi, \Psi)$ is a differential pre-Lie bialgebra if and only if it satisfies the conditions 1-3 in Definition 2.11 and that Ψ and Φ^* are admissible to the differential pre-Lie algebras $(A, \circ, \Phi)$ and $(A^*, \bullet, \Psi^*)$ respectively.*

Proof. By Corollary 2.18, the admissibility of Ψ to the differential pre-Lie algebra $(A, \circ, \Phi)$ determine the equations (2.13) and (2.14). Furthermore, for any $x \in A, u^*, v^* \in A^*$ we have

$$\begin{aligned}&\langle (id \otimes \Phi)\delta x - (\Psi \otimes id)\delta x - \delta\Phi(x), u^* \otimes v^*\rangle\\ &= \langle (id \otimes \Phi)\delta x, u^* \otimes v^*\rangle - \langle (\Psi \otimes id)\delta x, u^* \otimes v^*\rangle - \langle \delta\Phi(x), u^* \otimes v^*\rangle\\ &= \langle x, u^* \bullet \Phi^*(v^*)\rangle - \langle x, \Psi^*(u^*) \bullet v^*\rangle - \langle x, \Phi^*(u^* \bullet v^*)\rangle\\ &= \langle x, u^* \bullet \Phi^*(v^*) - \Psi^*(u^*) \bullet v^* - \Phi^*(u^* \bullet v^*)\rangle.\end{aligned}$$

Therefore, (2.15) leads to

$$u^* \bullet \Phi^*(v^*) - \Psi^*(u^*) \bullet v^* - \Phi^*(u^* \bullet v^*) = 0. \tag{2.24}$$

Similarly, we have

$$\begin{aligned}&\langle (\Phi \otimes id)\delta x - (id \otimes \Psi)\delta x - \delta\Phi(x), u^* \otimes v^*\rangle\\ &= \langle (\Phi \otimes id)\delta x, u^* \otimes v^*\rangle - \langle (id \otimes \Psi)\delta x, u^* \otimes v^*\rangle - \langle \delta\Phi(x), u^* \otimes v^*\rangle\\ &= \langle x, \Phi^*(u^*) \bullet v^*\rangle - \langle x, u^* \bullet \Psi^*(v^*)\rangle - \langle x, \Phi^*(u^* \bullet v^*)\rangle\\ &= \langle x, \Phi^*(u^*) \bullet v^* - u^* \bullet \Psi^*(v^*) - \Phi^*(u^* \bullet v^*)\rangle.\end{aligned}$$

Therefore, (2.16) leads to

$$\Phi^*(u^*) \bullet v^* - u^* \bullet \Psi^*(v^*) - \Phi^*(u^* \bullet v^*) = 0. \tag{2.25}$$

Hence (2.15) and (2.16) means that Φ^* is admissible to the differential pre-Lie algebra $(A^*, \bullet, \Psi^*)$. This ends the proof. □

The next proposition studies how admissibility behaves under scalar multiplication and inversion.

Proposition 2.20. *Let $(A,\circ,\Phi)$ be a differential pre-Lie algebra and $(l_\circ,r_\circ,\Theta,V)$ be a representation of $(A,\circ,\Phi)$ and let $\lambda\in\mathbb{K}$. Then*

1. *$\lambda\Theta$ is admissible to $(A,\circ,\Phi)$ on $(l_\circ,r_\circ,V)$ if and only if for all $x\in A,\ v\in V$,*
$$(1+\lambda)l_\circ(\Phi(x))v=(1+\lambda)r_\circ(\Phi(x))v=0.$$
2. *$-\Theta+\lambda id_V$ is admissible to $(A,\circ,\Phi)$ on $(l_\circ,r_\circ,V)$.*
3. *Suppose that Θ is an automorphism and $\lambda\neq 0$, then $\lambda\Theta^{-1}$ is admissible to $(A,\circ,\Phi)$ on $(l_\circ,r_\circ,V)$ if and only if for all $x\in A,\ v\in V$,*
$$l_\circ(\Phi(x))\Theta^{-1}(v)=\frac{1}{\lambda}\Theta(l_\circ(\Phi(x))v)\text{ and }r_\circ(\Phi(x))\Theta^{-1}(v)=\frac{1}{\lambda}\Theta(r_\circ(\Phi(x))v).$$

Proof. 1 By taking $\Lambda=\lambda\Theta$ in (2.22) and (2.23), we obtain,
$$\begin{aligned}&l_\circ(x)(\lambda\Theta)(v)-l_\circ(\Phi(x))v-(\lambda\Theta)(l_\circ(x)v)=\lambda l_\circ(x)\Theta(v)-l_\circ(\Phi(x))v-\lambda\Theta(l_\circ(x)v)\\&=\underbrace{\lambda(l_\circ(x)\Theta(v)-\Theta(l_\circ(x)v)+l_\circ(\Phi(x))v)}_{=0\text{ by (2.17)}}-(\lambda+1)l_\circ(\Phi(x))v=-(\lambda+1)l_\circ(\Phi(x))v,\end{aligned}$$
$$\begin{aligned}&r_\circ(x)(\lambda\Theta)(v)-r_\circ(\Phi(x))v-(\lambda\Theta)(r_\circ(x)v)=\lambda r_\circ(x)\Theta(v)-r_\circ(\Phi(x))v-\lambda\Phi(r_\circ(x)v)\\&=\underbrace{\lambda(r_\circ(x)\Theta(v)-\Theta(r_\circ(x)v)+r_\circ(\Phi(x))v)}_{=0\text{ by (2.18)}}-(\lambda+1)r_\circ(\Phi(x))v=-(\lambda+1)r_\circ(\Phi(x))v.\end{aligned}$$
Then $\lambda\Theta$ is admissible to $(A,\circ,\Phi)$ on $(l_\circ,r_\circ,V)$ if and only if $(\lambda+1)l_\circ(\Phi(x))v=(\lambda+1)r_\circ(\Phi(x))v=0$.

2 We prove that the axioms (2.22) and (2.23) hold for $\Lambda=-\Theta+\lambda id_V$,
$$\begin{aligned}&l_\circ(x)(-\Theta+\lambda id)(v)-l_\circ(\Phi(x))v-(-\Theta+\lambda id)(l_\circ(x)v)\\&=-l_\circ(x)\Theta(v)+\lambda l_\circ(x)(v)-l_\circ(\Phi(x))v+\Theta(l_\circ(x)v)-\lambda l_\circ(x)v\\&=\underbrace{-l_\circ(x)\Theta(v)-l_\circ(\Phi(x))v+\Theta(l_\circ(x)v)}_{=0\text{ by (2.17)}}=0,\end{aligned}$$
$$\begin{aligned}&r_\circ(x)(-\Theta+\lambda id)(v)-r_\circ(\Phi(x))v-(-\Theta+\lambda id)(r_\circ(x)v)\\&=-r_\circ(x)\Theta(v)+\lambda r_\circ(x)v-r_\circ(\Phi(x))v+\Theta(r_\circ(x)v)-\lambda r_\circ(x)v\\&=\underbrace{-r_\circ(x)\Theta(v)-r_\circ(\Phi(x))v+\Theta(r_\circ(x)v)}_{=0\text{ by (2.18)}}=0.\end{aligned}$$

3 We replace Λ by $\lambda\Theta^{-1}$ in (2.22) and (2.23) we obtain,
$$\begin{aligned}&l_\circ(x)(\lambda\Theta^{-1})(v)-l_\circ(\Phi(x))v-\lambda\Theta^{-1}(l_\circ(x)v)\\&=\Theta^{-1}\Big(\lambda\Theta(l_\circ(x))(\Theta^{-1}(v))-\Theta(l_\circ(\Phi(x))v)-\lambda l_\circ(x)(\Theta(\Theta^{-1}(v)))\Big)\\&=\Theta^{-1}\underbrace{\Big(\lambda\Theta l_\circ(x)(\Theta^{-1}(v))-\lambda l_\circ(\Phi(x))(\Theta^{-1}(v))-\lambda l_\circ(x)(\Theta(\Theta^{-1}(v)))\Big)}_{=0\text{ by (2.17)}}\\&+\lambda l_\circ(\Phi(x))(\Theta^{-1}(v))-\Theta(l_\circ(\Phi(x))v)\\&=\lambda l_\circ(\Phi(x))(\Theta^{-1}(v))-\Theta(l_\circ(\Phi(x))v),\end{aligned}$$
$$\begin{aligned}&r_\circ(x)(\lambda\Theta^{-1}(v))-r_\circ(\Phi(x))v-\lambda\Theta^{-1}(r_\circ(x)v)\\&=\Theta^{-1}\Big(\lambda\Theta r_\circ(x)(\Theta^{-1}(v))-\Theta(r_\circ(\Phi(x))v)-\lambda r_\circ(x)\Theta(\Theta^{-1}(v))\Big)\\&=\Theta^{-1}\underbrace{\Big(\lambda\Theta r_\circ(x)v)-\lambda r_\circ(x)(\Theta(\Theta^{-1}(v)))-\lambda r_\circ(\Phi(x))(\Theta^{-1}(v))\Big)}_{=0\text{ by (2.18)}}\end{aligned}$$

$$+\lambda r_\circ(\Phi(x))(\Theta^{-1}(v)) - \Theta(r_\circ(\Phi(x))v)$$
$$= \lambda r_\circ(\Phi(x))(\Theta^{-1}(v)) - \Theta(r_\circ(\Phi(x))v).$$

Then $\lambda\Theta^{-1}$ is admissible to $(A, \circ, \Phi)$ on $(l_\circ, r_\circ, V)$ if and only if

$$\lambda l_\circ(\Phi(x))(\Theta^{-1}(v)) - \Theta(l_\circ(\Phi(x))v) = \lambda r_\circ(\Phi(x))(\Theta^{-1}(v)) - \Theta(r_\circ(\Phi(x))v) = 0.$$

Since $\lambda \neq 0$, we can divide by λ and obtain that the admissibility condition is equivalent to

$$l_\circ(\Phi(x))\Theta^{-1}(v) = \frac{1}{\lambda}\Theta(l_\circ(\Phi(x))v), \quad r_\circ(\Phi(x))\Theta^{-1}(v) = \frac{1}{\lambda}\Theta(r_\circ(\Phi(x))v),$$

which proves the result. □

Taking the regular representation in Proposition 2.20 leads to the following statement.

Corollary 2.21. *Let $(A, \circ, \Phi)$ be a differential pre-Lie algebra and $\lambda \in \mathbb{K}$. Then*

1. *$\lambda\Phi$ is admissible to $(A, \circ, \Phi)$ if and only if for all $x, y \in A$,*

$$(1+\lambda)(\Phi(x) \circ y) = (1+\lambda)(x \circ \Phi(y)) = 0.$$

2. *$-\Phi + \lambda id_A$ is admissible to $(A, \circ, \Phi)$.*
3. *Suppose Φ is an automorphism and $\lambda \neq 0$. Then $\lambda\Phi^{-1}$ is admissible to $(A, \circ, \Phi)$ if and only if for all $x, y \in A$,*

$$\Phi(x) \circ \Phi^{-1}(y) = \frac{1}{\lambda}\Phi(y \circ \Phi(x)) \text{ and } \Phi^{-1}(x) \circ \Phi(y) = \frac{1}{\lambda}\Phi(x \circ \Phi(y)).$$

The following theorem, which is the main result of this section, provides a characterization of admissibility on semi-direct products and their duals.

Theorem 2.22. *Let $(A, \circ, \Phi)$ be a differential pre-Lie algebra and let $(l_\circ, r_\circ, V)$ be a representation of the pre-Lie algebra $(A, \circ)$. Let $\Psi : A \to A$ and $\Theta, \Lambda : V \to V$ be two linear maps. Then the following conditions are equivalent.*

1. *There is a differential pre-Lie algebra $(A \ltimes_{l_\circ, r_\circ} V, \Phi + \Theta)$ such that $\Psi + \Lambda$ is admissible to $(A \ltimes_{l_\circ, r_\circ} V, \Phi + \Theta)$.*
2. *There is a differential pre-Lie algebra $(A \ltimes_{l_\circ^* - r_\circ^*, -r_\circ^*} V^*, \Phi + \Lambda^*)$ such that $\Psi + \Theta^*$ is admissible to $(A \ltimes_{l_\circ^* - r_\circ^*, -r_\circ^*} V^*, \Phi + \Lambda^*)$.*
3. *The following conditions are satisfied:*
   (a) $(l_\circ, r_\circ, \Theta, V)$ is a representation of $(A, \circ, \Phi)$;
   (b) Ψ is admissible to $(A, \circ, \Phi)$;
   (c) Λ is admissible to $(A, \circ, \Phi)$ on $(l_\circ, r_\circ, V)$;
   (d) For $x \in A, v \in V$, we have

$$l_\circ(\Phi(x))u - \Lambda(l_\circ(x)u) - l_\circ(x)\Theta(u) = 0, \tag{2.26}$$

$$r_\circ(\Phi(x))u - \Lambda(r_\circ(x)u) - r_\circ(x)\Theta(u) = 0. \tag{2.27}$$

Proof. "$1 \Longleftrightarrow 3$" By Proposition 2.14, we have $(A \ltimes_{l_\circ, r_\circ} V, \circ', \Phi + \Theta)$ is a differential pre-Lie algebra if and only if $(l_\circ, r_\circ, \Theta, V)$ is a representation of the differential pre-Lie algebra $(A, \circ, \Phi)$. In the other hand, the linear operator $\Psi + \Lambda$ on $A \oplus V$is admissible to $(A \ltimes_{l_\circ, r_\circ}$

$V, \circ', \Phi+\Theta)$, then by Corollary 2.18, equations (2.13) and (2.14) hold. Furthermore, if A, Ψ and Φ are replaced by $A \oplus V, \Psi+\Lambda$ and $\Phi+\Theta$ in (2.13), we have

$$\begin{aligned}
(x_1+v_1)\circ'(\Psi+\Lambda)(x_2+v_2) &= (x_1+v_1)\circ'(\Psi(x_2)+\Lambda(v_2)) \\
&= x_1\circ\Psi(x_2)+l_\circ(x_1)\Lambda(v_2)+r_\circ(\Psi(x_2))v_1 \\
(\Phi+\Theta)(x_1+v_1)\circ'(x_2+v_2) &= (\Phi(x_1)+\Theta(v_1))\circ'(x_2+v_2) \\
&= \Phi(x_1)\circ x_2+l_\circ(\Phi(x_1))v_2+r_\circ(x_2)\Theta(v_1), \\
(\Psi+\Lambda)((x_1+v_1)\circ'(x_2+v_2)) &= (\Psi+\Lambda)(x_1\circ x_2+l_\circ(x_1)v_2+r_\circ(x_2)v_1) \\
&= \Psi(x_1\circ x_2)+\Lambda(l_\circ(x_1)v_2)+\Lambda(r_\circ(x_2)v_1).
\end{aligned}$$

We obtain,

$$\begin{aligned}
&x_1\circ\Psi(x_2)+l_\circ(x_1)\Lambda(v_2)+r_\circ(\Psi(x_2))v_1-\Phi(x_1)\circ x_2-l_\circ(\Phi(x_1))v_2 \\
&-r_\circ(x_2)\Theta(v_1)-\Psi(x_1\circ x_2)-\Lambda(l_\circ(x_1)v_2)-\Lambda(r_\circ(x_2)v_1)=0.
\end{aligned}\tag{2.28}$$

Then, (2.28) hold if and only if

⋆ equation (2.13) (corresponding $v_1=v_2=0$) hold, and
⋆ equation (2.22) (corresponding $x_2=v_1=0$) hold, and
⋆ equation (2.27) (corresponding $x_1=v_2=0$) hold.

Similarly, if A, Ψ and Φ are replaced by $A\oplus V, \Psi+\Lambda$ and $\Phi+\Theta$ in (2.14), we have

$$\begin{aligned}
((\Psi+\Lambda)(x_1+v_1))\circ'(x_2+v_2) &= (\Psi(x_1)+\Lambda(v_1))\circ'(x_2+v_2) \\
&= \Psi(x_1)\circ x_2+l_\circ(\Psi(x_1))v_2+r_\circ(x_2)\Lambda(v_1) \\
(x_1+v_1)\circ'(\Phi+\Theta)(x_2+v_2) &= (x_1+v_1)\circ'(\Phi(x_2)+\Theta(v_2)) \\
&= x_1\circ\Phi(x_2)+l_\circ(x_1)\Theta(v_2)+r_\circ(\Phi(x_2))v_1 \\
(\Psi+\Lambda)((x_1+v_1)\circ'(x_2+v_2)) &= (\Psi+\Lambda)(x_1\circ x_2+l_\circ(x_1)v_2+r_\circ(x_2)v_1) \\
&= \Psi(x_1\circ x_2)+\Lambda(l_\circ(x_1)v_2)+\Lambda(r_\circ(x_2)v_1).
\end{aligned}$$

We obtain,

$$\begin{aligned}
&\Psi(x_1)\circ x_2+l_\circ(\Psi(x_1))v_2+r_\circ(x_2)\Lambda(v_1)-x_1\circ\Phi(x_2)-l_\circ(x_1)\Theta(v_2) \\
&-r_\circ(\Phi(x_2))v_1-\Psi(x_1\circ x_2)-\Lambda(l_\circ(x_1)v_2)-\Lambda(r_\circ(x_2)v_1)=0.
\end{aligned}\tag{2.29}$$

Then, (2.29) hold if and only if

⋆ equation (2.14) (corresponding $v_1=v_2=0$) hold, and
⋆ equation (2.23) (corresponding $x_1=v_2=0$) hold, and
⋆ equation (2.26) (corresponding $x_2=v_1=0$) hold.

Hence, 1 holds if and only if

3a $(l_\circ, r_\circ, \Theta, V)$ is a representation of the differential pre-Lie algebra $(A,\circ,\Phi)$, and
3b equations (2.22) and (2.23) hold, that is, Ψ admissible to $(A,\circ,\Phi)$, and
3c equations (2.13) and (2.14) hold, that is, Λ is admissible to the differential pre-algebra $(A,\circ,\Phi)$ on $(l_\circ, r_\circ, V)$, and
3d equations (2.26) and (2.27) hold.

We conclude that 1 holds if and only if 3 holds.

"2 $\Longleftrightarrow$ 3" In 1, we take

$$V=V^*,\quad l_\circ=l_\circ^*-r_\circ^*,\quad r_\circ=-r_\circ^*,\quad \Lambda=\Theta^*,\quad \Theta=\Lambda^*,$$

then from the equivalence between 1 and 3, we have 2 holds if and only if 3a-3c and the following two equations hold

$$(l_\circ^* - r_\circ^*)(\Phi(x)v^* - (l_\circ^* - r_\circ^*)(x)\Lambda^*(v^*) - \Theta^*((l_\circ^* - r_\circ^*)(x)v^*) = 0, \tag{2.30}$$

$$r_\circ^*(\Phi(x))v^* - r_\circ^*(x)\Lambda^*(v^*) - \Theta^*(r_\circ^*(x)v^*) = 0, \tag{2.31}$$

where $x \in A$ and $v^* \in V^*$. Besides, for any $x \in A, u \in V$ and $v^* \in V^*$ we have

$$\langle (l_\circ^* - r_\circ^*)(\Phi(x))v^*, u\rangle = -\langle v^*, (l_\circ - r_\circ)(\Phi(x))u\rangle,$$

$$\langle (l_\circ^* - r_\circ^*)(x)\Lambda^*(v^*), u\rangle = -\langle \Lambda^*(v^*), (l_\circ - r_\circ)(x)u\rangle = -\langle v^*, \Lambda((l_\circ - r_\circ)(x)u)\rangle,$$

$$\langle \Theta^*((l_\circ^* - r_\circ^*)(x)v^*), u\rangle = \langle (l_\circ^* - r_\circ^*)(x)v^*, \Theta(u)\rangle = -\langle v^*, (l_\circ - r_\circ)(x)\Theta(u)\rangle,$$

we obtain

$$\begin{aligned}&\langle (l_\circ^* - r_\circ^*)(\Phi(x)v^* - (l_\circ^* - r_\circ^*)(x)\Lambda^*(v^*) - \Theta^*((l_\circ^* - r_\circ^*)(x)v^*), u\rangle\\ =& -\langle v^*, (l_\circ - r_\circ)(\Phi(x))u - \Lambda((l_\circ - r_\circ)(x)u) - (l_\circ - r_\circ)(x)\Theta(u)\rangle.\end{aligned}$$

Then, (2.30) holds if and only if (2.26)-(2.27) holds. Similarly, for any $x \in A, u \in V$ and $v^* \in V^*$ we have

$$\begin{aligned}\langle (r_\circ^*(\Phi(x))v^*, u\rangle &= -\langle v^*, r_\circ(\Phi(x))u\rangle,\\ \langle r_\circ^*(x)\Lambda^*(v^*), u\rangle &= -\langle \Lambda^*(v^*), r_\circ(x)u\rangle = -\langle v^*, \Lambda(r_\circ(x)u)\rangle,\\ \langle \Theta^*(r_\circ^*(x)v^*), u\rangle &= \langle r_\circ^*(x)v^*, \Theta(u)\rangle = -\langle v^*, r_\circ(x)\Theta(u)\rangle,\end{aligned}$$

we obtain

$$\langle r_\circ^*(\Phi(x))v^* - r_\circ^*(x)\Lambda^*(v^*) - \Theta^*(r_\circ^*(x)v^*), u\rangle = -\langle v^*, r_\circ(\Phi(x))u - \Lambda(r_\circ(x)u) - r_\circ(x)\Theta(u)\rangle.$$

Then, (2.31) holds if and only if (2.27) hold. Hence, 2 holds if and only if 3 holds. $\square$

3. Matched pairs, Manin triples and bialgebras of differential pre-Lie algebras

In this section, we introduce the notions of matched pair of differential pre-Lie algebras and the notions of a Manin triple of differential pre-Lie algebras. Next, we prove the differential pre-Lie bialgebras are characterized by generalizing matched pairs of pre-Lie algebras and Manin triples of pre-Lie algebras to the context of differential pre-Lie algebras.

3.1. Matched pairs and bialgebras of differential pre-Lie algebras. The following proposition recalls the notion of matched pairs of pre-Lie algebras and their relation with direct sum constructions.

PROPOSITION 3.1 ([1]). *Let $(A, \circ_A)$ and $(B, \circ_B)$ be two pre-Lie algebras. Suppose there are linear maps $l_A, r_A : A \to gl(B)$ and $l_B, r_B : B \to gl(A)$ such that the quadruple (l_A, r_A, B) is a representation of A, and (l_B, r_B, A) is a representation of B, satisfying, for any $x, y \in A, a, b \in B$, the following conditions:*

$$r_A(x)[a, b]_B = r_A(l_B(b)x)a - r_A(l_B(a)x)b + a \circ_B (r_A(x)b) - b \circ_B (r_A(x)a), \tag{3.1}$$

$$\begin{aligned}l_A(x)(a \circ_B b) = &-l_A(l_B(a)x - r_B(a)x)b + (l_A(x)a - r_A(x)a) \circ_B b\\ &+ r_A(r_B(b)x)a + a \circ_B (l_A(x)b),\end{aligned} \tag{3.2}$$

$$r_B(a)[x, y]_A = r_B(l_A(y)a)x - r_B(l_A(x)a)y + x \circ_A (r_B(a)y) - y \circ_A (r_B(a)x), \tag{3.3}$$

$$l_B(a)(x \circ_A y) = -l_B(l_A(x)a - r_A(x)a)y + (l_B(a)x - r_B(a)x) \circ_A y$$
$$+ r_B(r_A(y)a)x + x \circ_A (l_B(a)y), \tag{3.4}$$

where $[x, y]_A = x \circ_A y - y \circ_A x$ *and* $[a, b]_B = a \circ_B b - b \circ_B a$ *Then,* $(A, B, l_A, r_A, l_B, r_B)$ *is called a matched pair of pre-Lie algebras. Furthermore,* $(A, B, l_A, r_A, l_B, r_B)$ *is a matched pair of pre-Lie algebras if and only if* $(A \oplus B, \circ')$ *is a pre-Lie algebra where for all* $x, y \in A, a, b \in B$

$$(x + a) \circ' (y + b) = (x \circ_A y + l_B(a)y + r_B(b)x) + (a \circ_B b + l_A(x)b + r_A(y)a), \tag{3.5}$$

We denote this pre-Lie algebra by $(A \bowtie B, \circ')$ or $A \bowtie_{l_B, r_B}^{l_A, r_A} B$.

DEFINITION 3.2. A matched pair of differential pre-Lie algebras $(A, \circ_A, \Phi_A)$ and $(B, \circ_B, \Phi_B)$ is a 8-tuple $(A, B, \Phi_A, \Phi_B, l_A, r_A, l_B, r_B)$ where (l_A, r_A, Φ_B, B) is a representation of $(A, \circ_A, \Phi_A)$ and (l_B, r_B, Φ_A, A) is a representation of $(B, \circ_B, \Phi_B)$, and $(A, B, l_A, r_A, l_B, r_B)$ is a matched pair of pre-Lie algebras $(A, \circ_A)$ and $(B, \circ_B)$.

The next theorem extends matched pairs to the differential setting.

THEOREM 3.3. *Let* $(A, \circ_A, \Phi_A)$ *and* $(B, \circ_B, \Phi_B)$ *be two differential pre-Lie algebras. Then* $(A, B, \Phi_A, \Phi_B, l_A, r_A, l_B, r_B)$ *be a matched pair of the differential pre-Lie algebras* $(A, \circ_A, \Phi_A)$ *and* $(B, \circ_B, \Phi_B)$ *if and only if* $(A \oplus B, \circ', \Phi = \Phi_A + \Phi_B)$ *is a differential pre-Lie algebra by defining the multiplication in* $A \oplus B$ *for all* $x, y \in A, a, b \in B$ *by*

$$\begin{aligned} (x + a) \circ' (y + b) &:= (x \circ_A y + l_B(a)y + r_B(b)x) \\ &+ \; a \circ_B b + l_A(x)b + r_A(y)a), \end{aligned} \tag{3.6}$$
$$\Phi(x + a) := \Phi_A(x) + \Phi_B(a). \tag{3.7}$$

Proof. Suppose that $(A, B, \Phi_A, \Phi_B, l_A, r_A, l_B, r_B)$ be a matched pair of the differential pre-Lie algebras. Then by Proposition 3.1, we have $(A \bowtie B, \circ')$ is a pre-Lie algebra. Besides, for any $x, y \in A,\ a, b \in B$

$$\begin{aligned}
&\Phi((x + a) \circ' (y + b)) - \Phi(x + a) \circ' (y + b) - (x + a) \circ' \Phi(y + b)\\
&= (\Phi_A + \Phi_B)\Big(x \circ_A y + l_B(a)y + r_B(b)x + a \circ_B b + l_A(x)b + r_A(y)a\Big)\\
&- \Big(\Phi_A(x) \circ_A y + l_B(\Phi_B(a))y + r_B(b)\Phi_A(x) + \Phi_B(a) \circ_B b + l_A(\Phi_A(x))b + r_A(y)\Phi_B(a)\Big)\\
&- \Big(x \circ_A \Phi_A(y) + l_B(a)\Phi_A(y) + r_B(\Phi_A(b))x + a \circ_B \Phi_B(b) + l_A(x)\Phi_B(b) + r_A(\Phi_A(y))a\Big)\\
&= \Phi_A(x \circ_A y) + \Phi_A(l_B(a)y) + \Phi_A(r_B(b)x) + \Phi_B(a \circ_B b) + \Phi_B(l_A(x)b) + \Phi_B(r_A(y)a)\\
&- \Phi_A(x) \circ_A y - l_B(\Phi_B(a))y - r_B(b)\Phi_A(x) - \Phi_B(a) \circ_B b - l_A(\Phi_A(x))b - r_A(y)\Phi_B(a)\\
&- x \circ_B \Phi_A(y) - l_B(a)\Phi_A(y) - r_B(\Phi_A(b))x - a \circ_B \Phi_B(b) - l_A(x)\Phi_B(b) - r_A(\Phi_A(y))a\\
&= \underbrace{\Big(\Phi_A(x \circ_A y) - \Phi_A(x) \circ_A y - x \circ_A \Phi_A(y)\Big)}_{= 0 \text{ by } (1.3)} + \underbrace{\Big(\Phi_B(a \circ_B b) - \Phi_B(a) \circ_B b - a \circ_B \Phi_B(b)\Big)}_{= 0 \text{ by } (1.3)}\\
&+ \underbrace{\Big(\Phi_A(l_B(a)y) - l_B(\Phi_B(a))y - l_B(a)\Phi_A(y)\Big)}_{= 0 \text{ by } (2.17)} + \underbrace{\Big(\Phi_A(l_B(a)y) - l_B(\Phi_B(a))y - l_A(x)\Phi_B(b)\Big)}_{= 0 \text{ by } (2.17)}\\
&+ \underbrace{\Big(\Phi_A(r_B(b)\Phi_A(x) - r_B(b)\Phi_A(x) - r_B(\Phi_B(b))a\Big)}_{= 0 \text{ by } (2.18)} + \underbrace{\Big(\Phi_B(r_A(y)a) - r_A(y)\Phi_B(a) - r_A(\Phi_A(y))a\Big)}_{= 0 \text{ by } (2.18)} = 0.
\end{aligned}$$

Hence, Φ is a derivation on $(A \bowtie B, \circ')$,. This means that $(A \oplus B, \circ', \Phi)$ is a differential pre-Lie algebra.

Conversely, suppose that $(A \oplus B, \circ', \Phi_A + \Phi_B)$ is a differential pre-Lie algebra. Then By Proposition 3.1, we have $(A, B, l_A, r_A, l_B, r_B)$ is a matched pair of pre-Lie algebras

$(A, \circ_A)$ and $(B, \circ_B)$. Moreover, for any $x, y \in A,\ a, b \in B$

(3.8)

$$\Big(\Phi_A(x \circ_A y) - \Phi_A(x) \circ_A y - x \circ_A \Phi_A(y)\Big) + \Big(\Phi_B(a \circ_B b) - \Phi_B(a) \circ_B b - a \circ_B \Phi_B(b)\Big)$$
$$+ \Big(\Phi_A(l_B(a)y) - l_B(\Phi_B(a))y - l_B(a)\Phi_A(y)\Big) + \Big(\Phi_B(l_A(x)b) - l_A(\Phi_A(x))b - l_A(x)\Phi_B(b)\Big)$$
$$+ \Big(\Phi_A(r_B(b)\Phi_A(x) - r_B(b)\Phi_A(x) - r_B(\Phi_B(b))a\Big) + \Big(\Phi_B(r_A(y)a) - r_A(y)\Phi_B(a) - r_A(\Phi_A(y))a\Big) = 0.$$

Taking $x = 0$ and $b = 0$ (resp. $y = 0$ and $a = 0$) in the last equation (3.8) we obtain, (l_A, r_A, Φ_B, B) is a representation of $(A, \circ_A, \Phi_A)$ (resp. (l_B, r_B, Φ_A, A) is a representation of $(B, \circ_B, \Phi_B)$). Hence $(A, B, \Phi_A, \Phi_B, l_A, r_A, l_B, r_B)$ is a matched pair of the differential pre-Lie algebras $(A, \circ_A, \Phi_A)$ and $(B, \circ_B, \Phi_B)$. □

We denote this differential pre-Lie algebra by $(A \bowtie B, \circ', \Phi_A + \Phi_B)$ or $A \bowtie^{l_A, r_A, \Phi_A}_{l_B, r_B, \Phi_B} B$.

For a pre-Lie algebra $(A^*, \bullet)$, let $\delta : A \to \otimes^2 A$ be the dual map of $\bullet : \otimes^2 A^* \to A^*$, that is

$$\langle \delta x, a^* \otimes b^* \rangle = \langle x, a^* \bullet b^* \rangle.$$

Let us consider the four maps such that for all $x, u, v \in A,\ x^*, u^*, v^* \in A^*$,

$$\begin{array}{ll}
L^*_\circ : A \to gl(A^*), & \langle L^*_\circ(x)u^*, v\rangle = -\langle L_\circ(x)v, u^*\rangle = -\langle x \circ v, u^*\rangle, \\
R^*_\circ : A \to gl(A^*), & \langle R^*_\circ(x)u^*, v\rangle = -\langle R_\circ(x)v, u^*\rangle = -\langle v \circ x, u^*\rangle, \\
L^*_\bullet : A^* \to gl(A), & \langle L^*_\bullet(x^*)u, v^*\rangle = -\langle L_\bullet(x^*)v^*, u\rangle = -\langle x^* \bullet v^*, u\rangle, \\
R^*_\bullet : A^* \to gl(A), & \langle R^*_\bullet(x^*)u, v^*\rangle = -\langle R_\bullet(x^*)v^*, u\rangle = -\langle v^* \bullet x^*, u\rangle.
\end{array}$$

The following theorem characterizes pre-Lie bialgebras in terms of matched pairs.

Theorem 3.4 ([1]). *Let $(A, \circ)$ be a pre-Lie algebra. Suppose that there is a pre-Lie algebra structure "$\bullet$" on its dual space A^*. Then the triple (A, δ, ζ) is a pre-Lie bialgebra if and only if $(A, A^*, L^*_\circ - R^*_\circ, -R^*_\circ, L^*_\bullet - R^*_\bullet, -R^*_\bullet)$ is a matched pair of pre-Lie algebras.*

The next theorem generalizes the previous characterization to differential pre-Lie bialgebras.

Theorem 3.5. *Let $(A, \circ, \Phi)$ be a differential pre-Lie algebra. Suppose that there is a differential pre-Lie algebra structure "$\bullet$" on its dual space (A^*, Ψ^*). Then the quintuple $(A, \delta, \zeta, \Phi, \Psi)$ is a differential pre-Lie bialgebra if and only if $(A, A^*, \Phi, \Psi^*, L^*_\circ - R^*_\circ, -R^*_\circ, L^*_\bullet - R^*_\bullet, -R^*_\bullet)$ is a matched pair of differential pre-Lie algebras.*

Proof. If $(A, \delta, \zeta, \Phi, \Psi)$ is a differential pre-Lie bialgebra, then by Proposition 2.19 (A, δ, ζ) is a pre-Lie bialgebra and Ψ, Φ^* are admissible to $(A, \circ, \Phi)$ and $(A^*, \bullet, \Psi^*)$ respectively, this means that $(A^*, L^*_\circ - R^*_\circ, -R^*_\circ, \Psi^*)$ is a representation of $(A, \circ, \Phi)$ and $(A, L^*_\bullet - R^*_\bullet, -R^*_\bullet, \Phi)$ is a representation of $(A^*, \bullet, \Psi^*)$. Besides; by Theorem 3.4, $(A, A^*, L^*_\circ - R^*_\circ, -R^*_\circ, L^*_\bullet - R^*_\bullet, -R^*_\bullet)$ is a matched pair of pre-Lie algebras, Hence by Definition 3.2, $(A, A^*, \Phi, \Psi^*, L^*_\circ - R^*_\circ, -R^*_\circ, L^*_\bullet - R^*_\bullet, -R^*_\bullet)$ is a matched pair of differential pre-Lie algebras.

Conversely, if $(A, A^*, \Phi, \Psi^*, L^*_\circ - R^*_\circ, -R^*_\circ, L^*_\bullet - R^*_\bullet, -R^*_\bullet)$ is a matched pair of differential pre-Lie algebras, then by definition 3.2, $(A, A^*, L^*_\circ - R^*_\circ, -R^*_\circ, L^*_\bullet - R^*_\bullet, -R^*_\bullet)$ is a matched pair of pre-Lie algebras and Ψ, Φ^* are admissible to $(A, \circ, \Phi)$ and $(A^*, \bullet, \Psi^*)$ respectively. Hence by Theorem 3.4 and Proposition 2.19, $(A, \circ, \delta, \Phi, \Psi)$ is a differential pre-Lie bialgebra, □

3.2. Manin triples and bialgebras of differential pre-Lie algebras.

Definition 3.6. Let $(A, \circ)$ be a pre-Lie algebra, and $B : A \times A \to \mathbb{K}$ be a bilinear form on A. Then,

1. B is said to be nondegenerate if

$$A^{\perp} = \{x \in A \mid B(y, x) = 0, \forall y \in A\} = \{0_A\}; \tag{3.9}$$

2. B is said to be skew-symmetric if for all $x, y \in A$,

$$B(x, y) = -B(y, x); \tag{3.10}$$

3. B is said to be invariant if for all $x, y \in A$,

$$B(x \circ y, z) = -B(y, [x, z]), \tag{3.11}$$

where $[x, y] = x \circ y - y \circ x$.

Remark 3.7. If $B : A \times A \to \mathbb{K}$ be an invariant bilinear form on A. Then,

$$B(x \circ y, z) = -B(z \circ y, x). \tag{3.12}$$

In fact, by (3.11) and (2.1), for all $x, y, z \in A$,

$$B(x \circ y, z) = -B(y, [x, z]) = B(y, [z, x]) = -B(z \circ y, x).$$

Definition 3.8. A quadratic pre-Lie algebra is a pre-Lie algebra equipped with a non-degenerate skew-symmetric invariant bilinear form.

Definition 3.9. Let $(A, \circ)$ be a pre-Lie algebra. Suppose that $(A^*, \bullet)$ is a pre-Lie algebra on its dual space A^*. If there exists a pre-Lie algebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of A and A^* such that $(A, \circ)$ and $(A^*, \bullet)$ are pre-Lie subalgebras and the following skew-symmetric bilinear form B_s on $A \oplus A^*$ given by

$$B_s(x + a^*, y + b^*) = \langle a^*, y\rangle - \langle b^*, x\rangle,\ \forall x, y \in A,\ a^*, b^* \in A^*, \tag{3.13}$$

is invariant, then the quadruple $(A \oplus A^*, A, A^*, B_s)$ is called a standard Manin triple of pre-Lie algebras $(A, \circ)$ and $(A^*, \bullet)$.

Example 3.10. Let $A = \mathrm{span}\{e_1, e_2\}$ and $A^* = \mathrm{span}\{e_1^*, e_2^*\}$. Define

$$e_1 \circ e_2 = e_2, \quad \text{others } = 0, \qquad e_2^* \bullet e_1^* = -e_2^*, \quad \text{others } = 0.$$

On $A \oplus A^*$, define

$$(x + a^*) \star (y + b^*) = x \circ y + (L_\circ^* - R_\circ^*)(x)b^* - R_\circ^*(y)a^* + a^* \bullet b^*.$$

Then $(A \oplus A^*, A, A^*, B_s)$, with $B_s(x + a^*, y + b^*) = \langle a^*, y\rangle - \langle b^*, x\rangle$, is a standard Manin triple of pre-Lie algebras.

The following theorem provides a fundamental characterization of standard Manin triples of pre-Lie algebras in terms of matched pairs.

Theorem 3.11 ([15]). *Let $(A, \circ)$ be a pre-Lie algebra. Suppose that there is a pre-Lie algebra structure " $\bullet$ " on its dual space A^*. Then, there is a standard Manin triple of $(A, \circ)$ and $(A^*, \bullet)$ if and only if $(A, A^*, L_\circ^* - R_\circ^*, -R_\circ^*, L_\bullet^* - R_\bullet^*, -R_\bullet^*)$ is a matched pair of pre-Lie algebras.*

Definition 3.12. A differential quadratic pre-Lie algebra is a quadruple $(A, \circ, \Phi, B)$ where $(A, \circ, \Phi)$ is a differential pre-Lie algebra and $(A, \circ, B)$ is a quadratic pre-Lie algebra.

Let $(A, \circ, \Phi)$ be a differential pre-Lie algebra and B the nondegenerate bilinear form. We denote by $\widehat{\Phi}$ the adjoint linear transformation of Φ under B:

$$B(\Phi(x), y) = B(x, \widehat{\Phi}(y)),\ \ \forall x, y \in A. \tag{3.14}$$

We now describe how the dual representation of a differential quadratic pre-Lie algebra is induced via the adjoint operator associated to the invariant bilinear form.

Proposition 3.13. *Let $(A,\circ,\Phi,B)$ be a differential quadratic pre-Lie algebra. Then the quadruple $(L_\circ^*-R_\circ^*,-R_\circ^*,\widehat{\Phi}^*;A^*)$ is a representation of the differential pre-Lie algebra $(A,\circ,\Phi)$ that is equivalent to $(L_\circ,R_\circ,\Phi,A)$.*

Proof. For any $x,y,z\in A$, we have

$$\begin{aligned}
B\big(x,y\circ\widehat{\Phi}(z)-\Phi(y)\circ z-\widehat{\Phi}(y\circ z)\big) &= B(x,y\circ\widehat{\Phi}(z))-B(x,\Phi(y)\circ z)-B(x,\widehat{\Phi}(y\circ z))\\
\overset{(\text{by}(3.14))}{} &= B(x,y\circ\widehat{\Phi}(z))-B(x,\Phi(y)\circ z)-B(\Phi(x),y\circ z)\\
\overset{(\text{by }(3.11))}{} &= B(\widehat{\Phi}(z),[y,x])-B(z,[\Phi(y),x])-B(z,[y,\Phi(x)])\\
\overset{(\text{by}(3.14))}{} &= B(z,\Phi([y,x]))-B(z,[\Phi(y),x])-B(z,[y,\Phi(x)])\\
&= \underbrace{B(z,\Phi([y,x])-[\Phi(y),x]-[y,\Phi(x)])}_{=0\text{ by }(1.3)}=0,
\end{aligned}$$

$$\begin{aligned}
B\big(\widehat{\Phi}(x)\circ y-x\circ\Phi(y)-\widehat{\Phi}(x\circ y),z\big) &= B(\widehat{\Phi}(x)\circ y,z)-B(x\circ\Phi(y),z)-B(\widehat{\Phi}(x\circ y),z)\\
\overset{(\text{by }(3.14))}{} &= B(\widehat{\Phi}(x)\circ y,z)-B(x\circ\Phi(y),z)-B(x\circ y,\Phi(z))\\
\overset{(\text{by }(3.12))}{} &= -B(z\circ y,\widehat{\Phi}(x))+B(z\circ\Phi(y),x)+B(\Phi(z)\circ y,x)\\
\overset{(\text{by }(3.14))}{} &= -B(\Phi(z\circ y),x)+B(z\circ\Phi(y),x)+B(\Phi(z)\circ y,x)\\
&= \underbrace{-B(\Phi(z\circ y)-z\circ\Phi(y)-\Phi(z)\circ y,x)}_{=0\text{ by }(1.3)}=0.
\end{aligned}$$

Then we obtain

$$y\circ\widehat{\Phi}(z)-\Phi(y)\circ z-\widehat{\Phi}(y\circ z)=0,\widehat{\Phi}(x)\circ y-x\circ\Phi(y)-\widehat{\Phi}(x\circ y)=0.$$

These give equations (2.13) and (2.14) for $\Psi=\widehat{\Phi}$. Hence by Corollary 2.18 we have $(L_\circ^*-R_\circ^*,-R_\circ^*,\widehat{\Phi}^*;A^*)$ is a representation of $(A,\circ,\Phi)$.

Next, we define the linear map $f:A\to A^*$ by

$$f(x)y:=\langle f(x),y\rangle=B(x,y),\quad\forall x,y\in A.$$

The nondegeneracy of B gives the bijectivity of f. Also for $x,y,z\in A$, we have

$$\begin{aligned}
f(L_\circ(x)y)z&=B(x\circ y,z)\overset{(\text{by }(3.11))}{=}-B(y,[x,z])=-\langle f(y),[x,z]\rangle\\
&=-\langle f(y),x\circ z-z\circ x\rangle=-\langle f(y),(L_\circ-R_\circ)(x)z\rangle\\
&=\langle(L_\circ^*-R_\circ^*)(x)f(y),z\rangle=(L_\circ^*-R_\circ^*)(x)f(y)z,
\end{aligned}$$

$$\begin{aligned}
f(R_\circ(x)y)z&=B(y\circ x,z)\overset{(\text{by }(3.12))}{=}-B(z\circ x,y)\overset{(\text{by }(3.10))}{=}B(y,z\circ x)\\
&=\langle f(y),z\circ x\rangle=\langle f(y),R_\circ(x)z\rangle=-\langle R_\circ^*(x)f(y),z\rangle=(-R_\circ^*)(x)f(y)z,
\end{aligned}$$

$$f(\Phi(x))y=B(\Phi(x),y)=B(x,\widehat{\Phi}(y))=\langle f(x),\widehat{\Phi}(y)\rangle=\langle\widehat{\Phi}^*(f(x)),y\rangle=\widehat{\Phi}^*(f(x))y.$$

Hence $(L_\circ^*-R_\circ^*,-R_\circ^*,\widehat{\Phi}^*,A^*)$ is equivalent to $(L_\circ,R_\circ,\Phi,A)$ as representations of $(A,\circ,\Phi)$. $\square$

The following result shows that the equivalence of representations leads to the existence of a nondegenerate bilinear form defining a differential quadratic pre-Lie algebra.

Proposition 3.14. *Let* $(A, \circ, \Phi)$ *be a differential pre-Lie algebra and* $\Psi : A \to A$ *be a linear map that is admissible to* $(A, \circ, \Phi)$*. If the representation* $(L_\circ^* - R_\circ^*, -R_\circ^*, \widehat{\Phi}^*; A^*)$ *of* $(A, \circ, \Phi)$ *is equivalent to* $(L_\circ, R_\circ, \Phi, A)$*, then there exists a nondegenerate bilinear form* B *such that* $(A, \circ, \Phi, B)$ *is a differential quadratic pre-Lie algebra for which* $\widehat{\Phi} = \Psi$*.*

Proof. Suppose that the representations $(L_\circ^* - R_\circ^*, -R_\circ^*, \widehat{\Phi}^*; A^*)$ and $(L_\circ, R_\circ, \Phi, A)$ of $(A, \circ, \Phi)$ are equivalent. Let $f : A \to A^*$ the linear isomorphism giving by equivalence between these representations. Define a bilinear form B on A by

$$B(x, y) = \langle f(x), y\rangle, \quad \forall x, y \in A.$$

Then by a similar argument we give the differential quadratic pre-Lie algebra and $\widehat{\Phi} = \Psi$. □

Definition 3.15. Let $(A, \circ, \Phi)$ be a differential pre-Lie algebra. Suppose that $(A^*, \bullet, \Psi^*)$ is a differential pre-Lie algebra. A standard Manin triple of differential pre-Lie algebra associated to $(A, \circ, \Phi)$ and $(A^*, \bullet, \Psi^*)$ is a quintuple $(A \bowtie A^*, A, A^*, \Phi + \Psi^*, B_s)$ such that $(A \bowtie A^*, A, A^*, B_s)$ is a standard Manin triple associated to $(A, \circ)$ and $(A^*, \bullet)$ and $(A \bowtie A^*, \circ', \Phi + \Psi^*)$ is a differential pre-Lie algebra.

The next lemma determines the adjoint of the differential operator in a Manin triple and establishes its admissibility.

Lemma 3.16. *Let* $(A \bowtie A^*, A, A^*, \Phi+\Psi^*, B_s)$ *be a standard Manin triple of differential pre-Lie algebra associated to* $(A, \circ, \Phi)$ *and* $(A^*, \bullet, \Psi^*)$*. Then the adjoint* $\widehat{\Phi + \Psi^*}$ *of* $\Phi+\Psi^*$ *under* B *is* $\Psi + \Phi^*$*. Further* $\Psi + \Phi^*$ *is admissible to* $(A \bowtie A^*, \circ', \Phi + \Psi^*)$*.*

Proof. For any $x, y \in A, x^*, y^* \in A^*$, we have

$$\begin{aligned} B_s\big((\Phi + \Psi^*)(x + x^*), y + y^*\big) &= B\big(\Phi(x) + \Psi^*(x^*), y + y^*\big) \overset{\text{(by (3.13))}}{=} \langle \Psi^*(x^*), y\rangle - \langle y^*, \Phi(x)\rangle \\ &= \langle x^*, \Psi(y)\rangle - \langle \Phi^*(y^*), x\rangle = B_s(x + x^*, (\Psi + \Phi^*)(y + y^*)) \end{aligned}$$

Hence $\widehat{\Phi + \Psi^*} = \Psi + \Phi^*$. Furthermore, by Proposition 3.14, $\Psi + \Phi^*$ is admissible to $(A \bowtie A^*, \circ', \Phi + \Psi^*)$. □

Lemma 3.17. *Let* $(A \bowtie A^*, A, A^*, \Phi+\Psi^*, B_s)$ *be a standard Manin triple of differential pre-Lie algebra associated to* $(A, \circ, \Phi)$ *and* $(A^*, \bullet, \Psi^*)$*. Then* Ψ *is admissible to* $(A, \circ, \Phi)$ *and* Φ^* *is admissible to* $(A^*, \bullet, \Psi^*)$*.*

Proof. By Lemma 3.16, $(\Psi + \Phi^*)$ is admissible to $(A \bowtie A^*, A, A^*, \Phi + \Psi^*)$. Then by equations (2.22) and (2.23) where Λ and Φ are replaced respectively by $\Psi + \Phi^*$ and $\Phi + \Psi^*$, we obtain, for any $x, y \in A, x^*, y^* \in A^*$,

$$[x + x^*, (\Psi + \Phi)(y + y^*)]' - [(\Phi + \Psi^*)(x + x^*), y + y^*]' - (\Psi + \Phi^*)([x + x^*, y + y^*]') = 0, \tag{3.15}$$

$$[(\Psi + \Phi^*)(y + y^*), x + x^*]' - [y + y^*, (\Phi + \Psi^*)(x + x^*)]' - (\Psi + \Phi^*)([y + y^*, x + x^*]') = 0. \tag{3.16}$$

Now taking $x^* = y^* = 0$ (resp. $x = y = 0$) in equations (3.15) and (3.16), we obtain the admissibility of Ψ to $(A, [\cdot, \cdot], \Phi)$ (resp. the admissibility of Φ^* to $(A^*, \bullet, \Psi^*)$. □

This theorem establishes the fundamental link between differential Manin triples and matched pairs in the setting of differential pre-Lie algebras.

THEOREM 3.18. *Let $(A,\circ,\Phi)$ be a differential pre-Lie algebra. Suppose that there is a differential pre-Lie algebra structure " $\bullet$ " on its dual space (A^*,Ψ^*). Then, there is a standard Manin triple $(A\bowtie A^*,A,A^*,\Phi+\Psi^*,B_s)$ of $(A,\circ,\Phi)$ and $(A^*,\bullet,\Psi^*)$ if and only if $(A,A^*,\Phi,\Psi^*,L_\circ^*-R_\circ^*,-R_\circ^*,L_\bullet^*-R_\bullet^*,-R_\bullet^*)$ is a matched pair of differential pre-Lie algebras.*

Proof. Let $(A\bowtie A^*,A,A^*,\Phi+\Psi^*,B_s)$ be a standard Manin triple of differential pre-Lie algebra associated to $(A,\circ,\Phi)$ and $(A^*,\bullet,\Psi^*)$, this leads to $(A\bowtie A^*,A,A^*,B_s)$ be a standard Manin triple of pre-Lie algebra associated to $(A,\circ)$ and $(A^*,\bullet)$, Then by Theorem 3.11 $(A,A^*,L_\circ^*-R_\circ^*,-R_\circ^*,L_\bullet^*-R_\bullet^*,-R_\bullet^*)$ is a matched pair of pre-Lie algebras. Further, by lemma 3.16, $(A^*,L_\circ^*-R_\circ^*,-R_\circ^*,\Psi^*)$ is a representation of $(A,\circ,\Phi)$ and $(A,L_\bullet^*-R_\bullet^*,-R_\bullet^*,\Phi)$ is a representation of $(A^*,\bullet,\Psi^*)$. Hence $(A,A^*,\Phi,\Psi^*,L_\circ^*-R*_\circ,-R_\circ^*,L_\bullet^*-R_\bullet^*,-R_\bullet^*)$ is a matched pair of differential pre-Lie algebras.

Conversely, if $(A,A^*,\Phi,\Psi^*,L_\circ^*-R_\circ^*,-R_\circ^*,L_\bullet^*-R_\bullet^*,-R_\bullet^*)$ is a matched pair of differential pre-Lie algebras, this leads to $(A,A^*,L_\circ^*-R_\circ^*,-R_\circ^*,L_\bullet^*-R_\bullet^*,-R_\bullet^*)$ is a matched pair of pre-Lie algebras. Then by Theorem 3.11 $(A\bowtie A^*,A,A^*,B_s)$ be a standard Manin triple of pre-Lie algebra associated to $(A,\circ)$ and $(A^*,\bullet)$, Further, $(A\bowtie A^*,\circ',\Phi+\Psi^*)$ is a differential pre-Lie algebra. Hence $(A\bowtie A^*,A,A^*,\Phi+\Psi^*,B_s)$ is a standard Manin triple of differential pre-Lie algebra associated to $(A,\circ,\Phi)$ and $(A^*,\bullet,\Psi^*)$. □

This result provides an equivalence between three key structures: matched pairs, Manin triples, and differential pre-Lie bialgebras.

THEOREM 3.19. *Let $(A,\circ,\Phi)$ be a differential pre-Lie algebra. Suppose that there is a differential pre-Lie algebra structure " $\bullet$ " on its dual space (A^*,Ψ^*). Then the following assertions are equivalent.*

1. *The 8-tuple $(A,A^*,\Phi,\Psi^*,L_\circ^*-R_\circ^*,-R_\circ^*,L_\bullet^*-R_\bullet^*,-R_\bullet^*)$ is a matched pair of differential pre-Lie algebras.*
2. *There is a standard Manin triple $(A\bowtie A^*,A,A^*,\Phi+\Psi^*)$ of $(A,\circ,\Phi)$ and $(A^*,\bullet,\Psi^*)$.*
3. *The quintuple $(A,\delta,\zeta,\Phi,\Psi)$ is a differential pre-Lie bialgebra.*

Proof. Let $(A,\circ,\Phi)$ be a differential pre-Lie algebra and suppose that its dual space $(A^*,\bullet,\Psi^*)$. We prove that the three statements are equivalent.

$1\iff 3$: By Theorem 3.5, the quintuple $(A,\delta,\zeta,\Phi,\Psi)$ is a differential pre-Lie bialgebra if and only if

$$(A,A^*,\Phi,\Psi^*,L_\circ^*-R_\circ^*,-R_\circ^*,L_\bullet^*-R_\bullet^*,-R_\bullet^*)$$

is a matched pair of differential pre-Lie algebras. Hence, 1 and 3 are equivalent.

$1\iff 2$: By Theorem 3.18, there exists a standard Manin triple $(A\bowtie A^*,A,A^*,\Phi+\Psi^*)$ associated to $(A,\circ,\Phi)$ and $(A^*,\bullet,\Psi^*)$ if and only if $(A,A^*,\Phi,\Psi^*,L_\circ^*-R_\circ^*,-R_\circ^*,L_\bullet^*-R_\bullet^*,-R_\bullet^*)$ is a matched pair of differential pre-Lie algebras. Thus, 1 and 2 are equivalent.

Conclusion: Since $1\iff 3$ and $1\iff 2$, it follows that $1\iff 2\iff 3$. This completes the proof. □

4. Coboundary differential pre-Lie bialgebras

In this section, we study the coboundary differential pre-Lie bialgebras and show that they can be given by symmetric solutions of the Ψ-admissible $\mathcal{S}$-equation in a differential pre-algebra.

Thanks to [1], a pre-Lie bialgebra (A, δ, ζ) is called **coboundary** if there exists an $r \in A \otimes A$ such that

$$\delta(x) := \delta_r(x) := (L_\circ(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))(r), \quad \forall x \in A. \tag{4.1}$$

Definition 4.1. A differential pre-Lie bialgebra $(A, \delta, \zeta, \Phi, \Psi)$ is called **coboundary** if there exists an $r \in A \otimes A$ such that relation (4.1) holds.

Remark 4.2. Let $(A, \circ)$ be a pre-Lie algebra and $r \in A \otimes A$. Then by [1] the map δ defined by (4.1) induces a pre-Lie algebra structure on A^* such that (A, δ, ζ) is a pre-Lie bialgebra if and only if the following two conditions are satisfied for any $x, y \in A$:

$$[\Phi(x \circ y) - \Phi(x)\Phi(y)](r_{12} - r_{21}) = 0; \tag{4.2}$$

$$\Psi(x)[[r, r]] = 0, \tag{4.3}$$

where $[[r, r]] = r_{13} \circ r_{12} - r_{23} \circ r_{21} + [r_{23}, r_{12}] - [r_{13}, r_{21}] - [r_{13}, r_{23}]$, $\Psi(x) = L_\circ(x) \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes L_\circ(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ad}(x)$ and $\Phi(x) = L_\circ(x) \otimes \mathrm{id} + \mathrm{id} \otimes L_\circ(x)$ for any $x \in A$.

Hence in order for $(A, \delta, \zeta, \Phi, \Psi)$ to be a differential pre-Lie bialgebra, we only need to further require that (A^*, δ^*, Ψ^*) is a Φ^*-admissible differential pre-Lie algebra, that is, (A, δ, Ψ) is a differential pre-Lie coalgebra and (2.15), (2.16) hold.

Theorem 4.3. *Let $(A, \circ, \Phi)$ be a Ψ-admissible differential pre-Lie algebra and $r \in A \otimes A$. Define a linear map $\delta : A \to A \otimes A$ by* (4.1)*. Then the following results hold.*

1. Equation (2.12) *holds if and only if for any $x \in A$,*

$$(id \otimes ad(x))(id \otimes \Phi - \Psi \otimes id)(r) - (L_\circ(x) \otimes id)(id \otimes \Psi - \Phi \otimes id)(r) = 0. \tag{4.4}$$

2. Equation (2.15) *holds if and only if for any $x \in A$,*

$$(id \otimes ad(x) + L(x) \otimes id)(\Phi \otimes id - id \otimes \Psi)(r) = 0. \tag{4.5}$$

3. Equation (2.16) *holds if and only if for any $x \in A$,*

$$(id \otimes ad(x) + L_\circ(x) \otimes id)(id \otimes \Phi - \Psi \otimes id)(r) = 0. \tag{4.6}$$

Proof. 1 For any $r = \sum_i x_i \otimes y_i \in A \otimes A$ and $x \in A$, we have

$$\begin{aligned}
&(id \otimes ad(x))(id \otimes \Phi - \Psi \otimes id)(r) - (L_\circ(x) \otimes id)(id \otimes \Psi - \Phi \otimes id)(r)\\
&= \sum_i \Big(x_i \otimes [x, \Phi(y_i)] - \Psi(x_i) \otimes [x, y_i] - (x \circ x_i) \otimes \Psi(y_i) + x \circ \Phi(x_i) \otimes y_i\Big)\\
\text{(by (2.13),(2.14))} &= \sum_i \big(x_i \otimes (\Psi(x) \circ y_i - \Psi(x \circ y_i) + \Psi(y_i \circ x) - y_i \circ \Psi(x))\big)\\
&+ \sum_i \big(\Psi(x) \circ x_i - \Psi(x \circ x_i)) \otimes y_i\big) - \sum_i \big(\Psi(x_i) \otimes [x, y_i] - (x \circ x_i) \otimes \Psi(y_i)\big)\\
&= \sum_i \big(x_i \otimes ([\Psi(x), y_i] - \Psi([x, y_i]) + \Psi(y_i \circ x) - y_i \circ \Psi(x))\big)\\
&+ \sum_i \big((\Psi(x) \circ x_i - \Psi(x \circ x_i)) \otimes y_i\big) - \sum_i \big(\Psi(x_i) \otimes [x, y_i] - (x \circ x_i) \otimes \Psi(y_i)\big)\\
&= \sum_i \big(\Psi(x) \circ x_i) \otimes y_i + x_i \otimes [\Psi(x), y_i])\big)\\
&- \sum_i \big(\Psi(x \circ x_i) \otimes y_i - (x \circ x_i) \otimes \Psi(y_i) - \Psi(x_i) \otimes [x, y_i] - x_i \otimes \Psi([x, y_i])\big)\\
\text{(by (4.1))} &= \delta\Psi(x) - (\Psi \otimes id + id \otimes \Psi)\delta(x)
\end{aligned}$$

Hence (2.12) holds if and only if (4.4) holds. Items 2 and 3 can be proved by the same argument. □

This theorem gives a complete characterization of when the map δ induces a differential pre-Lie bialgebra structure.

THEOREM 4.4. *Let* $(A, \circ, \Phi)$ *be a* Ψ*-admissible differential pre-Lie algebra and* $r \in A \otimes A$*. Then the linear map* δ *defined by* (4.1) *induces a* Φ^**-admissible differential pre-Lie algebra* (A^*, δ^*, Ψ^*) *such that* $(A, \delta, \zeta, \Phi, \Psi)$ *is a differential pre-Lie bialgebra if and only if* (4.2)-(4.6) *are satisfied.*

Proof. Since the differential pre-Lie algebra $(A, \circ, \Phi)$ is Ψ-admissible. Then By Proposition 2.19 the quintuple $(A, \delta, \zeta, \Phi, \Psi)$ is a differential pre-Lie bialgebra if and only if the differential pre-Lie algebra (A^*, δ^*, Ψ^*) is a Φ^*-admissible. Hence, by Proposition 4.3 and Remark 4.2 $(A, \circ, \delta, \zeta, \Phi, \Psi)$ is a differential pre-Lie bialgebra if and only if (4.2)-(4.6) hold. □

As a consequence of Theorem 4.4, we obtain

COROLLARY 4.5. *Let* $(A, \circ, \Phi)$ *be a* Ψ*-admissible differential pre-Lie algebra and* $r \in A \otimes A$*. Then the linear map* δ *defined by* (4.1) *induces a* Φ^**-admissible differential pre-Lie algebra* (A^*, δ^*, Ψ^*) *such that* $(A, \delta, \zeta, \Phi, \Psi)$ *is a differential pre-Lie bialgebra if* (4.2) *and the following equations hold:*

$$- r_{12} \circ r_{13} + r_{12} \circ r_{23} + [r_{13}, r_{23}] = 0, \tag{4.7}$$

$$(\Phi \otimes \mathrm{id} - \mathrm{id} \otimes \Psi)(r) = 0, \tag{4.8}$$

$$(\Psi \otimes \mathrm{id} - \mathrm{id} \otimes \Phi)(r) = 0. \tag{4.9}$$

DEFINITION 4.6. Let $(A, \circ, \Phi)$ be a differential pre-Lie algebra. Suppose that $r \in A \otimes A$ and $\Psi : A \to A$ is a linear map. Then (4.7) with (4.8) and (4.9) is called the **Ψ-admissible $\mathcal{S}$-equations** in $(A, \circ, \Phi)$.

This proposition shows how symmetric solutions of the $\mathcal{S}$-equation naturally yield coboundary differential pre-Lie bialgebras.

PROPOSITION 4.7. *Let* $(A, \circ, \Phi)$ *be a* Ψ*-admissible differential Lie algebra and* $r \in A \otimes A$ *be an symmetric solution of the* Ψ*-admissible* $\mathcal{S}$*-equation in* $(A, \circ, \Phi)$*. Then* $(A, \delta, \zeta, \Phi, \Psi)$ *is a coboundary differential pre-Lie bialgebra, where the linear map* $\delta = \delta_r$ *is defined by* (4.1) *and* ζ *is the dual of the multiplication* $\circ$ *in* A*.*

Proof. Let $(A, \circ, \Phi)$ be a Ψ-admissible differential pre-Lie algebra and let $r \in A \otimes A$ be a symmetric solution of the Ψ-admissible $\mathcal{S}$-equations in $(A, \circ, \Phi)$. By Definition 4.6, the element r satisfies equations (4.7), (4.8), and (4.9). Moreover, since r is symmetric, it also satisfies (4.2). Besides, define the linear map $\delta = \delta_r$ by (4.1). Then, by Corollary 4.5, the map δ induces a Φ^*-admissible differential pre-Lie algebra structure (A^*, δ^*, Ψ^*) on the dual space A^*, and the quintuple $(A, \delta, \zeta, \Phi, \Psi)$ forms a differential pre-Lie bialgebra, where ζ is the dual map of the multiplication $\circ$.

Therefore, $(A, \delta, \zeta, \Phi, \Psi)$ is a coboundary differential pre-Lie bialgebra associated to the element r. This completes the proof. □

Remark 4.8. The above relations of Section 2 and Section 3 can be summarized into the following diagram:

$$
\begin{array}{ccc}
 & & \text{matched pairs of differential} \\
 & & \text{pre-Lie algebras} \\
 & & \Updownarrow \\
\text{admissible solutions} & & \text{differential} \\
\text{of } \mathcal{S}\text{−equation} & \Longrightarrow & \text{pre-Lie bialgebras} \\
 & & \Updownarrow \\
 & & \text{Manin triples of differential} \\
 & & \text{pre-Lie algebras}
\end{array} \tag{4.10}
$$

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