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HYERS-ULAM STABILITY OF FUNCTIONAL EQUATIONS WITH PARAMETERS

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Abstract. This paper explores the Hyers-Ulam stability of parameterized generalized Jensen additive and quadratic functional equations in β-homogeneous F-spaces, showing that approximately satisfying mappings have a unique exact approximating counterpart within a specific bound. The corresponding stability theorems are established with estimates. The results extend existing stability results to the parameterized case, enrich the theory of functional equations in β-homogeneous F-spaces, and provide a concise proof approach for similar equations.

1. Introduction

In 1940, Ulam [33] raised the question of functional equation stability: under what conditions does an approximate homomorphism have a nearby homomorphism? Hyers [2] later solved this for approximate additive mappings in Banach spaces, with subsequent extensions allowing unbounded Cauchy differences. Stabilities for quadratic equations, including those with involution, were also established.

Over the past few decades, the stability of functional equations has been extensively explored by numerous mathematicians (see [17]). Hyers was the first to examine the stability of the Cauchy equation [2], demonstrating that if f is a mapping from a normed space X to a Banach space Y satisfying $\|f(x+y)-f(x)-f(y)\| \leq \varepsilon$ for some $\varepsilon > 0$, then there exists a unique additive mapping $g : X \to Y$ such that $\|f(x)-g(x)\| \leq \varepsilon$.

Let us recall the standard definition of orthogonality by Rätz [11]:

Definition 1.1. Let X be a real linear space with $\dim X \geq 2$, and let $\perp$ denote a binary relation on X satisfying the following conditions:

(1) For all $x \in X$, $x \perp 0$ and $0 \perp x$;

(2) For any non-zero elements $x, y \in X \setminus \{0\}$, if $x \perp y$, then x and y are linearly independent;

(3) For any $x, y \in X$ with $x \perp y$, it holds that $\alpha x \perp \beta y$ for all scalars $\alpha, \beta \in \mathbb{R}$;

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(4) For every two-dimensional subspace $P \subseteq X$, every $x \in P$, and every $\lambda \in [0, \infty)$, there exists some $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

Such an ordered pair $(X, \perp)$ is referred to as an orthogonality space.

Orthogonal functional equations constitute a major nonlinear analysis topic, with extensive studies on their Hyers-Ulam stability in classical spaces. As summarized by Sikorska [13], numerous relevant open problems remain. In 2010, Fechner and Sikorska [37] investigated the stability of orthogonality and introduced an alternative definition of orthogonality, stated as follows:

Definition 1.2. Let X be an Abelian group, and let $\perp$ be a binary relation defined on X with the following properties:

(1) For any $x, y \in X$ with $x \perp y$, the relations $x \perp -y$, $-x \perp y$, and $2x \perp 2y$ hold;

(2) For each $x \in X$, there exists some $y \in X$ such that $x \perp y$ and $x + y \perp x - y$.

With the orthogonality relation $\perp$ defined above as a foundation, they studied an investigated into the following orthogonally quadratic equation:

$$f(x+y) = f(x) + f(y), \quad x \perp y.$$

Skof [8] was the first to study the stability of the quadratic equation. He proved that if f is a mapping from a normed space X to a Banach space Y satisfying

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon$$

for some $\epsilon > 0$, then there exists a unique quadratic function $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{\epsilon}{2}$. Cholewa [25] extended Skof's theorem by substituting X with an Abelian group G. Subsequently, Czerwik [29] generalized Skof's result in the vein of Hyers-Ulam. A number of mathematicians have carried out extensive investigations into the stability problem of functional equations (see [3, 4, 6, 30, 31, 34–36]). The authors presented and examined the Ulam stability issue pertaining to the relative Euler-Lagrange functional equation

$$f(\lambda x + \mu y) + f(\mu x - \lambda y) = \left(\lambda^2 + \mu^2\right)[f(x) + f(y)]$$

in the works [2, 28].

Drljević [9], Fochi [19], Moslehian [20, 21], and Szabó [32] generalized this result.

In recent years, the Ulam stability of diverse mathematical entities, such as functional equations and inequalities, differential, difference and integral equations, has drawn extensive attention from scholars. Researchers have delved into how these constructs maintain stability under perturbations, exploring the resilience of solutions and the existence of near - exact counterparts. This growing interest reflects the significance of Ulam stability in understanding the behavior of mathematical models across various fields, from basic functional relationships to complex dynamic systems described by differential and integral equations.

Suppose X is a real inner product space, and $f : X \to \mathbb{R}$ is a solution to the orthogonal Cauchy functional equation $f(x+y) = f(x) + f(y)$ where $\langle x, y \rangle = 0$. In light of the Pythagorean theorem, $f(x) = \|x\|^2$ constitutes a solution to this conditional equation. Naturally, this function does not satisfy the additivity equation in all cases. Hence, the orthogonal Cauchy equation is not equivalent to the classical Cauchy equation across the entire inner product space. This phenomenon might underscore the importance of studying the orthogonal Cauchy equation. Many authors have also established the Hyers-Ulam stability of special orthogonal quadratic functional equations (see [14], [15]).

R. Ger and J. Sikorska [28] conducted research on the orthogonal stability of the Cauchy functional equation $f(x+y) = f(x) + f(y)$. Specifically, they revealed that if f is a function mapping from an orthogonality space X to a real Banach space Y, and for all $x, y \in X$ with $x \perp y$, the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

holds for some $\epsilon > 0$, then there exists precisely one orthogonally additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\epsilon$ for all $x \in X$.

In the context of functional analysis, it is essential to highlight that all orthogonal spaces, along with normed linear spaces exhibiting isosceles orthogonality, satisfy these conditions. Nevertheless, spaces characterized solely by Pythagorean orthogonality no longer meet these requirements.

Despite the successful execution of numerous studies on stability, the non-linear structure inherent in infinite-dimensional F-spaces has constrained the availability of corresponding stability results. The non-linear nature of F-spaces holds substantial importance in functional analysis and other mathematical domains. A prime illustration is the $L^p([0,1])$ space with $0 < p < 1$, which constitutes an F-space (but not a Banach space) when furnished with the metric $d(f,g) = \int |f(x) - g(x)|^p dx$. For a comprehensive understanding of F-spaces, interested readers are directed to the literature [5, 23].

Definition 1.3. Let X be a linear space. A non-negative function $\|\cdot\|$ is termed an F-norm if it adheres to the subsequent conditions:
(1) $\|x\| = 0$ precisely when $x = 0$;
(2) $\|\lambda x\| = \|x\|$ for all scalars λ with $|\lambda| = 1$;
(3) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;
(4) $\|\lambda_n x\| \to 0$ as $\lambda_n \to 0$;
(5) $\|\lambda x_n\| \to 0$ as $x_n \to 0$;
(6) $\|\lambda_n x_n\| \to 0$ when both $\lambda_n \to 0$ and $x_n \to 0$.

The pair $(X, \|\cdot\|)$ is then defined as an F^*-space. An F-space is an F^*-space that is complete.

Building on such foundational work, subsequent scholars have conducted in-depth research on the stability of various functional equations under diverse spaces and conditions (see [10, 12, 24, 26, 27]). With regard to the stability of β-homogeneous, reference may be made to [12, 16, 26].

An F-norm is said to be β-homogeneous (where $\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and $t \in \mathcal{C}$ (see [7]).

In [18], Fu et al. investigated the stability of orthogonally Jensen additive and quadratic functional equations in F-spaces and quasi-Banach spaces. For a mapping $f : X \to Y$ from an Abelian group X to a β-homogeneous F-space Y, they established stability results for the conditions

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \varepsilon$$

and

$$\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\|\leq\varepsilon$$

which hold for all $x, y \in X$ with $x \perp y$.

The purpose of this study is to investigate the Hyers-Ulam stability of the following parameterized functional equations:

$$f(x+y)=M[f(x)+f(y)], \quad x\perp y$$

where $M \in \mathbb{R} \setminus \{\frac{1}{2}\}$. Here, $\perp$ is the generalized Cauchy orthogonality binary relation defined by:

(a) For any $x \in X$, $0 \perp x$ and $x \perp 0$;

(b) If $x, y \in X$ and $x \perp y$, then $x + y \perp x - y$.

Moreover, we focus on the following parameterized functional equation:

$$f(\lambda x+\lambda y)+f(\lambda x-\lambda y)=2\lambda^2[f(x)+f(y)], \quad x\perp y$$

where $\lambda \in \mathbb{Z}$ satisfies $\left(\frac{1}{\lambda^2}\right)^\beta + \left(\frac{2}{\lambda^2}\right)^\beta \leq 1$, with X being an Abelian group and Y being a β-homogeneous F-space as the underlying space setting. Here, $\perp$ is the binary relation satisfying the following properties:

(a') For any $x \in X$, $0 \perp x$ and $x \perp 0$;

(b') If $x, y \in X$ and $x \perp y$, then $-x \perp -y$, $x \perp -y$.

Although numerous stability results exist for functional equations, none addresses parameterized generalized Jensen additive-quadratic equations in β-homogeneous F-spaces. This paper studies their Hyers-Ulam stability to fill the research gap.

2. Stability of parameterized generalized Jensen additive functional equations

In the investigation of the stability of parameterized functional equations, we present the following lemma, which plays a crucial role in establishing the main results of this paper.

Lemma 2.1. *Let X be an Abelian group, and Y be a β-homogeneous F-space. For $\varepsilon \geq 0$, assume $f : X \to Y$ is a mapping such that for all $x, y \in X$, a constant $C > 0$, an arbitrary integer ω and real numbers α and γ, one has*

$$\|f(x)-\alpha f(\omega x)+\gamma f(-\omega x)\|\leq C. \tag{1}$$

Let $h(x, n) = \|f(x) - A_n f(\omega^n x) + B_n f(-\omega^n x)\|$ where the initial conditions are

$A_{n+1} = \alpha A_n + \gamma B_n$, $B_{n+1} = \gamma A_n + \alpha B_n$, $A_0 = 1$, $B_0 = 0$, $|\alpha+\gamma|^\beta + |\alpha-\gamma|^\beta < 1$, $\beta > 0$ and $g_n(x) = A_n f(\omega^n x) - B_n f(-\omega^n x)$, $n \in \mathbb{N}$. Then we have

(1) $|h(x, n+1) - h(x, n)| \leq C\left[\,|A_n|^\beta + |B_n|^\beta\right]$. Moreover,

$$h(x,n)\leq C\left(\sum_{n=1}^{\infty}\left[\,|A_n|^\beta+|B_n|^\beta\right]+1\right)\quad \text{for all } n\in\mathbb{N}.$$

(2) $\{g_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in X$. Therefore, there exists a mapping $g : X \to Y$, which is defined by

$$g(x):=\lim_{n\to\infty}g_n(x),$$

such that

$$\|f(x)-g(x)\| \leq C\left(\sum_{n=1}^{\infty}\left[\ |A_n|^\beta +\ |B_n|^\beta\right]+1\right) \quad \textit{for all } x \in X.$$

PROOF. First, we have

$$A_n=\frac{(\alpha+\gamma)^n+(\alpha-\gamma)^n}{2}, \quad B_n=\frac{(\alpha+\gamma)^n-(\alpha-\gamma)^n}{2}$$

where $A_{n+1}=\alpha A_n+\gamma B_n,\ B_{n+1}=\gamma A_n+\alpha B_n,\ A_0=1,\ B_0=0.$

So we can get

$$\sum_{n=0}^{\infty}\left(|A_n|^\beta+|B_n|^\beta\right)=\frac{1}{2^\beta}\sum_{n=0}^{\infty}\left[|(\alpha+\gamma)^n+(\alpha-\gamma)^n|^\beta+|(\alpha+\gamma)^n-(\alpha-\gamma)^n|^\beta\right]$$

and then

$$\begin{aligned}|A_n|^\beta+|B_n|^\beta &\leq (|\alpha+\gamma|^n+|\alpha-\gamma|^n)^\beta+||\alpha+\gamma|^n-|\alpha-\gamma|^n|^\beta\\ &\leq\ \max\left\{|\alpha+\gamma|^{n\beta},|\alpha-\gamma|^{n\beta}\right\}.\end{aligned}$$

Since $|\alpha+\gamma|^\beta+|\alpha-\gamma|^\beta<1,\ \max\left\{|\alpha+\gamma|^\beta,|\alpha-\gamma|^\beta\right\}<1.$

Let $r=\max\left\{|\alpha+\gamma|^\beta,|\alpha-\gamma|^\beta\right\}$. Then we have that $|A_n|^\beta+|B_n|^\beta\leq r^n$ and

$$\sum_{n=0}^{\infty}\left(|A_n|^\beta+|B_n|^\beta\right)\leq\sum_{n=0}^{\infty}r^n, \quad r<1.$$

Thus the series $\sum_{n=1}^{\infty}\left(|A_n|^\beta+|B_n|^\beta\right)$ converges.

Then, with the help of (1), through a simple estimate, we obtain

$$\begin{aligned}&\left\|f(x)-A_{n+1}f\left(\omega^{n+1}x\right)+B_{n+1}f\left(-\omega^{n+1}x\right)\right\|\\ &\leq\left\|f(x)-A_nf\left(\omega^nx\right)+B_nf\left(-\omega^{n+}x\right)\right\|+|A_n|^\beta\left\|f\left(\omega^nx\right)-\alpha f\left(\omega^{n+1}x\right)+\gamma f\left(-\omega^{n+1}x\right)\right\|\\ &\quad+|B_n|^\beta\left\|f\left(-\omega^nx\right)-\alpha f\left(-\omega^{n+1}x\right)+\gamma f\left(\omega^{n+1}x\right)\right\|\\ &\leq\|f(x)-A_nf\left(\omega^nx\right)+B_nf\left(-\omega^nx\right)\|+C\left[|A_n|^\beta+|B_n|^\beta\right],\end{aligned}$$

which implies

$$\begin{aligned}&\left\|f(x)-A_nf\left(\omega^{n+1}x\right)+B_nf\left(-\omega^{n+1}x\right)\right\|-\|f(x)-A_nf\left(\omega^nx\right)+B_nf\left(-\omega^nx\right)\|\\ &\leq C\left[|A_n|^\beta+|B_n|^\beta\right].\end{aligned}$$

Now let $h(x,n)=\|f(x)-A_nf\left(\omega^nx\right)+B_nf\left(-\omega^nx\right)\|$. Then we have $|h(x,n+1)-h(x,n)|\leq C\left(|A_n|^\beta+|B_n|^\beta\right)$, and so

$$\begin{aligned}h(x,n)\ &=\ \sum_{i=2}^{n}\big(h(x,i)-h(x,i-1)\big)+h(x,1)\\ &\leq\ C\left[\sum_{n=1}^{\infty}\left(|A_n|^\beta+|B_n|^\beta\right)+1\right].\end{aligned} \tag{2}$$

This means that

$$|f(x)-A_nf\left(\omega^nx\right)+B_nf\left(-\omega^nx\right)|\leq C\left[\sum_{n=1}^{\infty}\left[|A_n|^\beta+|B_n|^\beta\right]+1\right], \quad x\in X.$$

The next step is to prove that for each $x \in X$ the sequence

$$g_n(x) := A_n f\left(\omega^n x\right) - B_n f\left(-\omega^n x\right), \quad n \in \mathbb{N}$$

is convergent in Y. We have

$$\begin{aligned}\|g_n(x) - g_{n+1}(x)\| &= \big\| A_n \left(f\left(\omega^n x\right) - \alpha f\left(\omega^{n+1} x\right) + \gamma f\left(-\omega^{n+1} x\right)\right) \\ &\quad - B_n \left(f\left(-\omega^n x\right) - \alpha f\left(-\omega^{n+1} x\right) + \gamma f\left(\omega^{n+1} x\right)\right)\big\| \\ &\le C\left[|A_n|^\beta + |B_n|^\beta\right]\end{aligned}$$

for each $n \in \mathbb{N}$, which gives us that $\{g_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since Y is complete, the mapping $g : X \to Y$ can be defined as:

$$g(x) := \lim_{n\to\infty} g_n(x)$$

for all $x \in X$. Combining with (2) we have

$$\|f(x) - g(x)\| \le C\left(\sum_{n=1}^{\infty}\left[|A_n|^\beta + |B_n|^\beta\right] + 1\right)$$

for all $x \in X$. $\square$

REMARK 2.2. Since X is an Abelian group, without loss of generality, we consider that ω is an integer. Similarly, in the subsequent theorems, λ is required to be an integers for the same reason.

Next, we would like to show the main result of this section: since the parameterized generalized Jensen additive functional equation will be discussed next, we introduce the definition of generalized Cauchy orthogonality, stated as follows:

DEFINITION 2.3. Let X be an Abelian group, and let $\perp$ be a generalized Cauchy orthogonality binary relation on X with the properties:
(a) For any $x \in X$, $0 \perp x$ and $x \perp 0$;
(b) If $x, y \in X$ and $x \perp y$, then $x + y \perp x - y$.

THEOREM 2.4. *Let X be an Abelian group, and Y be a β-homogeneous F-space. For $\varepsilon \ge 0$, assume $f : X \to Y$ be a mapping such that for all $x, y \in X$ with $x \perp y$, $M \in \mathbb{R} \setminus \{\frac{1}{2}\}$, the following holds:*

$$\|f(x+y) - M[f(x) + f(y)]\| \le \varepsilon. \tag{3}$$

Then there exists a mapping $g : X \to Y$ such that

$$g(x+y) = M[g(x) + g(y)] \tag{4}$$

for all $x, y \in X$ with $x \perp y$ and

$$\|f(x) - g(x)\| \le \begin{cases} \left(\sum_{n=1}^{\infty}\left[\left(\frac{2^n+1}{2\cdot 4^n}\right)^\beta + \left(\frac{2^n-1}{2\cdot 4^n}\right)^\beta\right] + 1\right) \\ \cdot\left[1 + \left(\frac{3}{8}\right)^\beta + \left(\frac{1}{8}\right)^\beta\right] \frac{M + |1-2M|^\beta}{|1-M|^\beta|1-2M|^\beta}\varepsilon, M \ne 1, \\ \\ \left(\sum_{n=1}^{\infty}\left[\left(\frac{2^n+1}{2\cdot 4^n}\right)^\beta + \left(\frac{2^n-1}{2\cdot 4^n}\right)^\beta\right] + 1\right) \\ \cdot\frac{2^{\beta+2} + 3^{\beta+1} + 3}{8^\beta}\varepsilon, M = 1 \end{cases} \tag{5}$$

for all $x \in 2X = \{2x \mid x \in X\}$. Moreover, the mapping g is unique on the set $2X$.

Proof. For all $x \in X$, we have $0 \perp 0$, $0 \perp x$ and $x \perp 0$. So according to $0 \perp 0$ and (3), we have

$$\|f(0) - 2Mf(0)\| \leq \varepsilon.$$

Then we can obtain

$$\|f(0)\| \leq \frac{\varepsilon}{|1-2M|^{\beta}}. \tag{6}$$

(i) When $M \neq 1$, according to $x \perp 0$, we can write

$$\|f(x) - Mf(x) - Mf(0)\| \leq \varepsilon,$$

which implies

$$\|f(x)\| \leq \frac{M + |1-2M|^{\beta}}{|1-M|^{\beta}\,|1-2M|^{\beta}} \cdot \varepsilon,$$

so it is easy to get

$$\begin{aligned}\left\|f(x) - \frac{3}{8}f(2x) + \frac{1}{8}f(-2x)\right\| &\leq \|f(x)\| + \left\|\frac{3}{8}f(2x)\right\| + \left\|\frac{1}{8}f(-2x)\right\| \\ &\leq \left[1 + \left(\frac{3}{8}\right)^{\beta} + \left(\frac{1}{8}\right)^{\beta}\right] \frac{M + |1-2M|^{\beta}}{|1-M|^{\beta}\,|1-2M|^{\beta}}\varepsilon.\end{aligned}$$

According to Lemma 2.1, we can easily get

$$\begin{aligned}&\left\|f(x) - \frac{2^n+1}{2\cdot 4^n}f(2^n x) + \frac{2^n-1}{2\cdot 4^n}f(-2^n x)\right\| \\ &\leq \left(\sum_{n=1}^{\infty}\left[\left(\frac{2^n+1}{2\cdot 4^n}\right)^{\beta} + \left(\frac{2^n-1}{2\cdot 4^n}\right)^{\beta}\right] + 1\right) \\ &\quad \left[1 + \left(\frac{3}{8}\right)^{\beta} + \left(\frac{1}{8}\right)^{\beta}\right] \frac{M + |1-2M|^{\beta}}{|1-M|^{\beta}\,|1-2M|^{\beta}}\varepsilon\end{aligned} \tag{7}$$

for all $x \in X$ and $n \in \mathbb{N}$.

Moreover, for each $x \in X$, the sequence

$$g_n(x) := \frac{2^n+1}{2\cdot 4^n}f(2^n x) - \frac{2^n-1}{2\cdot 4^n}f(-2^n x), \quad n \in \mathbb{N}$$

is convergent in Y.

Hence, the mapping

$$g(x) := \lim_{n\to\infty} g_n(x)$$

is defined for each $x \in X$. Combining with (7), we have

$$\begin{aligned}\|f(x) - g(x)\| \leq &\left(\sum_{n=1}^{\infty}\left[\left(\frac{2^n+1}{2\cdot 4^n}\right)^{\beta} + \left(\frac{2^n-1}{2\cdot 4^n}\right)^{\beta}\right] + 1\right) \\ &\left[1 + \left(\frac{3}{8}\right)^{\beta} + \left(\frac{1}{8}\right)^{\beta}\right] \frac{M + |1-2M|^{\beta}}{|1-M|^{\beta}\,|1-2M|^{\beta}}\varepsilon\end{aligned}$$

for all $x \in X$.

In order to prove that g is orthogonally additive, observe first that for all $x, y \in X$ such that $x \perp y$ and $n \in \mathbb{N}$, $n > 1$, we have

$$\begin{aligned}
&\|g_n(x+y) - M\left[g_n(x) + g_n(y)\right]\| \\
&= \Bigg\| \frac{2^n+1}{2\cdot 4^n} f\left(2^n(x+y)\right) - \frac{2^n-1}{2\cdot 4^n} f\left(-2^n(x+y)\right) \\
&\quad - M\left[\frac{2^n+1}{2\cdot 4^n} f\left(2^n x\right) - \frac{2^n-1}{2\cdot 4^n} f\left(-2^n x\right)\right] - M\left[\frac{2^n+1}{2\cdot 4^n} f\left(2^n y\right) - \frac{2^n-1}{2\cdot 4^n} f\left(-2^n y\right)\right]\Bigg\| \\
&= \Bigg\| \frac{2^n+1}{2\cdot 4^n}\left[f\left(2^n(x+y)\right) - Mf\left(2^n x\right) - Mf\left(2^n y\right)\right] \\
&\quad - \frac{2^n-1}{2\cdot 4^n}\left[f\left(-2^n(x+y)\right) - Mf\left(-2^n x\right) - Mf\left(-2^n y\right)\right]\Bigg\| \\
&\le \left(\frac{2^n+1}{2\cdot 4^n}\right)^\beta \|f\left(2^n(x+y)\right) - Mf\left(2^n x\right) - Mf\left(2^n y\right)\| \\
&\quad + \left(\frac{2^n-1}{2\cdot 4^n}\right)^\beta \|f\left(-2^n(x+y)\right) - Mf\left(-2^n x\right) - Mf\left(-2^n y\right)\| \\
&\le \left[\left(\frac{2^n+1}{2\cdot 4^n}\right)^\beta + \left(\frac{2^n-1}{2\cdot 4^n}\right)^\beta\right]\varepsilon.
\end{aligned}$$

Moreover, letting $n \to \infty$, we get (4).

Now, we show the uniqueness of g. Assuming g' as another mapping satisfying (4) and (5), we have

$$\begin{aligned}
\|g(x) - g'(x)\| &\le \|g(x) - f(x)\| + \|g'(x) - f(x)\| \\
&\le 2\left(\sum_{n=1}^{\infty}\left[\left(\frac{2^n+1}{2\cdot 4^n}\right)^\beta + \left(\frac{2^n-1}{2\cdot 4^n}\right)^\beta\right] + 1\right) \\
&\quad \cdot \left[1 + \left(\frac{3}{8}\right)^\beta + \left(\frac{1}{8}\right)^\beta\right] \frac{M + |1-2M|^\beta}{|1-M|^\beta\,|1-2M|^\beta}\varepsilon
\end{aligned}$$

for all $x \in 2X$.

On the other hand, the mapping $g - g'$ satisfies (4) and thus, in particular, (3) with $\varepsilon = 0$. By applying (7) to $g - g'$, we see that

$$g(2x) - g'(2x) = \left(\frac{2^n+1}{2\cdot 4^n}\right)\left[g\left(2^{n+1}x\right) - g'\left(2^{n+1}x\right)\right] - \left(\frac{2^n-1}{2\cdot 4^n}\right)\left[g\left(-2^{n+1}x\right) - g'\left(-2^{n+1}x\right)\right]$$

and therefore

$$\begin{aligned}
&\|g(2x) - g'(2x)\| \\
&\le \left(\frac{2^n+1}{2\cdot 4^n}\right)^\beta \|g(2^{n+1}x) - g'(2^{n+1}x)\| + \left(\frac{2^n-1}{2\cdot 4^n}\right)^\beta \|g(-2^{n+1}x) - g'(-2^{n+1}x)\| \\
&\le 2\left(\sum_{n=1}^{\infty}\left[\left(\frac{2^n+1}{2\cdot 4^n}\right)^\beta + \left(\frac{2^n-1}{2\cdot 4^n}\right)^\beta\right] + 1\right) \cdot \left[1 + \left(\frac{3}{8}\right)^\beta + \left(\frac{1}{8}\right)^\beta\right] \frac{M + |1-2M|^\beta}{|1-M|^\beta\,|1-2M|^\beta}\varepsilon
\end{aligned}$$

for all $x \in X$.

(ii) When $M = 1$, by Definition 2.3, we get immediately there exists a $y \in X$ such that $x \perp y, (x+y) \perp (x-y)$. Moreover, we also have $\pm\frac{1}{2}x \perp \pm\frac{1}{2}y$, $\pm(x+y) \perp \pm(x-y)$. We can conclude that

$$\begin{aligned}
&\left\| f(x) - \frac{3}{8}f(2x) + \frac{1}{8}f(-2x) \right\| \\
&= \frac{1}{8^\beta} \left\| 8f(x) - 3f(2x) + f(-2x) \right\| \\
&= \frac{1}{8^\beta} \Bigg\| 3\left[f(2x) - f(x+y) - f(x-y)\right] + \left[f(-x+y) + f(-x-y) - f(-2x)\right] \\
&\quad + 3\left[f(x+y) - f(x) - f(y)\right] + 3\left[f(x-y) - f(x) - f(-y)\right] \\
&\quad + \left[f(-x) + f(y) - f(-x+y)\right] + \left[f(-x) + f(-y) - f(-x-y)\right] \\
&\quad + 2\left[f(y) - f\left(\frac{y-x}{2}\right) - f\left(\frac{y+x}{2}\right)\right] + 2\left[f(-y) - f\left(\frac{-y-x}{2}\right) - f\left(\frac{-y+x}{2}\right)\right] \\
&\quad + 2\left[f\left(\frac{y-x}{2}\right) + f\left(\frac{-y-x}{2}\right) - f(-x)\right] + 2\left[f\left(\frac{-y+x}{2}\right) + f\left(\frac{-y-x}{2}\right) - f(x)\right] \Bigg\| \\
&\le \frac{2^{\beta+2} + 3^{\beta+1} + 3}{8^\beta}\varepsilon, \quad x \in X.
\end{aligned}$$

According to Lemma 2.1, we can easily get

$$\begin{aligned}
&\left\| f(x) - \frac{2^n+1}{2\cdot 4^n}f(2^n x) + \frac{2^n-1}{2\cdot 4^n}f(-2^n x) \right\| \\
&\le \left(\sum_{n=1}^{\infty} \left[\left(\frac{2^n+1}{2\cdot 4^n}\right)^\beta + \left(\frac{2^n-1}{2\cdot 4^n}\right)^\beta \right] + 1 \right) \cdot \frac{2^{\beta+2} + 3^{\beta+1} + 3}{8^\beta}\varepsilon
\end{aligned} \tag{8}$$

for all $x \in X$ and $n \in \mathbb{N}$.

Moreover, for each $x \in X$, the sequence

$$g_n(x) := \frac{2^n+1}{2\cdot 4^n}f(2^n x) - \frac{2^n-1}{2\cdot 4^n}f(-2^n x), \quad n \in \mathbb{N}$$

is convergent in Y.

Hence, the mapping

$$g(x) := \lim_{n\to\infty} g_n(x)$$

is defined for each $x \in X$. Combining with (8), we have

$$\|f(x) - g(x)\| \le \left(\sum_{n=1}^{\infty} \left[\left(\frac{2^n+1}{2\cdot 4^n}\right)^\beta + \left(\frac{2^n-1}{2\cdot 4^n}\right)^\beta \right] + 1 \right) \frac{2^{\beta+2} + 3^{\beta+1} + 3}{8^\beta}\varepsilon$$

for all $x \in X$.

In order to prove that g is orthogonally additive, observe first that for all $x, y \in X$ such that $x \perp y$ and $n \in \mathbb{N}$, $n > 1$, similarly to (1), we have

$$\|g_n(x+y) - [g_n(x) + g_n(y)]\| \le \left[\left(\frac{2^n+1}{2\cdot 4^n}\right)^\beta + \left(\frac{2^n-1}{2\cdot 4^n}\right)^\beta \right] \varepsilon.$$

Moreover, letting $n \to \infty$, we get (4).

Now, we show the uniqueness of g. Assuming g' as another mapping satisfying (4) and (5), we have

$$\begin{aligned}\|g(x)-g'(x)\| &\le \|g(x)-f(x)\|+\|g'(x)-f(x)\| \\ &\le 2\left(\sum_{n=1}^{\infty}\left[\left(\frac{2^n+1}{2\cdot 4^n}\right)^{\beta}+\left(\frac{2^n-1}{2\cdot 4^n}\right)^{\beta}\right]+1\right)\cdot\frac{2^{\beta+2}+3^{\beta+1}+3}{8^{\beta}}\varepsilon\end{aligned}$$

for all $x\in 2X$.

On the other hand, the mapping $g-g'$ satisfies (4) and thus, in particular, (3) with $\varepsilon=0$. By applying (8) to $g-g'$, we see that

$$g(2x)-g'(2x)=\left(\frac{2^n+1}{2\cdot 4^n}\right)\left[g\left(2^{n+1}x\right)-g'\left(2^{n+1}x\right)\right]-\left(\frac{2^n-1}{2\cdot 4^n}\right)\left[g\left(-2^{n+1}x\right)-g'\left(-2^{n+1}x\right)\right]$$

and therefore

$$\begin{aligned}&\|g(2x)-g'(2x)\| \\ &\le\left(\frac{2^n+1}{2\cdot 4^n}\right)^{\beta}\|g(2^{n+1}x)-g'(2^{n+1}x)\|+\left(\frac{2^n-1}{2\cdot 4^n}\right)^{\beta}\|g(-2^{n+1}x)-g'(-2^{n+1}x)\| \\ &\le 2\frac{2^{\beta+2}+3^{\beta+1}+3}{8^{\beta}}\cdot\left(\sum_{n=1}^{\infty}\left[\left(\frac{2^n+1}{2\cdot 4^n}\right)^{\beta}+\left(\frac{2^n-1}{2\cdot 4^n}\right)^{\beta}\right]+1\right)\varepsilon\end{aligned}$$

for all $x\in X$.

Combining both inequalities, we can easily get the result. □

3. Stability of parameterized generalized quadratic functional equations

In this section, let X be an Abelian group and let $\perp$ be a binary relation defined on X with the properties:

(a') For any $x\in X$, $0\perp x$ and $x\perp 0$;

(b') If $x,y\in X$ and $x\perp y$, then $-x\perp -y$, $x\perp -y$.

THEOREM 3.1. *Let X be an Abelian group, and Y be a β-homogeneous F-space. For $\varepsilon\ge 0$, assume that $f:X\to Y$ is a mapping such that for all $x,y\in X$ with $x\perp y$, where $\lambda\in\mathbb{Z}$ satisfies $\left(\frac{1}{\lambda^2}\right)^{\beta}+\left(\frac{2}{\lambda^2}\right)^{\beta}\le 1$, the following holds:*

$$\left\|f(\lambda x+\lambda y)+f(\lambda x-\lambda y)-2\lambda^2\left[f(x)+f(y)\right]\right\|\le\varepsilon. \tag{9}$$

Then there exists a mapping $g:X\to Y$ such that $x\perp y$ implies

$$g(\lambda x+\lambda y)+g(\lambda x-\lambda y)=2\lambda^2\left[g(x)+g(y)\right] \tag{10}$$

and

$$\|f(x)-g(x)\|\le\left[\frac{3}{(2\lambda^2)^{\beta}}-\frac{2\lambda^2}{|4\lambda^2-8|^{\beta}}\right]\cdot\left(\sum_{n=1}^{\infty}\left[\left|\frac{2^n+1}{4^n\lambda^{2n}}\right|^{\beta}+\left|\frac{2^n-1}{4^n\lambda^{2n}}\right|^{\beta}\right]+1\right)\varepsilon \tag{11}$$

for all $x\in 2\lambda X=\{2\lambda x\mid x\in X\}$. Moreover, the mapping g is unique on the set $2\lambda X$.

PROOF. For all $x\in X$, we have $0\perp 0$, $0\perp x$ and $x\perp 0$. So according to $0\perp 0$ and (9), we have

$$\left\|f(0)+f(0)-4\lambda^2 f(0)\right\|\le\varepsilon.$$

Then we can obtain

$$\|f(0)\| \leq \frac{\varepsilon}{|2-4\lambda^2|^\beta}. \tag{12}$$

According to $x \perp 0$, we can write

$$\big\|f(\lambda x) + f(\lambda x) - 2\lambda^2[f(x) + f(0)]\big\| \leq \varepsilon.$$

It is easy to get

$$\big\|2f(\lambda x) - 2\lambda^2 f(x)\big\| \leq \left[\frac{-2\lambda^2}{|2-4\lambda^2|^\beta} + 1\right]\varepsilon.$$

Since $0 \perp -x$, we have

$$\big\|f(-\lambda x) + f(\lambda x) - 2\lambda^2[f(-x) + f(0)]\big\| \leq \varepsilon.$$

Combining with (12), we obtain

$$\big\|f(-\lambda x) + f(\lambda x) - 2\lambda^2 f(-x)\big\| \leq \left[\frac{-2\lambda^2}{|2-4\lambda^2|^\beta} + 1\right]\varepsilon.$$

So it is easy to get

$$\begin{aligned}
&\left\|f(x) - \frac{3}{2\lambda^2} f(\lambda x) + \frac{1}{2\lambda^2} f(-\lambda x)\right\| \\
&= \left\|\frac{1}{2\lambda^2}\left[\big(2f(\lambda x) - 2\lambda^2 f(x)\big) + \big(f(\lambda x) + f(-\lambda x) - 2\lambda^2 f(x)\big) \; - \big(2f(-\lambda x) - 2\lambda^2 f(-x)\big)\right]\right\| \\
&\leq \left[\frac{3}{(2\lambda^2)^\beta} - \frac{2\lambda^2}{|4\lambda^2 - 8|^\beta}\right]\varepsilon.
\end{aligned}$$

According to Lemma 2.1, we can easily get

$$\begin{aligned}
&\left\|f(x) - \frac{2^n+1}{4^n\lambda^{2n}} f((2\lambda)^n x) + \frac{2^n-1}{4^n\lambda^{2n}} f\left(-(2\lambda)^n x\right)\right\| \\
&\leq \left[\frac{3}{(2\lambda^2)^\beta} - \frac{2\lambda^2}{|4\lambda^2 - 8|^\beta}\right] \cdot \left(\sum_{n=1}^{\infty}\left[\left|\frac{2^n+1}{4^n\lambda^{2n}}\right|^\beta + \left|\frac{2^n-1}{4^n\lambda^{2n}}\right|^\beta\right] + 1\right)\varepsilon
\end{aligned} \tag{13}$$

for all $x \in X$ and $n \in \mathbb{N}$.

Moreover, for each $x \in X$, the sequence $\{g_n(x)\}$, defined by

$$g_n(x) := \frac{2^n+1}{4^n\lambda^{2n}} f\left((2\lambda)^n x\right) - \frac{2^n-1}{4^n\lambda^{2n}} f\left(-(2\lambda)^n x\right), \quad n \in \mathbb{N}$$

is convergent in Y.

Hence the mapping

$$g(x) := \lim_{n\to\infty} g_n(x)$$

is defined for all $x \in X$. Combining with (13), we have

$$\|f(x) - g(x)\| \leq \left[\frac{3}{(2\lambda^2)^\beta} - \frac{2\lambda^2}{|4\lambda^2 - 8|^\beta}\right] \cdot \left(\sum_{n=1}^{\infty}\left[\left|\frac{2^n+1}{4^n\lambda^{2n}}\right|^\beta + \left|\frac{2^n-1}{4^n\lambda^{2n}}\right|^\beta\right] + 1\right)\varepsilon, \quad \forall x \in X.$$

In order to prove that g is orthogonally additive, observe first that for $x, y \in X$ such that $x \perp y$ and $n \in \mathbb{N}$, $n > 1$, we have

$$\big\|g(\lambda x + \lambda y) + g(\lambda x - \lambda y) - 2\lambda^2\big[g(x) + g(y)\big]\big\|$$

$$
\begin{aligned}
&= \Bigg\| \Bigg[\frac{2^n+1}{4^n\lambda^{2n}} f\big((2\lambda)^n(\lambda x+\lambda y)\big) - \frac{2^n-1}{4^n\lambda^{2n}} f\big(-(2\lambda)^n(\lambda x+\lambda y)\big)\Bigg] \\
&\quad + \Bigg[\frac{2^n+1}{4^n\lambda^{2n}} f\big((2\lambda)^n(\lambda x-\lambda y)\big) - \frac{2^n-1}{4^n\lambda^{2n}} f\big(-(2\lambda)^n(\lambda x-\lambda y)\big)\Bigg] \\
&\quad - 2\lambda^2\Bigg[\Big(\frac{2^n+1}{4^n\lambda^{2n}} f((2\lambda)^n x) - \frac{2^n-1}{4^n\lambda^{2n}} f(-(2\lambda)^n x)\Big) \\
&\quad + \Big(\frac{2^n+1}{4^n\lambda^{2n}} f((2\lambda)^n y) - \frac{2^n-1}{4^n\lambda^{2n}} f(-(2\lambda)^n y)\Big)\Bigg]\Bigg\| \\
&= \Bigg\| \frac{2^n+1}{4^n\lambda^{2n}} \Bigg[f\big((2\lambda)^n(\lambda x+\lambda y)\big) + f\big((2\lambda)^n(\lambda x-\lambda y)\big) - 2\lambda^2\big(f((2\lambda)^n x) \\
&\quad + f((2\lambda)^n y)\big)\Bigg] - \frac{2^n-1}{4^n\lambda^{2n}} \Bigg[f\big(-(2\lambda)^n(\lambda x+\lambda y)\big) + f\big(-(2\lambda)^n(\lambda x-\lambda y)\big) \\
&\quad - 2\lambda^2\big(f(-(2\lambda)^n x) + f(-(2\lambda)^n y)\big)\Bigg]\Bigg\| \\
&\le \left|\frac{2^n+1}{4^n\lambda^{2n}}\right|^\beta \Big\| f\big((2\lambda)^n(\lambda x+\lambda y)\big) + f\big((2\lambda)^n(\lambda x-\lambda y)\big) - 2\lambda^2\big(f((2\lambda)^n x) \\
&\quad + f((2\lambda)^n y)\big)\Big\| + \left|\frac{2^n-1}{4^n\lambda^{2n}}\right|^\beta \Big\| f\big(-(2\lambda)^n(\lambda x+\lambda y)\big) + f\big(-(2\lambda)^n(\lambda x-\lambda y)\big) \\
&\quad - 2\lambda^2\big(f(-(2\lambda)^n x) + f(-(2\lambda)^n y)\big)\Big\| \\
&\le \left(\left|\frac{2^n+1}{4^n\lambda^{2n}}\right|^\beta + \left|\frac{2^n-1}{4^n\lambda^{2n}}\right|^\beta\right)\varepsilon.
\end{aligned}
$$

Moreover, letting $n \to \infty$, we get (10).

Now, we show the uniqueness of g. Assuming g' as another mapping satisfying (10) and (11), we have

$$
\begin{aligned}
\|g(x) - g'(x)\| &\le \|g(x) - f(x)\| + \|g'(x) - f(x)\| \\
&\le 2\left[\frac{3}{(2\lambda^2)^\beta} - \frac{2\lambda^2}{|4\lambda^2-8|^\beta}\right] \cdot \left(\sum_{n=1}^{\infty}\left[\left|\frac{2^n+1}{4^n\lambda^{2n}}\right|^\beta + \left|\frac{2^n-1}{4^n\lambda^{2n}}\right|^\beta\right] + 1\right)\varepsilon
\end{aligned}
$$

for all $x \in 2\lambda X$.

On the other hand, the mapping $g - g'$ satisfies (10) and thus, in particular, (9) with $\varepsilon = 0$. By applying (13) to $g - g'$, we see that

$$
\begin{aligned}
&g(2\lambda x) - g'(2\lambda x) \\
&= \frac{2^n+1}{4^n\lambda^{2n}}\left[g\left((2\lambda)^{n+1}x\right) - g'\left((2\lambda)^{n+1}x\right)\right] - \frac{2^n-1}{4^n\lambda^{2n}}\left[g\left(-(2\lambda)^{n+1}x\right) - g'\left(-(2\lambda)^{n+1}x\right)\right]
\end{aligned}
$$

and therefore

$$
\|g(2\lambda x) - g'(2\lambda x)\|
$$

$$\leq \left|\frac{2^n+1}{4^n\lambda^{2n}}\right|^\beta \left\|g\left((2\lambda)^{n+1}x\right) - g'\left((2\lambda)^{n+1}x\right)\right\| + \left|\frac{2^n-1}{4^n\lambda^{2n}}\right|^\beta \left\|g\left(-(2\lambda)^{n+1}x\right) - g'\left(-(2\lambda)^{n+1}x\right)\right\|$$

$$\leq 2\left[\frac{3}{(2\lambda^2)^\beta} - \frac{2\lambda^2}{|4\lambda^2-8|^\beta}\right]\cdot\left(\sum_{n=1}^{\infty}\left[\left|\frac{2^n+1}{4^n\lambda^{2n}}\right|^\beta + \left|\frac{2^n-1}{4^n\lambda^{2n}}\right|^\beta\right]+1\right)\varepsilon.$$

for $x \in X$.

Combining these two inequalities, we can easily get the result. $\square$

4. Conclusion

This paper investigates the Hyers-Ulam stability of parameterized generalized Jensen additive and quadratic functional equations in β-homogeneous F-spaces. The corresponding stability theorems are established and estimates are derived. The work extends existing results to the parameterized case and enriches the stability theory of functional equations in β-homogeneous F-spaces.

5. Future work

In future research, we can extend our results to more general functional equations, study their hyperstability, and apply the theory to the stability analysis of dynamic systems, optimization problems, and engineering mathematical models.

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