Korean J. Math. **34** (2026), No. 3, pp. 409–422
https://dx.doi.org/10.11568/kjm.2026.34.3.409

A DOUBLE INERTIAL EXTRAGRADIENT METHOD FOR FINITE FAMILIES OF MAPPINGS AND ITS APPLICATION TO VARIATIONAL INEQUALITIES

Prashant Patel* and Rahul Shukla

Abstract. This paper investigates the problem of finding a common solution to a variational inequality problem involving a pseudo-monotone operator and a fixed point problem for a finite family of quasi-nonexpansive mappings in real Hilbert spaces. We introduce a new double inertial Mann-type extragradient algorithm that incorporates a non-monotonic self-adaptive step size, eliminating the need for prior knowledge of the operator's Lipschitz constant. A weak convergence theorem is established for the proposed iterative method under standard assumptions. Our work extends and generalizes recent results in the literature, particularly those of Singh and Chandok (2024), to the more general setting of a finite family of mappings.

1. Introduction

Equilibrium is a foundational concept across numerous disciplines such as physics, economics, operations research, and engineering. Variational inequality theory has emerged as a powerful and unifying framework for modeling and solving a wide array of equilibrium problems [4, 7]. With its aid, one can formulate diverse problems, qualitatively analyze them in terms of existence and uniqueness of solutions, perform sensitivity analysis, and develop efficient computational algorithms with rigorous convergence guarantees. This framework elegantly encapsulates many well-known mathematical programming problems, including systems of nonlinear equations, optimization problems, complementarity problems, and fixed point problems as special cases [5, 13, 14, 19].

Let Σ be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, and let $\mathcal{E}$ be a nonempty, closed, and convex subset of Σ. The classical Variational Inequality Problem (VIP), in the sense of Stampacchia [19], is to find a point $\zeta^* \in \mathcal{E}$ such that

$$\langle G(\zeta^*), \zeta - \zeta^* \rangle \geq 0, \quad \forall \zeta \in \mathcal{E}. \tag{1}$$

The VIP (1) signifies that the operator $G(\zeta^*)$ is normal to the feasible set $\mathcal{E}$ at the solution point ζ^*.

Received February 7, 2026. Revised May 31, 2026. Accepted June 13, 2026.

2010 Mathematics Subject Classification: 47H09, 47H10.

Key words and phrases: Variational Inequality Problem, Quasi-nonexpansive mapping, Fixed Point Problem.

∗ Corresponding author.

One of the most fundamental methods for solving the VIP is the projected gradient method. However, its convergence relies on restrictive assumptions, such as the strong monotonicity of the underlying mapping G. To mitigate this strong requirement, Korpelevich [8] introduced the Extragradient Method (EGM) for monotone VIPs in Euclidean spaces. The EGM requires two projections onto the feasible set $\mathcal{E}$ per iteration. In many practical scenarios, the projection onto an arbitrary closed convex set can be computationally expensive. This limitation has spurred significant research into reducing the number of projections. Two prominent approaches are the Tseng's Extragradient Method (TEGM) [23], which replaces the second projection with a cost-effective gradient step, and the Subgradient Extragradient Method (SEGM) [2], which projects onto a readily computable half-space. Recently, Patel and Shukla presented shrinking projection method to approximate common hierarchical fixed points of a countable family of strict pseudocontractive mappings using the metric projection [15].

To accelerate the convergence of iterative schemes, the inertial technique, inspired by a discretization of second-order dissipative dynamical systems [1, 6, 16], has been widely adopted. This technique, often interpreted as a momentum term, has been successfully incorporated into various algorithms to improve their convergence rate [3, 10, 12, 17, 20, 21].

Another problem of significant interest in nonlinear analysis is the Fixed Point Problem (FPP). Finding a common element of the solution set of a VIP and the fixed point set of a nonlinear mapping is particularly important, as constraints in many mathematical models can be naturally expressed as either a VIP or an FPP. Takahashi and Toyoda [20] pioneered an iterative method for finding a common solution of a VIP for inverse strongly monotone mappings and an FPP for nonexpansive mappings. Since then, numerous algorithms have been developed under milder assumptions [18, 21].

Recently, Singh and Chandok [18] proposed a novel Double Inertial Mann-Type Extragradient Method (DIMTEM). Their algorithm incorporates a double inertial extrapolation step and a non-monotonic self-adaptive step size rule that does not require prior knowledge of the Lipschitz constant of the operator. They established a weak convergence theorem for finding a common solution of a VIP involving a pseudo-monotone mapping and an FPP for a quasi-nonexpansive mapping. Furthermore, they proved strong convergence for VIPs with strongly pseudo-monotone mappings. Numerical experiments demonstrated the computational efficiency and advantages of their method over several existing algorithms.

Inspired and motivated by the work of Singh and Chandok [18], we extend their results to the more general and challenging case of a finite family of quasi-nonexpansive mappings. Our main contributions are as follows: We propose a new iterative algorithm that combines a double inertial Tseng's extragradient method with a Mann-type iteration for a finite family of mappings. We prove a weak convergence theorem for our algorithm under standard conditions, assuming the underlying operator is pseudo-monotone and Lipschitz continuous. We maintain the desirable features of the original DIMTEM, including a non-monotonic self-adaptive step size and the relaxation that one of the inertial parameters can be chosen close to 1.

This paper is organized as follows. In Sect. 2, we recall essential definitions and preliminary lemmas required for the convergence analysis. Section 3 is devoted to presenting our algorithm and proving the main weak convergence theorem. In Section

4, we provide numerical examples to illustrate the behavior of our proposed method. Finally, we give a brief conclusion.

2. Preliminaries

Suppose Σ is a real Hilbert space and $\mathcal{E}$ is a convex and closed subset of Σ. We denote the set of fixed points of the mapping G by $F(G)$. The mapping $G : \Sigma \to \Sigma$ is called

(1) monotone on Σ, if $\langle G(\zeta) - G(\eta), \zeta - \eta\rangle \geq 0, \forall \zeta, \eta \in \Sigma$.
(2) k-strongly pseudo-monotone on Σ if there exists a $k > 0$ such that $\langle G(\zeta), \eta-\zeta\rangle \geq 0$ implies $\langle G(\eta), \eta - \zeta\rangle \geq k\|\zeta - \eta\|^2,\ \forall \zeta, \eta, \in \Sigma$.
(3) pseudo-monotone on Σ if $\langle G(\zeta), \eta - \zeta\rangle \geq 0$ implies $\langle G(\eta), \eta - \zeta\rangle \geq 0\ \forall \zeta, \eta, \in \Sigma$.
(4) G is said to be nonexpansive if

$$\|G(\zeta) - G(\eta)\| \leq \|\zeta - \eta\|, \forall \zeta, \eta \in \Sigma.$$

(5) G is said to be quasi-nonexpansive if

$$\|G(\zeta) - \zeta^\dagger\| \leq \|\zeta - \zeta^\dagger\|, \forall \zeta \in \Sigma, \zeta^\dagger \in F(G).$$

(6) G is said to be L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|G(\zeta) - G(\eta)\| \leq L\|\zeta - \eta\|, \forall \zeta, \eta \in \Sigma.$$

Lemma 2.1. *The following are true in a Hilbert space* Σ

(1) $\|\zeta + \eta\|^2 = \|\zeta\|^2 + \|\eta\|^2 + 2\langle \zeta, \eta\rangle, \forall \zeta, \eta \in \Sigma$,
(2) $\|\zeta + \eta\|^2 \leq \|\zeta\|^2 + 2\langle \eta, \zeta + \eta\rangle, \forall \zeta, \eta \in \Sigma$,
(3) $\|\alpha\zeta+\beta\eta\|^2 = \alpha(\alpha+\beta)\|\zeta\|^2+\beta(\alpha+\beta)\|\eta\|^2-\alpha\beta\|\zeta-\eta\|^2, \forall \zeta, \eta \in \Sigma$, and $\alpha, \beta \in \mathbb{R}$.

Lemma 2.2. [9] *Let* $\{\zeta_n\}$, $\{\eta_n\}$, $\{\varpi_n\}$ *be sequences in* $[0, \infty)$ *such that*

$$\zeta_{n+1} \leq \zeta_n + \varpi_n(\zeta_n - \zeta_{n-1}) + \eta_n, \forall n \geq 1, \sum_{n=1}^{\infty} \eta_n < \infty,$$

and $\exists$ *a real number* ϖ *with* $0 \leq \varpi_n \leq \varpi < 1$, $\forall\ n \in \mathbb{N}$. *Then following hold:*

(1) $\sum_{n=1}^{\infty} [\zeta_n - \zeta_{n-1}]_+ < \infty$, *where* $[\zeta]_+ = \max\{\zeta, 0\}$;
(2) $\exists\ \zeta^\dagger \in [0, \infty)$ *such that* $\lim_{n\to\infty} \zeta_n = \zeta^\dagger$.

Lemma 2.3. [11] *Suppose* $\mathcal{E} \neq \emptyset$ *is a subset of* Σ *and* $\{\zeta_n\}$ *is a sequence in* Σ *such that the following conditions hold:*

(1) $\forall \zeta \in \mathcal{E}$, $\lim_{n\to\infty} \|\zeta_n - \zeta\|$ *exists;*
(2) *every sequentially weak cluster point of* $\{\zeta_n\}$ *is in* $\mathcal{E}$.

Then the sequence $\{\zeta_n\}$ *converges weakly to a point in* $\mathcal{E}$.

Lemma 2.4. [22] *Suppose* $\mathcal{E} \neq \emptyset$ *is a convex and closed subset of* Σ. *Suppose* $G : \Sigma \to \Sigma$ *is pseudo-monotone and* L*-Lipschitz continuous mapping which satisfies*

$$\text{whenever } \{\zeta_n\} \in \mathcal{E}, \zeta_n \rightharpoonup \zeta^*, \text{ one has } \|G(\zeta^*)\| \leq \liminf_{n\to\infty} \|G(\zeta_n)\|.$$

Suppose $\{\alpha_n\}$ *and* $\{\delta_n\}$ *are two real sequences in* $(0, 1)$ *such that* $\{\delta_n\} \subset (c, 1 - \alpha_n)$ *for some* $c > 0$ *and* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Suppose* $\{c_n\}$ *is any sequence generated* *by* $c_{n+1} = (1-\alpha_n-\delta_n)c_n + \delta_n P_{T_n}(c_n - \tau_n G(b_n))$, *where* $b_n = P_{\mathcal{E}}(c_n - \tau_n c_n)$ *and* τ_n *is a monotonic step size. If there exists a subsequence* $\{c_{n_k}\}$ *converging weakly to* $\zeta^* \in \Sigma$ *and* $\lim_{n\to\infty} \|c_{n_k} - b_{n_k}\| = 0$ *then* $\zeta^* \in VI(\mathcal{E}, G)$.

3. Main Result

Condition 3.1. (C1) *The mapping* $G : \Sigma \to \Sigma$ *is pseudo-monotone and L-Lipschitz continuous on* Σ *and* G *satisfies the following condition: whenever* $\{\zeta_n\} \subset \Sigma$ *and* $\zeta_n \rightharpoonup \zeta^\dagger$, *one has* $\|G(\zeta^\dagger)\| \leq \liminf_{n\to\infty} \|G(\zeta_n)\|$.

(C2) $\Phi_i : \Sigma \to \Sigma$ *is a finite family of quasi-nonexpansive mappings.*

(C3) *The solution set* $VI(\mathcal{E}, G) \bigcap_{i=1}^{m} F(\Phi_i)$ *is nonempty.*

(C4) $0 \leq \lambda_n \leq \lambda_{n+1} \leq 1$.

(C5) $0 \leq \mu < \min\left\{\frac{\varepsilon-\sqrt{2\varepsilon}}{\varepsilon}, \theta_1\right\}, \varepsilon \in (2,\infty)$.

(C6) $0 < \beta \leq \beta_n \leq \beta_{n+1} < \frac{1}{1+\varepsilon}, \varepsilon \in (2,\infty)$.

Algorithm 3.2. *Choose* $\rho \in (0,1)$, $\sigma_1 > 0$ *and suppose* $\{\nu_n\}$ *is a sequence of nonnegative real numbers such that* $\sum_{n=1}^{\infty} \nu_n < \infty$. *Suppose* $\zeta_0, \zeta_1 \in \Sigma$ *are arbitrarily chosen, set* $n = 1$.

Step 1:

$$\begin{cases} \varpi_n = \zeta_n + \mu(\zeta_n - \zeta_{n-1}), \\ \eta_n = \zeta_n + \lambda_n(\zeta_n - \zeta_{n-1}), \\ \xi_n = P_{\mathcal{E}}(\eta_n - \sigma_n G(\eta_n)), \end{cases} \tag{2}$$

update the step size σ_{n+1} *as*

$$\sigma_{n+1} = \begin{cases} \min\left\{\sigma_n + \nu_n, \frac{\rho\|\eta_n - \xi_n\|}{\|G(\eta_n) - G(\xi_n)\|}\right\}, if G(\eta_n) \neq G(\xi_n); \\ \sigma_n + \nu_n, \text{otherwise.} \end{cases} \tag{3}$$

Step 2:

$$\zeta_{n+1} = (1-\beta_n)\varpi_n + \beta_n \Phi_i(\omega_n), \tag{4}$$

where $\omega_n = \xi_n - \sigma_n(G(\xi_n) - G(\eta_n))$. *Set* $n = n+1$ *and return to Step 1.*

For each $n \geq 1$, *the index* $i = i(n) \in \{1, 2, \dots, m\}$ *is chosen according to the cyclic control sequence* $i(n) = ((n-1) \bmod m) + 1$. *Thus,* i *depends on the iteration number* n, *and the mappings* $\Phi_1, \Phi_2, \dots, \Phi_m$ *are applied successively in a periodic manner.*

Remark 3.1. (R1) The step size σ_n has lower bound $\min\left\{\frac{\rho}{L}, \sigma_1\right\}$ and the upper bound $\sigma_1 + D$, where $D = \sum_{n=1}^{\infty} \nu_n$. Moreover,

$$\|G(\eta_n) - G(\xi_n)\| \leq \frac{\rho}{\sigma_{n+1}} \|\eta_n - \xi_n\|. \tag{5}$$

(R2) The parameter λ_n can be 1. Also, when ε increases, we see μ approaching because $\lim_{n\to\infty} \frac{\varepsilon-\sqrt{2\varepsilon}}{\epsilon} = 1$. Consequently, we can choose μ as close as to 1.

THEOREM 3.3. *Suppose condition 3.1 satisfies and $(I - \Phi_i)$ is demiclosed at zero. Then the sequence $\{\zeta_n\}$ generated by algorithm 3.2 converges weakly to a point in $VI(\mathcal{E}, G) \bigcap_{i=1}^{m} F(\Phi_i)$.*

Proof. Suppose $\zeta^\dagger \in VI(\mathcal{E}, G) \bigcap_{i=1}^{m} F(\Phi_i)$ then $\langle G(\zeta^\dagger), \xi_n - \zeta^\dagger \rangle \geq 0$ and since the mapping G is pseudo-monotone, we get

$$\langle G(\xi_n), \xi_n - \zeta^\dagger \rangle \geq 0. \tag{6}$$

Using Lemma 2.1, we get

$$\begin{aligned}
\|\omega_n - \zeta^\dagger\|^2 &= \|\xi_n - \sigma_n(G(\xi_n) - G(\eta_n)) - \zeta^\dagger\|^2 \\
&= \|\xi_n - \zeta^\dagger\|^2 + \sigma_n^2\|G(\xi_n) - G(\eta_n)\|^2 - 2\sigma_n\langle \xi_n - \zeta^\dagger, G(\xi_n) - G(\eta_n)\rangle \\
&= \|\xi_n - \eta_n + \eta_n - \zeta^\dagger\|^2 + \sigma_n^2\|G(\xi_n) - G(\eta_n)\|^2 - 2\sigma_n\langle \xi_n - \zeta^\dagger, G(\xi_n) - G(\eta_n)\rangle \\
&= \|\xi_n - \eta_n\|^2 + \|\eta_n - \zeta^\dagger\| + 2\langle \xi_n - \eta_n, \eta_n - \zeta^\dagger\rangle + \sigma_n^2\|G(\xi_n) - G(\eta_n)\|^2 \\
&\quad - 2\sigma_n\langle \xi_n - \zeta^\dagger, G(\xi_n) - G(\eta_n)\rangle \\
&= \|\xi_n - \eta_n\|^2 + \|\eta_n - \zeta^\dagger\| + 2\langle \xi_n - \eta_n, \eta_n - \zeta_n\rangle + 2\langle \xi_n - \eta_n, \xi_n - \zeta^\dagger\rangle \\
&\quad + \sigma_n^2\|G(\xi_n) - G(\eta_n)\|^2 - 2\sigma_n\langle \xi_n - \zeta^\dagger, G(\xi_n) - G(\eta_n)\rangle \\
&= \|\eta_n - \zeta^\dagger\| - \|\xi_n - \eta_n\|^2 + 2\langle \xi_n - \eta_n, \xi_n - \zeta^\dagger\rangle + \sigma_n^2\|G(\xi_n) - G(\eta_n)\|^2 \\
&\quad - 2\sigma_n\langle \xi_n - \zeta^\dagger, G(\xi_n) - G(\eta_n)\rangle. \qquad (7)
\end{aligned}$$

Since $\xi_n = P_{\mathcal{E}}(\eta_n - \sigma_n G(\eta_n))$, we get

$$\langle \xi_n - \eta_n + \sigma_n G(\eta_n), \xi_n - \zeta^\dagger\rangle \leq 0.$$

Moreover

$$\langle \xi_n - \eta_n, \xi_n - \zeta^\dagger\rangle \leq -\sigma_n\langle G(\eta_n), \xi_n - \zeta^\dagger\rangle. \tag{8}$$

Now, if we combine (7), (8) and (5), we will have

$$\begin{aligned}
\|\omega_n - \zeta^\dagger\|^2 &\leq \|\eta_n - \zeta^\dagger\|^2 - \|\eta_n - \xi_n\|^2 - 2\sigma_n\langle G(\eta_n), \xi_n - \zeta^\dagger\rangle \\
&\quad + \sigma_n^2\|G(\xi_n) - G(\eta_n)\|^2 - 2\sigma_n\langle \xi_n - \zeta^\dagger, G(\xi_n) - G(\eta_n)\rangle \\
&= \|\eta_n - \zeta^\dagger\|^2 - \|\eta_n - \xi_n\|^2 + \sigma_n^2\|G(\xi_n) - G(\eta_n)\|^2 - 2\sigma_n\langle \xi_n - \zeta^\dagger, G(\xi_n)\rangle \\
&\leq \|\eta_n - \zeta^\dagger\|^2 - \|\eta_n - \xi_n\|^2 + \frac{\sigma_n^2}{\sigma_{n+1}^2}\rho^2\|\eta_n - \xi_n\|^2 - 2\sigma_n\langle \xi_n - \zeta^\dagger, G(\xi_n)\rangle \\
&\leq \|\eta_n - \zeta^\dagger\|^2 - \left(1 - \frac{\sigma_n^2}{\sigma_{n+1}^2}\rho^2\right)\|\eta_n - \xi_n\|^2 - 2\sigma_n\langle \xi_n - \zeta^\dagger, G(\xi_n)\rangle. \qquad (9)
\end{aligned}$$

Moreover

$$\|\omega_n - \zeta^\dagger\|^2 \leq \|\eta_n - \zeta^\dagger\|^2 - \left(1 - \frac{\sigma_n^2}{\sigma_{n+1}^2}\rho^2\right)\|\eta_n - \xi_n\|^2. \tag{10}$$

Since $\lim_{n\to\infty}\left(1 - \frac{\sigma_n^2}{\sigma_{n+1}^2}\rho^2\right) = 1 - \rho^2 > 0$, then $\exists\ M \geq 1$ such that

$$\|\omega_n - \zeta^\dagger\| \leq \|\eta_n - \zeta^\dagger\| \text{ for all } n \geq M. \tag{11}$$

Again we can also have

$$\begin{aligned}
\|\zeta_{n+1}-\zeta^\dagger\|^2 &= \|(1-\beta_n)\varpi_n+\beta_n\Phi_i(\omega_n)-\zeta^\dagger\|^2\\
&= (1-\beta_n)\|\varpi_n-\zeta^\dagger\|^2+\beta_n\|\Phi_i(\omega_n)-\zeta^\dagger\|^2-(1-\beta_n)\beta_n\|\Phi_i(\omega_n)-\varpi_n\|^2\\
&\le (1-\beta_n)\|\varpi_n-\zeta^\dagger\|^2+\beta_n\|\omega_n-\zeta^\dagger\|^2-(1-\beta_n)\beta_n\|\Phi_i(\omega_n)-\varpi_n\|^2\\
&\le (1-\beta_n)\|\varpi_n-\zeta^\dagger\|^2+\beta_n\|\eta_n-\zeta^\dagger\|^2-(1-\beta_n)\beta_n\|\Phi_i(\omega_n)-\varpi_n\|^2 \qquad (12)
\end{aligned}$$

for all $n \ge M$. Now using (4) to the above equation (12), we get

$$\begin{aligned}
\|\zeta_{n+1}-\zeta^\dagger\|^2 &\le (1-\beta_n)\|\varpi_n-\zeta^\dagger\|^2+\beta_n\|\eta_n-\zeta^\dagger\|^2-(1-\beta_n)\beta_n\frac{1}{\beta_n^2}\|\Phi_i(\omega_n)-\varpi_n\|^2\\
&= (1-\beta_n)\|\zeta_n+\mu(\zeta_n-\zeta_{n-1})-\zeta^\dagger\|^2+\beta_n\|\zeta_n+\lambda_n(\zeta_n-\zeta_{n-1})-\zeta^\dagger\|^2\\
&-\frac{(1-\beta_n)}{\beta_n}\|\zeta_{n+1}-\zeta_n-\mu(\zeta_n-\zeta_{n-1})\|^2\\
&\le (1-\beta_n)\left[(1+\mu)\|\zeta_n-\zeta^\dagger\|^2-\mu\|\zeta_{n-1}-\zeta^\dagger\|^2+\mu(1+\mu)\|\zeta_n-\zeta_{n-1}\|^2\right]\\
&+\beta_n\left[(1+\lambda_n)\|\zeta_n-\zeta^\dagger\|^2-\lambda_n\|\zeta_{n-1}-\zeta^\dagger\|^2+\lambda_n(1+\lambda_n)\|\zeta_n-\zeta_{n-1}\|^2\right]\\
&-\frac{(1-\beta_n)}{\beta_n}\left[(1-\mu)\|\zeta_{n+1}-\zeta_n\|^2+(\mu^2-\mu)\|\zeta_n-\zeta_{n-1}\|^2\right]\\
&= \left((1-\beta_n)(1+\mu)+\beta_n(1+\lambda_n)\right)\|\zeta_n-\zeta^\dagger\|^2\\
&-\left(\mu(1-\beta_n)+\beta_n\lambda_n\right)\|\zeta_{n-1}-\zeta^\dagger\|^2\\
&+\left((1-\beta_n)\mu(1+\mu)+\beta_n\lambda_n(1+\lambda_n)-\frac{(1-\beta_n)(\mu^2-\mu)}{\beta_n}\right)\|\zeta_n-\zeta_{n-1}\|^2\\
&-\frac{(1-\mu)(1-\beta_n)}{\beta_n}\|\zeta_{n+1}-\zeta_n\|^2\\
&= \left(1+\beta_n\lambda_n+\mu(1-\beta_n)\right)\|\zeta_n-\zeta^\dagger\|^2-\left(\beta_n\lambda_n+\mu(1-\beta_n)\right)\|\zeta_{n-1}-\zeta^\dagger\|^2\\
&-a_n\|\zeta_{n+1}-\zeta_n\|^2+b_n\|\zeta_n-\zeta_{n-1}\|^2, \qquad (13)
\end{aligned}$$

here $a_n = \dfrac{(1-\mu)(1-\beta_n)}{\beta_n}$, and $b_n=(1-\beta_n)\mu(1+\mu)+\beta_n\lambda_n(1+\lambda_n)-\dfrac{(1-\beta_n)(\mu^2-\mu)}{\beta_n}$.

Assume

$$\Gamma_n = \|\zeta_n-\zeta^\dagger\|^2-\left(\beta_n\lambda_n+\mu(1-\beta_n)\right)\|\zeta_{n-1}-\zeta^\dagger\|^2+b_n\|\zeta_n-\zeta_{n-1}\|^2.$$

Thus

$$\begin{aligned}
\Gamma_{n+1}-\Gamma_n &= \|\zeta_{n+1}-\zeta_n\|^2-\left(\beta_{n+1}\lambda_{n+1}+\mu(1-\beta_{n+1})\right)\|\zeta_n-\zeta^\dagger\|^2+b_{n+1}\|\zeta_{n+1}-\zeta_n\|^2\\
&-\|\zeta_n-\zeta^\dagger\|^2+\left(\beta_n\lambda_n+\mu(1-\beta_n)\right)\|\zeta_{n-1}-\zeta^\dagger\|^2-b_n\|\zeta_n-\zeta_{n-1}\|^2\\
&\le \left(\beta_n\lambda_n+\mu(1-\beta_n)-\beta_{n+1}\lambda_{n+1}-\mu(1-\beta_{n+1})\right)\|\zeta_n-\zeta^\dagger\|^2-a_n\|\zeta_{n+1}-\zeta_n\|^2\\
&+b_{n+1}\|\zeta_{n+1}-\zeta_n\|^2\\
&= \left(\beta_n(\lambda_n-\mu)-(\lambda_{n+1}-\mu)\beta_{n+1}\right)\|\zeta_n-\zeta^\dagger\|^2-a_n\|\zeta_{n+1}-\zeta_n\|^2\\
&+b_{n+1}\|\zeta_{n+1}-\zeta_n\|^2. \qquad (14)
\end{aligned}$$

Using Condition 3.1, we have $\lambda_n \le \lambda_{n+1}, \beta_n \le \beta_{n+1}$ and $0 \le \mu \le \lambda_1 \le \lambda_n$. It gives $\lambda_n-\mu \ge 0$ and $\lambda_{n+1}-\mu \ge 0$, which further gives $\lambda_n-\mu \le \lambda_{n+1}-\mu$ and

$(\lambda_n - \mu)\beta_n \leq (\lambda_{n+1} - \mu)\beta_{n+1}$ $\forall\, n \geq 1$. From the above equation (14) we get

$$\Gamma_{n+1} - \Gamma_n \leq -a_n\|\zeta_{n+1} - \zeta_n\|^2 + b_{n+1}\|\zeta_{n+1} - \zeta_n\|^2 = -(a_n - b_{n+1})\|\zeta_{n+1} - \zeta_n\|^2. \tag{15}$$

Now,

$$\begin{aligned} a_n - b_{n+1} &= \frac{(1-\beta_n)(1-\mu)}{\beta_n} - (1-\beta_{n+1})\mu(1+\mu) - \beta_{n+1}\lambda_{n+1}(1+\lambda_{n+1}) \\ &\quad + \frac{(1-\beta_{n+1})(\mu^2-\mu)}{\beta_{n+1}} \\ &\geq \varepsilon(1-\mu) + \varepsilon(\mu^2-\mu) - (1-\beta_{n+1})\mu(1+\mu) - 2\beta_{n+1} \\ &> \varepsilon(1-\mu) + \varepsilon(\mu^2-\mu) - 2(1-\beta_{n+1}) - 2\beta_{n+1} \\ &= \varepsilon(1-\mu) + \varepsilon(\mu^2-\mu) - 2 \\ &= \varepsilon\mu^2 - 2\varepsilon\mu + \varepsilon - 2. \end{aligned} \tag{16}$$

Since from the Condition 3.1 $\mu < \frac{\varepsilon - \sqrt{2\varepsilon}}{\varepsilon}$. Using this condition in (16), we get $\varepsilon\mu^2 - 2\varepsilon\mu + \varepsilon - 2 > 0$. From (15) and (16), we get

$$\Gamma_{n+1} - \Gamma_n \leq -(\varepsilon\mu^2 - 2\varepsilon\mu + \varepsilon - 2)\|\zeta_{n+1} - \zeta_n\|^2,$$

let $a = \varepsilon\mu^2 - 2\varepsilon\mu + \varepsilon - 2$ then

$$\Gamma_{n+1} - \Gamma_n \leq -a\|\zeta_{n+1} - \zeta_n\|^2. \tag{17}$$

This gives us the the sequence $\{\Gamma_n\}$ is non increasing and

$$\begin{aligned} \Gamma_n &= \|\zeta_n - \zeta^\dagger\|^2 - (\beta_n\lambda_n + \mu(1-\beta_n))\|\zeta_{n-1} - \zeta^\dagger\|^2 + b_n\|\zeta_n - \zeta_{n-1}\|^2 \\ &\geq \|\zeta_n - \zeta^\dagger\|^2 - (\beta_n\lambda_n + \mu(1-\beta_n))\|\zeta_{n-1} - \zeta^\dagger\|^2. \end{aligned} \tag{18}$$

It gives

$$\begin{aligned} \|\zeta_n - \zeta^\dagger\|^2 &\leq (\beta_n\lambda_n + \mu(1-\beta_n))\|\zeta_{n-1} - \zeta^\dagger\|^2 + \Gamma_n \\ &\leq \left(\frac{1}{1+\varepsilon} + \mu(1-\beta)\right)\|\zeta_{n-1} - \zeta^\dagger\|^2 + \Gamma_n \\ &\leq \left(\frac{1}{1+\varepsilon} + \mu(1-\beta)\right)\|\zeta_{n-1} - \zeta^\dagger\|^2 + \Gamma_1 \\ &\ \vdots \\ &\leq \Upsilon^n\|\zeta_0 - \zeta^\dagger\|^2 + (1 + \Upsilon + \Upsilon^2 + \ldots + \Upsilon^{n-1})\Gamma_1 \\ &\leq \Upsilon^n\|\zeta_0 - \zeta^\dagger\|^2 + \frac{\Gamma_1}{1-\Upsilon}, \end{aligned} \tag{19}$$

here $\Upsilon = \frac{1}{1+\varepsilon} + \mu(1-\beta) \in (0,1)$. Hence the sequence $\{\|\zeta_n - \zeta^\dagger\|\}$ is bounded and so the sequence $\{\zeta_n\}$. Moreover

$$\begin{aligned} \Gamma_{n+1} &= \|\zeta_{n+1} - \zeta^\dagger\|^2 - (\beta_{n+1}\lambda_{n+1} + \mu(1-\beta_{n+1}))\|\zeta_n - \zeta^\dagger\|^2 + b_{n+1}\|\zeta_{n+1} - \zeta_n\|^2 \\ &\geq -(\beta_{n+1}\lambda_{n+1} + \mu(1-\beta_{n+1}))\|\zeta_n - \zeta^\dagger\|^2. \end{aligned} \tag{20}$$

From (19) and (20), we get

$$\begin{aligned} -\Gamma_{n+1} &\leq (\beta_{n+1}\lambda_{n+1} + \mu(1-\beta_{n+1}))\|\zeta_n - \zeta^\dagger\|^2 \\ &\leq \Upsilon\|\zeta_n - \zeta^\dagger\|^2 \end{aligned}$$

$$\leq \Upsilon^{n+1}\|\zeta_0 - \zeta^\dagger\|^2 + \Upsilon\frac{\Gamma_1}{1-\Upsilon}. \tag{21}$$

Also from (17), we get

$$\Gamma_{n+1} - \Gamma_n \leq -a\|\zeta_{n+1} - \zeta_n\|^2$$
$$a\|\zeta_{n+1} - \zeta_n\|^2 \leq \Gamma_n - \Gamma_{n+1}. \tag{22}$$

It gives

$$a\sum_{n=1}^{t}\|\zeta_{n+1} - \zeta_n\|^2 \leq \sum_{n=1}^{t}(\Gamma_n - \Gamma_{n+1}) \tag{23}$$
$$= (\Gamma_1 - \Gamma_{t+1}) = \Upsilon^{n+1}\|\zeta_0 - \zeta^\dagger\|^2 + \frac{\Gamma_1}{1-\Upsilon}. \tag{24}$$

Thus

$$\sum_{n=1}^{\infty}\|\zeta_{n+1} - \zeta_n\|^2 \leq \frac{\Gamma_1}{a(1-\Upsilon)} < \infty. \tag{25}$$

We get

$$\lim_{n\to\infty}\|\zeta_{n+1} - \zeta_n\| = 0. \tag{26}$$

Now we prove that $\lim_{n\to\infty}\|\zeta_n - \zeta^\dagger\|$ exists. Using (13), we get

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^\dagger\|^2 &\leq (1 + \beta_n\lambda_n + \mu(1-\beta_n))\|\zeta_n - \zeta^\dagger\|^2 - (\beta_n\lambda_n + \mu(1-\beta_n))\|\zeta_{n-1} - \zeta^\dagger\|^2 \\
&\quad - a_n\|\zeta_{n+1} - \zeta^\dagger\|^2 + b_n\|\zeta_n - \zeta_{n-1}\|^2 \\
&\leq \|\zeta_n - \zeta^\dagger\|^2 + (\beta_n\lambda_n + \mu(1-\beta_n))\|\zeta_n - \zeta^\dagger\|^2 \\
&\quad - (\beta_n\lambda_n + \mu(1-\beta_n))\|\zeta_{n-1} - \zeta^\dagger\|^2 + b_n\|\zeta_n - \zeta_{n-1}\|^2 \\
&= \|\zeta_n - \zeta^\dagger\|^2 + (\beta_n\lambda_n + \mu(1-\beta_n))(\|\zeta_n - \zeta^\dagger\|^2 - \|\zeta_{n-1} - \zeta^\dagger\|^2) \\
&\quad + b_n\|\zeta_n - \zeta_{n-1}\|^2. \qquad (27)
\end{aligned}$$

As

$$\begin{aligned}
b_n &= (1-\beta_n)\mu(1+\mu) + \beta_n\lambda_n(1+\lambda_n) - \frac{(1-\beta_n)(\mu^2-\mu)}{\beta_n} \\
&\leq (1-a)\mu(1+\mu) + \frac{2}{1+\varepsilon} + \frac{(1-\alpha)(\mu-\mu^2)}{\alpha}.
\end{aligned}$$

We also have

$$\beta_n\lambda_n + \mu(1-\beta_n) \leq \frac{1}{1+\varepsilon} + \mu(1-\beta) < 1.$$

Since $\mu < \frac{\varepsilon}{(1+\varepsilon)(1-\beta)}$, for $\beta \in (0,1)$ and $\varepsilon \in (2,\infty)$. Now using Lemma 2.2 in (27), we get $\lim_{n\to\infty}\|\zeta_n - \zeta^\dagger\|$ exists and let $\lim_{n\to\infty}\|\zeta_n - \zeta^\dagger\| = p$. Now

$$\|\zeta_{n+1} - \varpi_n\| = \|\zeta_{n+1} - \zeta_n - \mu(\zeta_n - \zeta_{n-1})\|$$
$$\leq \|\zeta_{n+1} - \zeta_n\| + \mu\|\zeta_n - \zeta_{n-1}\|. \tag{28}$$

It gives

$$\lim_{n\to\infty}\|\zeta_{n+1} - \varpi_n\| = 0.$$

We can also get

$$\begin{aligned}\|\zeta_{n+1}-\eta_n\| &= \|\zeta_{n+1}-\zeta_n-\lambda_n(\zeta_n-\zeta_{n-1})\|\\ &\le \|\zeta_{n+1}-\zeta_n\|+\lambda_n\|\zeta_n-\zeta_{n-1}\|.\end{aligned}$$

It gives

$$\lim_{n\to\infty}\|\zeta_{n+1}-\eta_n\|=0.$$

Now

$$\begin{aligned}\|\zeta_{n+1}-\varpi_n\| &= \|(1-\beta_n)\varpi_n+\beta_n\Phi_i(\omega_n)-\varpi_n\|\\ &= \|\beta_n(\Phi_i(\omega_n)-\varpi_n)\|\\ &\ge \beta\|\Phi_i(\omega_n)-\varpi_n\|.\end{aligned}$$

It gives

$$\lim_{n\to\infty}\|\Phi_i(\omega_n)-\varpi_n\|=0.$$

Further, we can also have

$$\begin{aligned}\|\eta_n-\varpi_n\| &= \|\zeta_n+\lambda_n(\zeta_n-\zeta_{n-1})-\zeta_n-\mu(\zeta_n-\zeta_{n-1})\|\\ &\le \lambda_n\|\zeta_n-\zeta_{n-1}\|+\mu\|\zeta_n-\zeta_{n-1}\|\\ &\le \|\zeta_n-\zeta_{n-1}\|+\mu\|\zeta_n-\zeta_{n-1}\|.\end{aligned}$$

It gives

$$\lim_{n\to\infty}\|\eta_n-\varpi_n\|=0.$$

Now from (12), we get

$$\begin{aligned}\|\zeta_{n+1}-\zeta^\dagger\|^2 &\le (1-\beta_n)\|\varpi_n-\zeta^\dagger\|^2+\beta_n\|\omega_n-\zeta^\dagger\|^2\\ &= (1-\beta_n)\|\varpi_n-\eta_n\|^2+(1-\beta_n)\|\eta_n-\zeta^\dagger\|^2\\ &\quad +2(1-\beta_n)\langle\varpi_n-\eta_n,\eta_n-\zeta^\dagger\rangle+\beta_n\|\omega_n-\zeta^\dagger\|^2.\end{aligned}\tag{29}$$

Simplifying the above equation

$$\begin{aligned}\beta_n\|\omega_n-\zeta^\dagger\|^2 &\ge \|\zeta_{n+1}-\zeta^\dagger\|^2-(1-\beta_n)\|\varpi_n-\eta_n\|^2-(1-\beta_n)\|\eta_n-\zeta^\dagger\|^2\\ &\quad -2(1-\beta_n)\langle\varpi_n-\eta_n,\eta_n-\zeta^\dagger\rangle\end{aligned}$$

$$\begin{aligned}\|\omega_n-\zeta^\dagger\|^2 &\ge \frac{\|\zeta_{n+1}-\zeta^\dagger\|^2-\|\eta_n-\zeta^\dagger\|^2}{\beta_n}+\|\eta_n-\zeta^\dagger\|^2\\ &\quad -\frac{1-\beta_n}{\beta_n}\|\varpi_n-\eta_n\|^2-2\frac{1-\beta_n}{\beta_n}\|\varpi_n-\eta_n\|\|\eta_n-\zeta^\dagger\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|\zeta_n-\zeta^\dagger\|=p$ it gives $\lim_{n\to\infty}\|\eta_n-\zeta^\dagger\|=p$ and $\lim_{n\to\infty}\|\varpi_n-\eta_n\|=0$. Thus we get

$$\lim_{n\to\infty}\|\omega_n-\zeta^\dagger\|^2\ge\lim_{n\to\infty}\|\eta_n-\zeta^\dagger\|^2=p^2.\tag{30}$$

From (11) we also have

$$\lim_{n\to\infty}\|\omega_n-\zeta^\dagger\|^2\le\lim_{n\to\infty}\|\eta_n-\zeta^\dagger\|^2=p^2.\tag{31}$$

Applying (30) and (31) in (10), we get

$$\left(1-\frac{\sigma_n^2}{\sigma_{n+1}^2}\rho^2\right)\|\eta_n-\xi_n\|^2\le\|\eta_n-\zeta^\dagger\|^2-\|\omega_n-\zeta^\dagger\|^2.$$

Applying limit $n \to \infty$ to the above equation we get

$$\lim_{n\to\infty} \|\eta_n - \xi_n\| = 0. \tag{32}$$

Thus

$$\begin{aligned}\lim_{n\to\infty} \|\omega_n - \xi_n\| &= \lim_{n\to\infty} \sigma_n \|G(\xi_n) - G(\eta_n)\| \\ &\leq \lim_{n\to\infty} \sigma_n L \|\xi_n - \eta_n\| = 0.\end{aligned}$$

Also

$$\begin{aligned}\lim_{n\to\infty} \|\Phi_i(\omega_n) - \eta_n\| &= \lim_{n\to\infty} \|\Phi_i(\omega_n) - \varpi_n + \varpi_n - \eta_n\| \\ &\leq \lim_{n\to\infty} \|\Phi_i(\omega_n) - \varpi_n\| + \lim_{n\to\infty} \|\varpi_n - \eta_n\| = 0.\end{aligned}$$

Moreover

$$\begin{aligned}\|\Phi_i(\omega_n) - \omega_n\| &= \|\Phi_i(\omega_n) - \eta_n + \eta_n - \omega_n\| \\ &\leq \|\Phi_i(\omega_n) - \eta_n\| + \|\eta_n - \omega_n\|.\end{aligned}$$

It gives

$$\lim_{n\to\infty} \|\Phi_i(\omega_n) - \omega_n\| = 0. \tag{33}$$

We can also get $\lim\limits_{n\to\infty} \|\omega_n - \eta_n\| = 0$ and $\lim\limits_{n\to\infty} \|\omega_n - \varpi_n\| = 0$. Since the sequence $\{\zeta_n\}$ is bounded so there exists a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that $\{\zeta_{n_k}\}$ converges weakly to a point $\zeta^* \in \Sigma$. Thus we have $\{\eta_{n_k}\}$ and $\{\omega_{n_k}\}$ also converges weakly to $\zeta^* \in \Sigma$. Using the demiclosedness of $(I - \Phi_i)$, from (33), we get $\zeta^* \in \bigcap\limits_{i=1}^{m} F(\Phi_i)$. Since $\|\eta_{n_k} - \omega_{n_k}\| = 0$, using Lemma 2.4, we get $\zeta^\dagger \in VI(\mathcal{E}, G)$. Therefore we have proved that $\zeta^\dagger \in VI(\mathcal{E}, G) \bigcap\limits_{i=1}^{m} F(\Phi_i)$, $\lim\limits_{n\to\infty} \|\zeta_n - \zeta^\dagger\|$ exists and each sequential weak cluster point of the sequence $\{\zeta_n\}$ is in $VI(\mathcal{E}, G) \bigcap\limits_{i=1}^{m} F(\Phi_i)$. So using Lemma 2.3 we get that the sequence $\{\zeta_n\}$ converges weakly to a point in $VI(\mathcal{E}, G) \bigcap\limits_{i=1}^{m} F(\Phi_i)$. □

4. Examples

In this section, first we present an example which is a quasi-nonexpansive mapping but fails to satisfy the nonexpansive condition and second we present an example which satisfies all the conditions of Theorem 3.3. Later, we also present the convergence behaviour of our algorithm 3.2 for different choices of initial guesses and different values of the sequences $\{\beta_n\}$, $\{\lambda_n\}$, $\{\nu_n\}$ with $\sigma_1 = 0.8, \mu = 0.5, \rho = 0.6$.

Example 4.1. Suppose $\Sigma = \mathbb{R}^2$ and $\mathcal{E} = \{\zeta = (\zeta_1, \zeta_2) \in [0,1] \times [0,1]\}$ be the subset of Σ with the norm $\|\zeta\| = \|(\zeta_1, \zeta_2)\| = (|\zeta_1|^2 + |\zeta_2|^2)^{1/2}$. Let $\Phi : \mathcal{E} \to \mathcal{E}$ is defined by

$$\Phi(\zeta_1, \zeta_2) = \begin{cases} (1 - \zeta_1, 1 - \zeta_2), & (\zeta_1, \zeta_2) \in \left[0, \frac{1}{2}\right] \times [0,1], \\ \frac{1}{3}(1 + \zeta_1, 1 + \zeta_2), & (\zeta_1, \zeta_2) \in \left(\frac{1}{2}, 1\right] \times [0,1]. \end{cases}$$

Clearly, only $\left(\frac{1}{2}, \frac{1}{2}\right)$ is the fixed point of mapping Φ. So we have

$$\|\Phi(\zeta) - \zeta^{\dagger}\| = \left\| \frac{1}{2} - \zeta_1, \frac{1}{2} - \zeta_2 \right\| \leq \|\zeta - \zeta^{\dagger}\|,$$

and Φ is a quasi-nonexpansive mapping. On the other hand, for $\zeta = (0,0)$, $\eta = \left(\frac{51}{100}, \frac{25}{100}\right)$, we have

$$\|\Phi(\zeta) - \Phi(\eta)\| = 0.766 > 0.567 = \|\zeta - \eta\|.$$

Therefore Φ is not a nonexpansive mapping.

EXAMPLE 4.2. Suppose $\Sigma = \mathbb{R}$ and $\Phi_1 : \Sigma \to \Sigma$ be a mapping defined as

$$\Phi_1(\zeta) = \begin{cases} \frac{\zeta}{2} \cos \frac{1}{\zeta}, \zeta \neq 0, \\ 0, \zeta = 0. \end{cases}$$

Clearly, only 0 is the fixed point of above mapping Φ_1. So we have

$$\|\Phi_1(\zeta) - \zeta^{\dagger}\| = \|\Phi_1(\zeta) - 0\| = \left|\frac{\zeta}{2}\right| \left|\cos \frac{1}{\zeta}\right| \leq \frac{|\zeta|}{2} < |\zeta| = \|\zeta - \zeta^{\dagger}\|,$$

and hence Φ_1 is a quasi-nonexpansive mapping. For $\zeta = \frac{2}{3\pi}$, $\eta = \frac{1}{\pi}$,

$$\|\Phi_1(\zeta) - \Phi_1(\eta)\| = \frac{1}{2\pi} > \frac{1}{3\pi} = \|\zeta - \eta\|.$$

Hence Φ_1 is not a nonexpansive mapping. Now define $\Phi_2 : \Sigma \to \Sigma$ be a mapping defined as

$$\Phi_2(\zeta) = \begin{cases} \frac{1}{2}\zeta, \zeta \geq 0, \\ 2\zeta, \zeta < 0. \end{cases}$$

Clearly, only 0 is the fixed point of above mapping Φ_2 and it is also a quasi-nonexpansive mapping. But if we take $\zeta = 1, \eta = -1$, then Φ_2 is not a nonexpansive mapping

$$\|\Phi_2(\zeta) - \Phi_2(\eta)\| = \frac{5}{2} > 2 = \|\zeta - \eta\|.$$

We have $\bigcap\limits_{i=1}^{2} F(\Phi_i) = \{0\}$. Now define a mapping $G : \Sigma \to \Sigma$ as $G(\zeta) = \zeta + \sin(\zeta)$. The mapping G is monotone and 2-Lipschitz continuous for all values of $\mathbb{R}$. Since example 4.2 satisfies all the conditions of Theorem 3.3, so the sequence $\{\zeta_n\}$ generated by Algorithm 3.2 converges weakly to a point in $VI(\mathcal{E}, G) \bigcap\limits_{i=1}^{2} F(\Phi_i) = VI(\mathcal{E}, G) \bigcap \{0\}$.

Now we present the convergence behaviour of our algorithm 3.2 with $\Sigma = \mathbb{R}$, $G : \Sigma \to \Sigma$ defined as $G(\zeta) = \zeta + \sin(\zeta)$, $\Phi : \Sigma \to \Sigma$ defined as $\Phi(\zeta) = \begin{cases} \frac{1}{2}\zeta, \zeta \geq 0, \\ 2\zeta, \zeta < 0. \end{cases}$ for different choices of initial guesses and different values of the sequences $\{\beta_n\}$, $\{\lambda_n\}$, $\{\nu_n\}$ with $\sigma_1 = 0.8, \mu = 0.5, \rho = 0.6$.

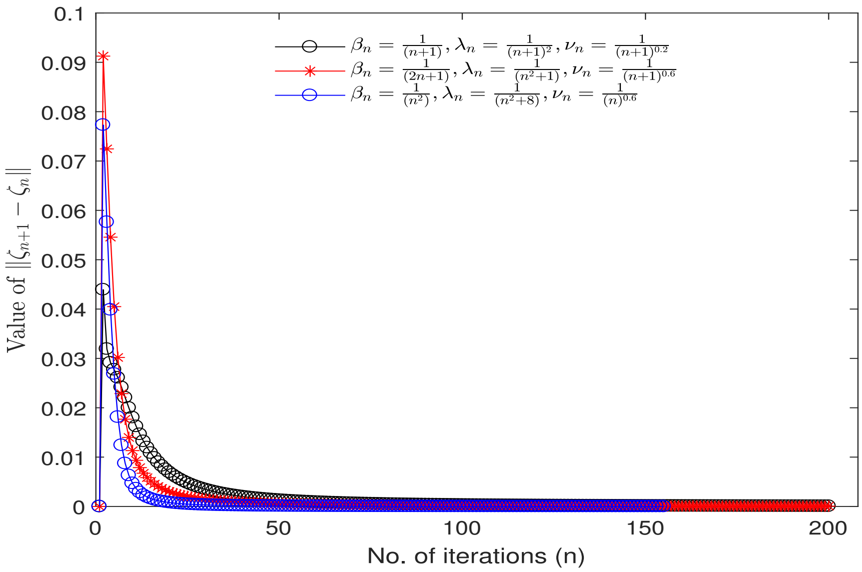

FIGURE 1. Influence of Coefficients.

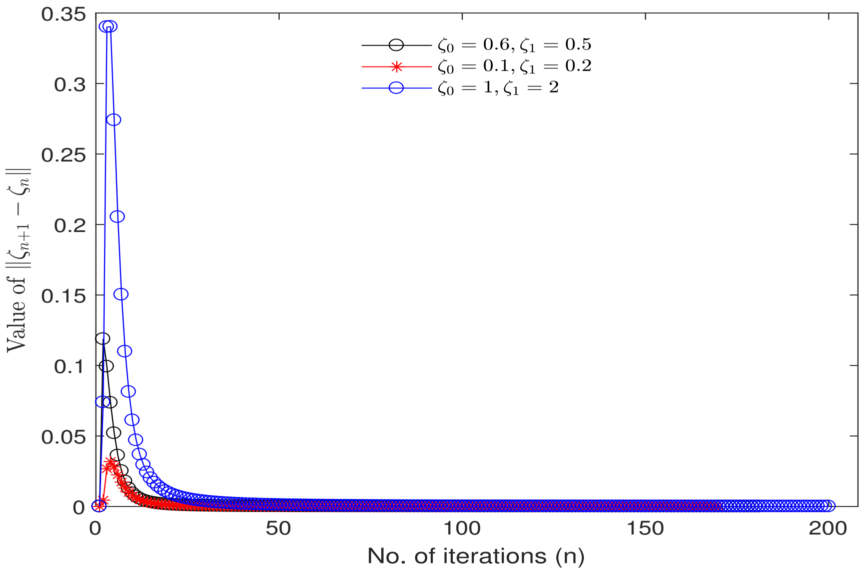

FIGURE 2. Influence of initial guesses.

Conclusion

In this paper, we have introduced a new double inertial Mann-type extragradient algorithm and proved that the sequence generated by our algorithm converges weakly to a common solution to a variational inequality problem and a fixed point of a finite family of quasi-nonexpansive mappings. In this way, we have extended and generalized results which are already present in the literature. We have also presented a couple of non trivial examples and convergence behaviour of our algorithm for different choices of coefficients and initial guesses.

Data Availability

There is no any availability of data.

References

[1] F. Alvarez and H. Attouch, *An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping*, Set-Valued Anal. **9** (1-2) (2001), 3–11.
https://doi.org/10.1023/A:1011253113155

[2] Y. Censor, A. Gibali, and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl. **148** (2) (2011), 318–335.
https://doi.org/10.1007/s10957-010-9757-3

[3] M. Dilshad, F. M. Alamrani, A. Alamer, E. Alshaban, and M. G. Alshehri, *Viscosity-type inertial iterative methods for variational inclusion and fixed point problems*, AIMS Math. **9** (7) (2024), 18553–18573.
https://doi.org/10.3934/math.2024903

[4] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York (2003).

[5] D. Filali, M. Dilshad, M. Akram, M. F. Khan, and S. S. Irfan, *Hybrid inertial viscosity-type forward-backward splitting algorithms for variational inclusion problems*, AIMS Math. **10** (12) (2025), 28829–28860.
https://doi.org/10.3934/math.20251269

[6] D. Filali, M. Dilshad, A. F. Y. Alfaifi, and M. Akram, *Hybrid inertial self-adaptive iterative methods for split variational inclusion problems*, Axioms **14** (5) (2025), 373.
https://doi.org/10.3390/axioms14050373

[7] P. T. Harker, *A variational inequality approach for the determination of oligopolistic market equilibrium*, Math. Programming **30** (1) (1984), 105–111.
https://doi.org/10.1007/BF02591802

[8] G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Matekon **12** (1976), 747–756.

[9] P. E. Maingé, *Convergence theorems for inertial KM-type algorithms*, J. Comput. Appl. Math. **219** (1) (2008), 223–236.
https://doi.org/10.1016/j.cam.2007.07.021

[10] F. O. Nwawuru, O. K. Narain, M. Dilshad, and J. N. Ezeora, *Splitting method involving two-step inertial for solving inclusion and fixed point problems with applications*, Fixed Point Theory Algorithms Sci. Eng. **2025** (2025), Paper No. 8, 28 pp.
https://doi.org/10.1186/s13663-025-00781-w

[11] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (4) (1967), 591–597.
https://doi.org/10.1090/S0002-9904-1967-11761-0

[12] P. Patel and R. Shukla, *Solving fixed point problems and variational inclusions using viscosity approximations*, Aust. J. Math. Anal. Appl. **21** (2) (2024), Art. 8, 13 pp.

[13] P. Patel and R. Shukla, *Viscosity approximation for split equality generalized mixed equilibrium problems with semigroups of nonexpansive mappings*, Abstr. Appl. Anal. **2024** (2024), Art. ID 2812752, 13 pp.
https://doi.org/10.1155/aaa/2812752

[14] P. Patel and R. Shukla, *Viscosity approximation methods for generalized modification of the system of equilibrium problem and fixed point problems of an infinite family of nonexpansive mappings*, Math. Stat. **12** (4) (2024), 339–347.
https://doi.org/10.13189/ms.2024.120405

[15] P. Patel and R. Shukla, *Hierarchical fixed point results for a countable family of strict pseudo-contractive mappings in Hadamard manifolds*, European J. Pure Appl. Math. **18** (2) (2025), 5951.
https://doi.org/10.29020/nybg.ejpam.v18i2.5951

[16] B. T. Polyak, *Some methods of speeding up the convergence of iteration methods*, USSR Comput. Math. Math. Phys. **4** (5) (1964), 1–17.

[17] Y. Shehu, O. S. Iyiola, and S. Reich, *A modified inertial subgradient extragradient method for solving variational inequalities*, Optim. Eng. **23** (1) (2022), 421–449.
https://doi.org/10.1007/s11081-020-09593-w

[18] W. Singh and S. Chandok, *Mann-type extragradient algorithm for solving variational inequality and fixed point problems*, Comput. Appl. Math. **43** (4) (2024), Paper No. 259, 25 pp.
https://doi.org/10.1007/s40314-024-02785-5

[19] G. Stampacchia, *Formes bilinéaires coercitives sur les ensembles convexes*, C. R. Acad. Sci. Paris **258** (1964), 4413–4416.

[20] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2) (2003), 417–428.
https://doi.org/10.1023/A:1025407607560

[21] D. V. Thong and D. V. Hieu, *Mann-type algorithms for variational inequality problems and fixed point problems*, Optimization **69** (10) (2020), 2305–2326.
https://doi.org/10.1080/02331934.2019.1692207

[22] D. V. Thong, J. Yang, Y. J. Cho, and T. M. Rassias, *Explicit extragradient-like method with adaptive stepsizes for pseudomonotone variational inequalities*, Optim. Lett. **15** (6) (2021), 2181–2199.
https://doi.org/10.1007/s11590-020-01678-w

[23] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim. **38** (2) (2000), 431–446.
https://doi.org/10.1137/S0363012998338806

Prashant Patel
Department of Mathematics, School of Advanced Sciences, VIT-AP University
Inavolu, Beside AP Secretariat, Amaravati, 522237, Andhra Pradesh, India
E-mail: prashant.patel9999@gmail.com, prashant.p@vitap.ac.in

Rahul Shukla
Department of Mathematical Sciences & Computing, Mthatha 5117,
Walter Sisulu University, South Africa
E-mail: rshukla@wsu.ac.za